Combinatorics of colored factorizations, flow polytopes and of matrices over finite fields

by

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Abstract

In the first part of this thesis we study factorizations of the permutation $(1, 2, \ldots, n)$ into k factors of given cycle type. Using representation theory, Jackson obtained for each k an elegant formula for counting these factorizations according to the number of cycles of each factor. For the case $k = 2$, Bernardi gave a bijection between these factorizations and tree-rooted maps; certain graphs embedded on surfaces with a distinguished spanning tree. This type of bijection also applies to all k and we use it to show a symmetry property of a refinement of Jackson's formula first exhibited in the case $k = 2, 3$ by Morales and Vassilieva.

We then give applications of this symmetry property. First, we study the mixing properties of permutations obtained as a product of two uniformly random permutations of fixed cycle types. For instance, we give an exact formula for the probability that elements $1, 2, \ldots, k$ are in distinct cycles of the random permutation of $\{1, 2, \ldots, n\}$ obtained as product of two uniformly random n-cycles. Second, we use the symmetry to give a short bijective proof of the number of planar trees and cacti with given vertex degree distribution calculated by Goulden and Jackson.

In the second part we establish the relationship between volumes of flow polytopes associated to signed graphs and the Kostant partition function. A special case of this relationship, namely, when the graphs are signless, has been studied combinatorially by Postnikov and Stanley and by Baldoni and Vergne using residues. As a special family of flow polytopes, we study the Chan-Robbins-Yuen polytope whose volume is the product of the consecutive Catalan numbers. We introduce generalizations of this polytope and give intriguing conjectures about their volume.

In the third part we consider the problem of finding the number of matrices over a finite field with a certain rank and with support that avoids a subset of the entries. These matrices are a q-analogue of permutations with restricted positions (i.e., rook placements). Extending a result of Haglund, we show that when the set of entries is a skew Young diagram, the numbers, up to a power of $q - 1$, are polynomials with nonnegative coefficients. We apply this result to the case when the set of entries is the Rothe diagram of a permutation. We end by giving conjectures connecting invertible matrices whose support avoids a Rothe diagram and Poincaré polynomials of the strong Bruhat order.

Thesis Supervisor: Alexander Postnikov
Title: Associate Professor

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This thesis is dedicated to the memory of Manuel Caro and Nicolas E. Del Castillo.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

One of the most fundamental structures in enumerative and algebraic combinatorics are permutations. In each of the three parts of this thesis we study different aspects of them; see Figure 1-1. In Chapter 2 we study the number of factorizations of permutations into k factors. In the other two chapters we view permutations as permutation matrices. In Chapter 3 we study the volumes of certain polytopes called flow polytopes. An important example of these polytopes is a convex hull of certain permutation matrices. And in Chapter 4 we view a $n \times n$ permutation matrix as a placement of n non-attacking rooks and look at a q-analogue of these objects: invertible matrices over finite fields. Next we give an overview of the problems we study and the main results in each chapter.

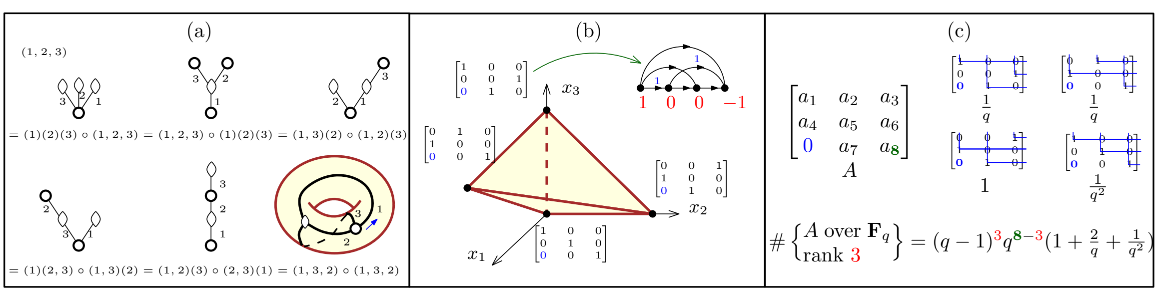


Figure 1-1: Examples of (a) factorizations of the cycle $(1, 2, 3)$ viewed as graphs embedded in surfaces, (b) a projection of a type A_3 Chan-Robbins-Yuen polytope, a flow polytope that is a convex hull of four 3×3 permutation matrices, and (c) using those four permutation matrices to count invertible 3×3 matrices over $\mathbf{F}_q$ with the entry $(3, 1)$ forced to be zero.

1.1 Colored factorizations of permutations and their symmetries

In Chapter 2 we study the enumeration of factorizations of a fixed n-cycle in the symmetric group $\mathfrak{S}_n$ into k permutations, each with a given number of cycles or a specific cycle type. That is, given positive integers $q_1, \ldots, q_k \geq 0$, let $k^{(n)}_{q_1,\ldots,q_k}$ be the number of factorizations $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k$ of the long cycle $(1, 2, \ldots, n)$, where π_t is a permutation with q_t cycles. This number is called a **connection coefficient** of $\mathfrak{S}_n$. With an additional transitivity condition, these numbers also count certain graphs, called *constellations*, embedded on orientable surfaces [36] (see Figure 1-1 (a)).

These connection coefficients can be computed using representation theory of $\mathfrak{S}_n$ (see (2.2.11)), but except for a few cases there are no explicit or cancellation-free expressions for them. We enumerate them indirectly using a coloring argument: for each permutation we color the cycles allowing repeated colors and then count the resulting *colored* factorizations by the number of colors used. Such coloring arguments have been widely studied for related problems (for example by Lass [38], Goulden-Nica [26], Schaeffer-Vassilieva [55], and Bernardi [6]). Using a very close analogue of a bijection of Bernardi [6] we show that these colored factorizations are in bijection with *tree-rooted constellations*, certain graphs embedded on surfaces with a distinguished spanning tree. These structures are more manageable to count and have interesting symmetry properties.

Explicitly, let $\mathcal{C}^{(n)}_{p_1,\ldots,p_k}$ be the set of *colored k-factorizations*. These are factorizations $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k$ of $(1, 2, \ldots, n)$ where the cycles of π_t are colored with $\{1, 2, \ldots, p_t\}$ (all colors are used but two cycles of π_t can have the same color). One can obtain $\#\mathcal{C}^{(n)}_{p_1,\ldots,p_k}$ from $k^{(n)}_{q_1,\ldots,q_k}$ and vice versa via linear integral equations involving Stirling numbers.

Using irreducible characters of $\mathfrak{S}_n$, Jackson [32] showed a surprising simple relation between $\#\mathcal{C}^{(n)}_{p_1,\ldots,p_k}$ and certain k-tuples of sets with empty intersection.

Theorem 2.1.4 (Jackson [32])**.** *The number of colored k-factorizations in $\mathcal{C}^1_{p_1,\ldots,p_k}(n)$ satisfies*

$$\#\mathcal{C}^{(n)}_{p_1,\ldots,p_k} = n!^{k-1} \cdot M^{(n-1)}_{p_1-1,\ldots,p_k-1}, \tag{1.1.1}$$

where $M^{(n)}_{p_1,\ldots,p_k}$ is the number of n-tuples $(R_1, \ldots, R_n)$ of strict subsets R_t of $[k]$ such that each integer $t \in [k]$ appears in exactly p_t of the subsets $R_1, \ldots, R_n$.

This formula had resisted a combinatorial approach except for the nontrivial cases $k = 2$ and $k = 3$ proven bijectively by Schaeffer and Vassilieva in [55] and [54] respectively. In [10, 9], joint work with O. Bernardi, we prove this result combinatorially. In Chapter 2 we focus on the first part of this proof: the *symmetry of colored factorizations.*

Symmetry: Next we consider a refinement of the colored factorizations in $\mathcal{C}^{(n)}_{p_1,\ldots,p_k}$ by the number of elements of $\{1, 2, \ldots, n\}$ in cycles of π_t with the same color. That is, if $\gamma^{(t)}$ is a composition of n with p_t parts, then $\mathcal{C}_{\gamma^{(1)},\cdots,\gamma^{(k)}}$ is the subset of $\mathcal{C}^{(n)}_{p_1,\cdots,p_k}$ with colored factorizations where $\gamma^{(t)}_j$ elements of π_t are colored j.

In [46, 48], joint work with E. Vassilieva, we extended the construction in [55, 54] to give Bijective formulas for $\#\mathcal{C}_{\gamma^{(1)},\dots,\gamma^{(k)}}$ for $k=2$ and $k=3$. For example, when $k=2$ we have: $\#\mathcal{C}_{\gamma^{(1)},\gamma^{(2)}} = n(n-p_1)!(n-p_2)!/(n+1-p_1-p_2)!$. Notice that $\#\mathcal{C}_{\gamma^{(1)},\gamma^{(2)}}$ only depends on n and the number of parts of $\gamma^{(1)}$ and $\gamma^{(2)}$. In Section 2.3 (based in [10, Sec. 3,4]) we show that this kind of symmetry holds for $\#\mathcal{C}_{\gamma^{(1)},\dots,\gamma^{(k)}}$ in general:

Theorem 2.1.6 (Symmetry of colored factorizations)**.** *Let $\gamma^{(1)},\delta^{(1)},\dots,\gamma^{(k)},\delta^{(k)}$ be compositions of n. If for every $t\in[k]$ the length of the compositions $\gamma^{(t)}$ and $\delta^{(t)}$ are equal, then $\#\mathcal{C}_{\gamma^{(1)},\dots,\gamma^{(k)}} = \#\mathcal{C}_{\delta^{(1)},\dots,\delta^{(k)}}$.*

This symmetry can be shown algebraically (see Section 2.2) but this approach provides little insight. In Section 2.3 we use an analogue of a bijection of Bernardi [6] to exhibit this symmetry elucidating why colored factorizations have this property.

In the rest of this chapter we present the following applications of this symmetry result:

1. We can compute $\#\mathcal{C}^1_{\gamma^{(1)},\dots,\gamma^{(k)}}$ for small k like $2,3$ choosing easy compositions $\gamma^{(t)}$ like *hooks* $(1^{p_t-1}, n+1-p_t)$ (see Section 2.4). It also implies that $\#\mathcal{C}^1_{\gamma^{(1)},\dots,\gamma^{(k)}} = \#\mathcal{C}^{(n)}_{p_1,\dots,p_k}/\prod_{t=1}^k \binom{n-1}{p_t-1}$, since there are $\binom{n-1}{p_t-1}$ compositions of n with p_t parts.
2. Another application of the symmetry is to easily enumerate planar objects like rooted trees and cacti [8]. Then by looking at hooks for special values of p_t, one can easily recover classical numbers like the Narayana numbers, and the number of rooted planar k-cacti calculated in [24] (see Section 2.5).
3. Lastly, we use the bijections and symmetries to study **separation probabilities** for products of permutations. The archetypal question can be stated as follows: *In the symmetric group $\mathfrak{S}_n$, what is the probability that the elements $1,2,\dots,r$ are in distinct cycles of the product of two n-cycles chosen uniformly randomly?* The answer is very elegant: the probability is $\frac{1}{r!}$ if $n-r$ is odd and $\frac{1}{r!}+\frac{2}{(r-2)!(n-r+1)(n+r)}$ if $n-r$ is even. This result was originally conjectured by Bóna [13] for $r=2$ and n odd and subsequently Du and Stanley proved for all r and n and proposed additional conjectures [59]. In Section 2.6 (based on [7], joint work with O. Bernardi, R.X. Du, and R.P. Stanley) we prove this result and the conjectures (Corollary 2.6.15) using the symmetry and the formula for $\#\mathcal{C}_{\gamma^{(1)},\gamma^{(2)}}$.

1.2 Flow polytopes and the Kostant partition function

In Chapter 3, based on [44] joint work with K. Mészáros, we study the volumes of a family of polytopes called **flow polytopes** of graphs.

Given a collection X of m vectors and a vector $\mathbf{a}$ all in $\mathbb{Z}^n$, let $k_X(\mathbf{a})$ be the number of ways of writing $\mathbf{a}$ as an $\mathbb{N}$-linear combination of vectors in X. The function $k_X(\mathbf{a})$ is called a **vector partition function**. The most salient property of vector partition functions is that they are piecewise (quasi) polynomial [17, 62]. Also, this function can be interpreted as the number of lattice points of a polytope $\mathcal{F}_X(\mathbf{a}) = \{\mathbf{u} \in \mathbb{R}^m \mid M_X\mathbf{u} = \mathbf{a}, u_i \geq 0\}$, where M_X is the $n \times m$ matrix whose columns are the vectors in X.

An important case is when X is the set of positive roots of a root system, for example for type A_{n-1} we have $X = \{e_i - e_j \mid 1 \leq i < j \leq n\}$ (where e_i is the ith standard vector), and for other types like D_n we have $X = \{e_i \pm e_j \mid 1 \leq i < j \leq n\}$. In this case, the partition function is called the **Kostant partition function**. These partition functions are very useful in representation theory for calculations of multiplicities of weights and tensor products.

If G is a directed acyclic graph on n vertices, let X_G be the multiset of vectors $e_i - e_j$ for each edge (i, j) in G with $i < j$. Note that X_G is also a subset of the positive roots of A_{n-1} mentioned above. The polytope $\mathcal{F}_G(\mathbf{a}) := \mathcal{F}_{X_G}(\mathbf{a})$ is called a **flow polytope** since it consists of nonnegative real flows on the directed edges of G such that the **netflow** on vertex i is a_i.

Postnikov and Stanley [53, 58] and Baldoni-Vergne [2, 4, 3] showed that when $\mathbf{a}$ is the highest root $e_1 - e_n$ then the volume of $\mathcal{F}_G(e_1 - e_n)$ is given by the value of the partition function $k_G(\mathbf{a}')$ where $\mathbf{a}'$ only depends on G. This means that the volume of a flow polytope is given by the number of lattice points/integer flows of a very similar flow polytope.

One natural question to ask is whether there is an analogue of this result for other classical root systems. For this setting, we work with *signed* graphs G that have negative edges $(i, j, -)$ corresponding to the roots $e_i - e_j$ $(i < j)$ and positive edges $(i, j, +)$ corresponding to the roots $e_i + e_j$ $(i < j)$. For this graph, the flow polytope $\mathcal{F}_G(\mathbf{a})$ is defined accordingly. We show that when $\mathbf{a} = 2e_1$, the highest type C root, then the volume of $\mathcal{F}_{G^\pm}(2e_1)$ is given by a variant of the partition function:

Theorem 3.6.16. *Given a signed graph G with vertices $\{1, 2, \ldots, n\}$ then the volume of the flow polytope $\mathcal{F}_G(2e_1)$ is*

$$\mathrm{vol}(\mathcal{F}_G(2e_1)) = k_G^{dyn}(0, d_2, \ldots, d_{n-1}, d_n),$$

where d_i is the number of incoming negative edges to vertex i minus one, and k_G^{dyn} has the following generating series:

$$\sum_{\mathbf{a} \in \mathbb{Z}^n} k_G^{dyn}(\mathbf{a})\mathbf{x}^{\mathbf{a}} = \prod_{\text{edge } (i,j,-) \text{ in } G} (1 - x_i x_j^{-1})^{-1} \prod_{\text{edge } (i,j,+) \text{ in } G} (1 - x_i - x_j)^{-1}. \quad (1.2.1)$$

Note that k_G^{dyn} is not the Kostant partition function for X_G. However, k_G^{dyn} can be interpreted as counting certain integer flows on G called **dynamic**.

An intriguing application of the result by Postnikov-Stanley and Baldoni-Vergne is the case when G is the complete graph K_n on n vertices. The polytope $\mathcal{F}_{K_n}(e_1 - e_n)$ is called the *Chan-Robbins-Yuen (CRY) polytope* [14, 15]. It is the convex hull of certain permutation matrices (see Figure 1-1(b)) and its volume is given by the value of the partition function k_{K_n} at $(0, 1, 2, \ldots, n-2, -\binom{n-1}{2})$. This value is the captivating product $Cat_1 Cat_2 \ldots Cat_{n-3}$ of Catalan numbers $Cat_k = \frac{1}{k+1}\binom{2k}{k}$. This was proved analytically by Zeilberger in [64] using an identity closely related to Selberg's integral (see Lemma 3.7.1) and no combinatorial proof is known.

In Section 3.6.2 we look at the flow polytope $\mathcal{F}_{K_n^{\pm}}(2e_1)$ where $K_n^{\pm}$ is the complete signed graph with all edges $(i, j, \pm)$. We call this the *type D Chan-Robbins-Yuen ($CRYD_n$) polytope.* By Theorem 3.6.16, its volume is given by $k^{dyn}_{K_n^{\pm}}(0, 1, 2, 3, \ldots, n-1)$. Computer calculations up to $n = 8$ suggest that this volume is as interesting as that of the CRY polytope:

Conjecture 3.7.10. *The volume of* $\mathcal{F}_{K_n^{\pm}}(2e_1)$ *is* $2^{(n-2)^2} Cat_1 Cat_2 \ldots Cat_{n-2}$.

1.3 Counting matrices over finite fields with restricted positions

In Chapter 4 we study certain q-analogues of permutations with restricted positions or equivalently of placements of non-attacking rooks. It is based on [33], joint work with A.J. Klein, J.B. Lewis and [39], joint work with R. I. Liu, J.B. Lewis, G. Panova, S. V Sam, and Y. X Zhang.

The q-analogue of permutations we work with are invertible $n \times n$ matrices over the finite field $\mathbf{F}_q$ with q elements [60]. Then the analogue of permutations with restricted positions are matrices over $\mathbf{F}_q$ with some entries required to be zero. Specifically, given a subset S of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, let $\mathrm{mat}_q(n, S, r)$ be the number of $n \times n$ matrices over $\mathbf{F}_q$ with rank r and whose support avoids S. In [39] we showed that $\mathrm{mat}_q(n, S, r)/(q-1)^r$ is indeed an enumerative q-analogue of permutations with restricted positions by showing that when it is given by a (quasi) polynomial its value at $q = 1$ counts the placements of r non-attacking rooks on the complement of S.

When $S = \varnothing$ then $\mathrm{mat}_q(n, \emptyset, n)$ is the number of $n \times n$ invertible matrices over $\mathbf{F}_q$ which is $q^{\binom{n}{2}}(q-1)^n \prod_{i=1}^n (1 + q + \cdots + q^{i-1})$. In this case we obtain a polynomial of the form $(q-1)^n f(q)$ where $f(q)$ has nonnegative integer coefficients. Interestingly, $\mathrm{mat}_q(n, S, r)$ is not always a polynomial in q. Stembridge [61] (following a suggestion of Kontsevich) found a set S when $n = 7$ such that $\mathrm{mat}_q(7, S, 7)$ is two distinct polynomials depending on whether q is even or odd. Also, even if $\mathrm{mat}_q(n, S, r)$ is a polynomial, it might have negative coefficients (after dividing by $(q-1)^r$). For example when $n = 3$ and S is the diagonal $\{(1, 1), (2, 2), (3, 3)\}$ then $\mathrm{mat}_q(3, \{(1, 1), (2, 2), (3, 3)\}, 3) = (q-1)^3(q^3 + 2q^2 - q)$. A general polynomial formula for the number of invertible matrices over $\mathbf{F}_q$ with zero diagonal was also found in [39], answering a question by Stanley. (This is a q-analogue of derangements.)

From these examples, one natural question we study is

Question 4.1.2. *What families of sets S are there such that* $\mathrm{mat}_q(n, S, r)/(q-1)^r$ *is (i) not a polynomial in q, (ii) a polynomial in q, or (iii) a polynomial in q with nonnegative integer coefficients?*

For instance, Haglund [27] showed that one obtains polynomials with nonnegative integer coefficients when S is given by the shape of a partition λ (see Figure 1-1(c)). In this case the answer is given by the Garsia-Remmel q-rook numbers [21]: a generating polynomial of inversions over placements of r non-attacking rooks. In Section 4.4 we extended Haglund's result to the case when S is the complement of a skew shape λ/μ.

Corollary 4.4.6. *For skew shapes $S_{\lambda/\mu}$, $\mathrm{mat}_q(n, \overline{S_{\lambda/\mu}}, r) = (q-1)^r f(q)$ where $f(q)$ is a polynomial with nonnegative integer coefficients.*

Also, by a suggestion of Postnikov, in Section 4.5 we consider the family of sets given by **Rothe diagrams** of permutations which are important in the study of Schubert polynomials [41]. These are defined as follows: if w is a permutation in $\mathfrak{S}_n$, we view it as the word $w_1 w_2 \cdots w_n$ where $w_i = w(i)$. Let I_w be the subset of $\{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$ of entries (i, w_i) and those below and to the right of (i, w_i). The complement R_w of I_w is called the Rothe diagram of w. Computational evidence up to $n \leq 6$ suggests that $\mathrm{mat}_q(n, R_w, r)/(q-1)^r$ is a polynomial with nonnegative integer coefficients.

Conjecture 4.5.1. *If R_w is the Rothe diagram of a permutation w in $\mathfrak{S}_n$, then $\mathrm{mat}_q(n, R_w, r)/(q-1)^r$ is a polynomial with nonnegative integer coefficients.*

By Theorem 4.4.6, we can chip away some cases of this conjecture by finding criteria on permutations for their Rothe diagrams to be skew shapes (after permuting rows and columns). For instance, Lascoux-Schützenberger [37] showed that R_w, up to permuting rows and columns, is the shape of a partition if and only if w avoids the pattern 2143 (*i.e.* there is no sequence $i < j < k < l$ such that $w_j < w_i < w_l < w_k$). We found an analogous criterion for the case of skew shapes (see Theorem 4.5.4).

Theorem 4.5.4. *The Rothe diagram R_w of a permutation w, up to permuting its rows and columns, is the diagram of a skew shape if and only if w can be* decomposed *as $a_1 a_2 \ldots a_k b_1 b_2 \ldots b_{n-k}$ where $a_i < b_j$ and both $a_1 a_2 \ldots a_k$ and $b_1 b_2 \ldots b_{n-k}$ are* 2143 *avoiding.*

For such permutations, $\mathrm{mat}_n(n, R_w, n)/(q-1)^n$ is given by q-rook numbers which are of the form $\sum_{\text{some permutations } u} q^{\text{inversions}(u)}$. A similar type of polynomials are the **Poincaré polynomials** $P_w(q)$ of w. These are given by $\sum_{u \geq w} q^{\text{inversions}(u)}$ where the sum is over all permutations u greater than w in the **strong Bruhat order** of $\mathfrak{S}_n$. Computational evidence up to $n \leq 7$ suggests that there are necessary and sufficient conditions on w for $\mathrm{mat}_q(n, R_w, w)/(q-1)^n$ to be given by a Poincaré polynomial.

Conjecture 4.6.6. *Fix a permutation w in $\mathfrak{S}_n$ and let R_w be its Rothe diagram. We have that $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is coefficient-wise less than or equal to $q^{\binom{n}{2} - \mathrm{inv}(w)} P_w(q)$. We have equality if and only if w avoids the patterns* 1324, 24153, 31524, *and* 426153.

Interestingly, the reverse of the four patterns above, 4231, 35142, 42513, and 351624, have appeared in related contexts in a conjecture of Postnikov [52] proved by Hultman-Linusson-Shareshian-Sjöstrand [30], and in work by Gasharov-Reiner [22]. This suggests interesting connections.

Chapter 2

Colored factorizations of permutations

In this chapter, Section 2.3 is from [10, Secs. 3,4] and Section 2.6 is from [8], both joint work with O. Bernardi. Section 2.6 is from [7], joint work with O. Bernardi, R.X. Du and R.P. Stanley.

2.1 Background on Jackson's formula

2.1.1 Partitions, compositions and permutations

We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. A **composition** of an integer n is a tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ of positive integer summing to n. We then say that γ has **size** $|\gamma| = n$ and **length** $\ell(\gamma) = p$. An **integer partition** is a composition such that the **parts** γ_i are in weakly decreasing order. We use the notation $\lambda \models n$ (resp. $\lambda \vdash n$) to indicate that λ is a composition (resp. integer partition) of n. We sometimes write integer partitions in multiset notation: writing $\lambda = [1^{n_1(\lambda)}, 2^{n_2(\lambda)}, \ldots, j^{n_j(\lambda)}]$ means that λ has $n_i(\lambda)$ parts equal to i. Also $Aut_\lambda = \prod_i n_i(\lambda)!$.

We denote by $\mathfrak{S}_n$ the symmetric group on $[n]$. Given a partition λ of n, we denote by $\mathcal{C}_\lambda$ the set of permutations in $\mathfrak{S}_n$ with cycle type λ. It is well known that $\#\mathcal{C}_\lambda = n!/z_\lambda$ where $z_\lambda = \prod_i i^{n_i(\lambda)} n_i(\lambda)!$.

2.1.2 Factorizations of a long cycle and Jackson's formula

Definition 2.1.1. *Given k partitions $\lambda^{(1)}, \ldots, \lambda^{(k)}$ of n, we let $k^{(n)}_{\lambda^{(1)} \ldots \lambda^{(k)}}$ be the number of ordered factorizations in $\mathfrak{S}_n$ of the long cycle $(1, 2, \ldots, n)$ as a product $\pi_1 \circ \cdots \circ \pi_k$ of k permutations where π_t has cycle type $\lambda^{(t)}$. These numbers are called the* connection coefficients *of $\mathfrak{S}_n$. Also for positive integers $q_1, \ldots, q_k$, let $k^{(n)}_{q_1,\ldots,q_k} = \sum_{\lambda^{(i)} \vdash n, \ell(\lambda^{(i)}) = q_i, i=1,\ldots,k} k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$.*

By the general theory of group representations, the **connection coefficients** $k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ can be expressed in terms of the characters of the symmetric group, but this expression involves cancellation and is not really explicit even for $k = 2$ (see (2.2.11)).

However, Jackson established in [32] a remarkable formula for the generating function of factorizations counted according to the number of cycles of the factors, namely,

$$\sum_{\pi_1\circ\cdots\circ\pi_k=(1,2,\ldots,n)} \prod_{t=1}^{k} x_t^{c(\pi_i)} = \sum_{1\le p_1,\ldots,p_k\le n} \prod_{t=1}^{k} \binom{x_t}{p_t} n!^{k-1} M^{(n-1)}_{p_1-1,\ldots,p_k-1} \tag{2.1.2}$$

where $c(\pi)$ is the number of cycles of the permutation π, and $M^{(n)}_{p_1,\ldots,p_k}$ is the coefficient of $t_1^{p_1}\cdots t_k^{p_k}$ in the polynomial $(\prod_{i=1}^k(1+t_i)-\prod_{i=1}^k t_i)^n$.

Jackson's formula can equivalently be stated in terms of *colored factorizations.*

Definition 2.1.3. *Given positives integers $p_1,\ldots,p_k$, a $(p_1,\ldots,p_k)$*-**colored factorization** *of $(1,2,\ldots,n)$ is a tuple $(\pi_1,\ldots,\pi_k,\phi_1,\ldots,\phi_k)$, where $\pi_1,\ldots,\pi_k$ are permutations of $[n]$ such that $\pi_1\circ\cdots\circ\pi_k=(1,2,\ldots,n)$ and for all $t\in[k]$, c_t is a* surjective *mapping from $[n]$ to $[p_t]$ such that $c_t(a)=c_t(b)$ if a,b are in the same cycle of π_t. In other words, the mapping c_t can be seen as a coloring of the cycles of the permutation π_t with colors in $[p_t]$ and we want* all the colors *to be used. Let $c^{(n)}_{p_1,\ldots,p_k}$ be the number of such colored factorizations.*

It is easy to see that (2.1.2) is equivalent to the following theorem.

Theorem 2.1.4 (Jackson's counting formula [32])**.** *The number $C^{(n)}_{p_1,\ldots,p_k}$ of $(p_1,\ldots,p_k)$-colored factorizations of the permutation $(1,2,\ldots,n)$ is*

$$c^{(n)}_{p_1,\ldots,p_k} = n!^{k-1} M^{(n-1)}_{p_1-1,\ldots,p_k-1}, \tag{2.1.5}$$

where $M^{(n)}_{p_1,\ldots,p_k} = [t_1^{p_1}\cdots t_k^{p_k}](\prod_{i=1}^k(1+t_i)-\prod_{i=1}^k t_i)^n$ is the cardinality of the set $\mathcal{M}^{(n)}_{p_1,\ldots,p_k}$ of n-tuples $(R_1,\ldots,R_n)$ of strict subsets R_t of $[k]$ such that each integer $t\in[k]$ appears in exactly p_t of the subsets $R_1,\ldots,R_n$.

The original proof of Theorem 2.1.4 in [32] is based on the representation theory of the symmetric group. An explicit version of this algebraic proof will be given in Section 2.2. Bijections explaining the cases $k=2,3$ were subsequently given by Schaeffer and Vassilieva [55, 54]. The case $k=2$ of Theorem 2.1.4 is actually closely related to the celebrated Harer-Zagier formula [28], which was proved bijectively by Goulden and Nica [26]. In Section 2.3 we shall give a bijection based on the construction by Bernardi in [6] which extends the results in [55, 54] to arbitrary k (however, for a general k, this bijection does not directly imply Theorem 2.1.4).

We now consider a refined enumeration problem. Let $\gamma^{(1)},\ldots,\gamma^{(k)}$ be compositions of n, where $\gamma^{(t)}=(\gamma_1^{(t)},\gamma_2^{(t)},\ldots,\gamma_{p_t}^{(t)})$. We say that a $(p_1,\ldots,p_k)$-colored factorization $(\pi_1,\ldots,\pi_k,\phi_1,\ldots,\phi_k)$ has **color-compositions** $(\gamma^{(1)},\ldots,\gamma^{(k)})$ if the permutation π_t has $\gamma_i^{(t)}$ elements colored i (i.e. $\gamma_i^{(t)}=|\phi_t^{-1}(i)|$) for all $t\in[k]$ and all $i\in[p_t]$. Let $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ be the number of colored factorizations of color-compositions $(\gamma^{(1)},\ldots,\gamma^{(k)})$. In Section 2.3 we shall prove bijectively the following surprising symmetry property.

Theorem 2.1.6 (Symmetry property)**.** *Let $\gamma^{(1)},\delta^{(1)},\ldots,\gamma^{(k)},\delta^{(k)}$ be compositions of n. If for every $t\in[k]$ the length of the compositions $\gamma^{(t)}$ and $\delta^{(t)}$ are equal, then $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}=c_{\delta^{(1)},\ldots,\delta^{(k)}}$.*

Given that there are $\binom{n-1}{\ell-1}$ compositions of n with ℓ parts, the symmetry property together with Theorem 2.1.4 gives the following refined formula.

Corollary 2.1.7. *For any compositions $\gamma^{(1)}, \ldots, \gamma^{(k)}$ of n where $\ell(\gamma^{(t)}) = p_t$, the number of colored factorizations of color-compositions $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ is*

$$c_{\gamma^{(1)},\ldots,\gamma^{(k)}} = \frac{n!^{k-1} M^{(n-1)}_{p_1-1,\ldots,p_k-1}}{\prod_{t=1}^{k} \binom{n-1}{p_t-1}}, \tag{2.1.8}$$

We will prove (2.1.8) explicitly using the algebraic approach of Jackson in Section 2.2. Then in Section 2.3 we will present the bijection from [10] which proves Theorem 2.1.6: the symmetry of colored factorizations. We will then apply this symmetry to prove combinatorially (2.1.8) in [46, 48] for the cases $k = 2, 3$ (Section 2.4), to compute planar k-cacti (Section 2.5), and to compute certain separation probabilities (Section 2.6).

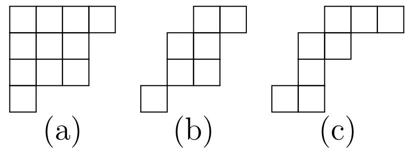


Figure 2-1: Examples of (a) a Young diagram of shape $\lambda = 4331$, (b) a skew Young diagram of shape $\lambda/\mu = 4331/211$, and (c) a skew Young diagram of shape $\lambda/\mu = 5322/211$ that is also a rim hook.

2.2 Algebraic proof Jackson's formula

In this section we give an algebraic proof of Jackson's formula (2.1.2). Specifically, we prove Corollary 2.1.7. We are interested in counting certain factorizations $\pi_1 \circ \cdots \circ \pi_k$ of permutations with respect to number of cycles (or colors) of each permutation π_t. In terms of generating series we encode such factorizations with monomials $x_1^{c(\pi_1)} \cdots x_k^{c(\pi_k)}$ (or $\binom{x_1}{p_1} \cdots \binom{x_k}{p_k}$). We are also interested in counting such factorizations with respect to cycle type (or number of elements in $[n]$ with the same color). In order to encode this extra information we use symmetric functions. We give some background on symmetric functions and on irreducible characters of $\mathfrak{S}_n$.

2.2.1 Symmetric functions and irreducible characters of $\mathfrak{S}_n$

Before discussing symmetric functions we need more terminology related to partitions. A **Young diagram** of the partition λ is a finite collection of cells arranged in rows of $\lambda_1, \lambda_2, \ldots$ cells that are left-justified and organized from top to bottom (this is called *English notation* for tableaux). The diagram of a **skew Young diagram** of partitions λ and μ such that $\lambda_i \geq \mu_i$ is the set theoretic difference of the Young diagrams of λ and μ. See Figure 2-1(a)-(c) for examples of Young diagrams and skew Young diagrams. We say a skew shape λ/μ is **connected** if the union of cells is a connected set and each cell of the shape shares and edge with another cell. See Figure 2-1(b),(c) for examples of skew shapes that are not connected and connected, respectively.

We shall consider symmetric functions in an infinite number of variables $\mathbf{x} = \{x_1, x_2, \ldots\}$. For any sequence of nonnegative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ we denote $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}$. Let Λ be the ring of symmetric functions over $\mathbb{Q}$. For more background on this ring see [57, Ch. 7.] and [40, Ch 1.]. If $\{v_\lambda(\mathbf{x})\}$ is a basis of Λ, for an element $f(\mathbf{x})$ in Λ, we denote by $[p_\lambda(\mathbf{x})]f(\mathbf{x})$ the coefficient of $v_\lambda(\mathbf{x})$ of the decomposition of $f(\mathbf{x})$ in this basis. Next we mention three important bases of this ring that we will use in Part 1 of this thesis.

For an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we denote by $m_\lambda(\mathbf{x})$ the **monomial symmetric function** indexed by λ. That is, $m_\lambda(\mathbf{x}) = \sum_\alpha \mathbf{x}^\alpha$ where the sum is over all the distinct sequences α whose positive parts are $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ (in any order).

We denote by $p_\lambda(\mathbf{x})$ the **power symmetric function** indexed by λ (see e.g. [57]). That is, $p_\lambda(\mathbf{x}) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(\mathbf{x})$ where $p_k(\mathbf{x}) = \sum_{i \geq 1} x_i^k$.

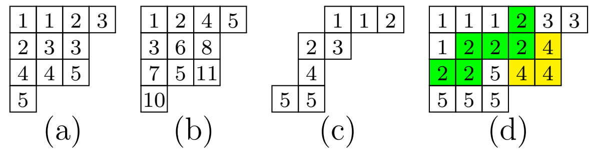


Figure 2-2: Examples, in *English notation*, of (a) a semistandard Young tableau (SSYT) of shape 4331 and type $(2, 2, 3, 2, 2)$, (b) a standard Young tableau (SYT) of shape 4331, (c) a semistandard skew Young tableau of shape 5433/211 and type $(2, 2, 1, 1, 2)$, and (d) a rim-hook tableau of shape 6553 of type $(4, 6, 2, 3, 4)$ and height 3. The rim-hooks with the value $i = 1, 2, \ldots, 5$ are in color.

Proposition 2.2.1 ([57, Prop.7.7.1, Cor. 7.7.2])**.** *If we expand $p_\lambda(\boldsymbol{x})$ in terms of the monomial basis $\{m_\lambda(\boldsymbol{x})\}$ as*

$$p_\lambda(\boldsymbol{x}) = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu(\boldsymbol{x}), \tag{2.2.2}$$

then $R_{\lambda\mu}$ is the number of ordered set partitions $(B_1, \ldots, B_k)$ of the set $[\ell(\lambda)]$ such that $\mu_j = \sum_{i \in B_j} \lambda_i$ for $j = 1, \ldots, k$. Moreover, $R_{\lambda\mu} = 0$ unless $\lambda \leq \mu$ in the dominance order and $R_{\lambda\lambda} = \prod_j n_j(\lambda)!$ hence $\{p_\lambda(\boldsymbol{x})\}$ is a $\mathbb{Q}$-basis for Λ.

Before we introduce the next basis, we give an identity relating monomial symmetric functions and power symmetric functions.

Proposition 2.2.3 ([57, Prop. 7.7.4.])**.**

$$\begin{aligned} \sum_\lambda m_\lambda(\boldsymbol{x}) y^{|\lambda|} &= \prod_{i \geq 1} (1 - x_i y)^{-1} \\ &= \exp\Big(\sum_{j \geq 1} \frac{1}{j} p_j(\boldsymbol{x}) y^j \\ &= \sum_\lambda \frac{1}{z_\lambda} p_\lambda(\boldsymbol{x}) y^{|\lambda|}. \end{aligned}$$

To introduce the next basis we need the following objects: given a partition λ, a **(standard) semistandard Young tableau**, or (SYT) SSYT in short, T of shape λ is a function from the cells of the Young diagram of λ to the positive integers such that the values are (strictly) weakly increasing on the rows and strictly increasing on the columns. The type $\alpha = (\alpha_1, \alpha_2, \ldots)$ of a tableau T indicates that the tableau has α_i cells with the value i. One generalization of SSYT is to consider **skew semistandard tableau** of skew shapes λ/μ which are functions from the cells of the skew Young diagram of λ/μ to the positive integers such that the values are weakly increasing in the rows and strictly increasing on the columns. See Figure 2-2 for illustrations of tableaux.

We denote by $s_\lambda(\mathbf{x})$ the **Schur function** indexed by λ. It is defined to be $s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T$ where the sum is over **semistandard Young tableau** (SSYT) of shape λ

7	5	4	1
5	3	2	
4	2	1	
1			

Figure 2-3: Example of the hook-length formula to compute the number of SYT of shape 4331. The numbers in blue in each cell c are the hook-lengths $h(c)$. The number of SYT of shape $\lambda = 4331$ is $11!/(7 \cdot 5^2 \cdot 4^2 \cdot 3 \cdot 2^2) = 1188$.

and type μ and $\mathbf{x}^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ where α_i is the number of entries i in T. It is not clear from the definition that $s_\lambda(\mathbf{x})$ is a symmetric function but this is in fact true (see [57, Thm. 7.10.2]), moreover $\{s_\lambda(\mathbf{x})\}$ is a $\mathbb{Z}$-basis of Λ. Then expanding $s_\lambda(\mathbf{x})$ in the monomial basis gives

$$s_\lambda(\mathbf{x}) = \sum_\mu K_{\lambda\mu} m_\mu(\mathbf{x}), \tag{2.2.4}$$

where $K_{\lambda\mu}$ is the number of SSYT of shape λ and type μ. This number is called the **Kostka number**. If $\mu = 1^n$ and instead we count SYT, we write $f_\lambda := K_{\lambda 1^n}$ to denote the number of SYT of shape λ with n entries. There is no general explicit formula for the Kostka numbers, however there is a celebrated product formula for f_λ, the **hook-length formula**, of Frame, Robinson and Thrall [19].

$$f_\lambda = \frac{n!}{\prod_{c \in \lambda} h(c)}, \tag{2.2.5}$$

where $h(c)$ is the **hook-length** of the cell c of the Young diagram λ, this is the total number of cells in the same row and east of c (including c) and in the same column and south of c. See Figure 2-3 for an example of this formula to compute the number of SYT of shape $\lambda = 4331$.

Next, we give the relation between power sum symmetric functions in terms of Schur functions. This relation is called the **Murnaghan-Nakayama** rule. To state this rule we need to define rim-hook tableau. A **rim-hook** B is a connected skew shape with no 2×2 square (see Figure 2-1(c) for an example). A **rim-hook tableau** of shape λ/μ is a function from the cells of a skew shape λ/μ to the positive integers such that the values are weakly increasing on the rows *and* columns, and the set of cells with the value i form a rim-hook. The notion of type for these tableaux is the same as before. The **height** $ht(T)$ of a rim-hook tableau T is the number of rows of T minus one. See Figure 2-2(d) for an example of a rim-hook tableau of height 3.

Theorem 2.2.6 ([57, Cor. 7.17.4])**.** *For a composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ we have*

$$p_\alpha(\boldsymbol{x}) = \sum_\lambda \chi_\alpha^\lambda s_\lambda(\boldsymbol{x}), \tag{2.2.7}$$

where $\chi_\alpha^\lambda = \sum_T (-1)^{ht(T)}$ and the sum is over rim-hook tableaux of shape λ and type α.

Example 2.2.8. *If $\alpha = (n)$ then to compute χ_n^λ we have to consider rim-hook tableaux of shape λ and type (n). These tableaux are just* **hooks** $(n-a, 1^a)$ *whose cells have ones. The height of such hooks is a. Thus*

$$p_{(n)}(\boldsymbol{x}) = \sum_{a=1}^{n} (-1)^a s_{(n-a,1^a)}(\boldsymbol{x}).$$

Remark 2.2.9. *If we view $\chi^\lambda(\mu)$ in (2.2.7) as a function over partitions $\mu \vdash n$ and define $\chi^\lambda : \mathfrak{S}_n \to \mathbb{C}$ to be $\chi^\lambda(\pi) = \chi^\lambda_\mu$ where μ is the cycle type of π, then we obtain the* **irreducible characters** *of $\mathfrak{S}_n$.* ■

We will use a formula in terms of characters for connection coefficients like $k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ defined in Section 2.1.2.

Theorem 2.2.10 (Fröbenius)**.** *For partitions $\mu, \lambda^{(1)}, \ldots, \lambda^{(k)}$ of n, let $k^{\mu}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ be the number of ordered factorizations $\pi_1 \circ \cdots \circ \pi_k$ of a fixed permutation π in $\mathfrak{S}_n$ with cycle type μ where π_t has cycle type $\lambda^{(t)}$ for $t = 1, \ldots, k$ then*

$$k^{\mu}_{\lambda^{(1)},\ldots,\lambda^{(k)}} = \frac{\prod_{t=1}^{k} \#\mathcal{C}_{\lambda^{(i)}}}{\#\mathfrak{S}_n} \sum_{\theta \vdash n} \frac{1}{f_\theta^{k-1}} \chi^\theta_\mu \prod_{t=1}^{k} \chi^\theta_{\lambda^{(t)}}, \tag{2.2.11}$$

where f_θ is the number of SYT of shape θ.

We are interested in the case when $\mu = n$. In this case (2.2.11) simplifies considerably.

Corollary 2.2.12.

$$k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}} = \frac{n^{k-1}}{\prod_{t=1}^{k} z_{\lambda^{(t)}}} \sum_{a=0}^{n-1} (-1)^a (a!(n-1-a)!)^{k-1} \chi^{(n-a,1^a)}_{\lambda^{(1)}} \cdots \chi^{(n-a,1^a)}_{\lambda^{(k)}}. \tag{2.2.13}$$

Proof. In the case $\mu = n$, χ^θ_n is zero if θ is not a hook (see Example 2.2.8). We obtain the desired formula from (2.2.11) and from the hook-length formula (2.2.5) $f_{(n-a,1^a)} = \frac{n!}{n(n-1-a)!a!}$ for this case. □

To conclude this background section we mention how can we specialize symmetric functions $p_\lambda(\mathbf{x})$ and $m_\lambda(\mathbf{x})$ to obtain $t^{\ell(\lambda)}$ and $\binom{t}{\ell(\lambda)}$. Let $f \in \Lambda$ be a symmetric function, then $ps^1_n(f) := f(\underbrace{1, \ldots, 1}_{n})$ is the evaluation of the **principal specialization** of order n of f (see [57, Prop.7.8.3]). Then $ps^1_n(p_\lambda) = n^{\ell(\lambda)}$ and $ps^1_n(m_\lambda) = \binom{n}{n_1(\lambda), n_2(\lambda), \ldots}$ where $n_i(\lambda)$ is the number of i-parts of λ. Since n is arbitrary then we can extend this to a specialization $ps_{poly} : \Lambda \to \mathbb{Q}[t]$ defined on the power sum symmetric functions by $ps_{poly}(p_\lambda) = t^{\ell(\lambda)}$. It then follows that $ps_{poly}(m_\lambda) = \binom{t}{\ell(\lambda)}\binom{\ell(\lambda)}{n_1(\lambda), n_2(\lambda), \ldots}$.

2.2.2 Generating functions for (colored) factorizations

Recall that for partitions $\lambda^{(1)}, \ldots, \lambda^{(k)}$ of n, $k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ is the number of factorizations of $(1, 2, \ldots, n)$ as a product $\pi_1 \circ \cdots \circ \pi_k$ of k permutations where π_t has cycle type $\lambda^{(t)}$ and for positive integers $q_1, \ldots, q_k$, $k^{(n)}(q_1, \ldots, q_k) = \sum_{\lambda^{(t)} \vdash n, \ell(\lambda^{(t)}) = q_t, t=1,\ldots,k} k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$. Also for compositions $\gamma^{(1)}, \ldots, \gamma^{(k)}$ of n, $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ is the number of colored factorizations of color-compositions $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ and for positive integers $p_1, \ldots, p_k$, $c^{(n)}_{p_1,\ldots,p_k}$ is the number of $(p_1, \ldots, p_k)$-colored factorizations.

If we consider the generating series $\sum_{q_i} k^{(n)}_{q_1,\ldots,q_k} t_1^{q_1} \cdots t_k^{q_k}$ and evaluate t_i at a nonnegative integer n_i, then $k^{(n)}_{q_1,\ldots,q_k} n_1^{q_1} \cdots n_k^{q_k}$ counts the number of factorizations $\pi_1 \circ \cdots \circ \pi_k$ of $(1, 2, \ldots, n)$ where each of the q_i cycles of π_i has been colored with *some* of the colors in $[n_i]$. Then $c^{(n)}_{p_1,\ldots,p_k}$ is the number of such factorizations where p_i colors of $[n_i]$ are actually used. It immediately follows that

$$\sum_{q_i=1}^{n} k^{(n)}_{q_1,\ldots,q_k} t_1^{q_1} \cdots t_k^{q_k} = \sum_{p_i=1}^{n} c_{p_1,\ldots,p_k} \binom{t_1}{p_1} \cdots \binom{t_k}{p_k}. \tag{2.2.14}$$

There is an analogue relation between the generating series of $k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ and of $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}$. Instead of using the bases $\{t_i^m\}$ and $\{\binom{t_i}{m}\}$ we use the bases $\{p_\lambda(\mathbf{x}^{(t)})\}$ and $\{m_\lambda(\mathbf{x}^{(t)})\}$ of symmetric functions:

$$\sum_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)} \vdash n} k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}} \prod_{t=1}^{k} p_{\lambda^{(t)}}(\mathbf{x}^{(t)}) = \sum_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)} \vdash n} c_{\lambda^{(1)},\ldots,\lambda^{(k)}} \prod_{t=1}^{k} m_{\lambda^{(t)}}(\mathbf{x}^{(t)}). \tag{2.2.15}$$

One can go from (2.2.15) to (2.2.14) by doing a principal specialization ps_{poly} as defined in Section 2.2.1. Also (2.2.15) is equivalent to the relation

$$c_{\mu^{(1)},\ldots,\mu^{(k)}} = \sum_{\lambda^{(t)} \leq \mu^{(t)}} k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}} \prod_{t=1}^{k} R_{\lambda^{(t)}\mu^{(t)}}, \tag{2.2.16}$$

where $\leq$ is the **dominance order** and $R_{\lambda\mu}$ are the change of basis coefficients between the power symmetric functions and the monomial symmetric functions (see (2.2.2)).

2.2.3 Changing basis from power sums to monomial sums

Recall that (2.1.8) in Corollary 2.1.7 states that $c_{\gamma^{(1)},\ldots,\gamma^{(k)}} = n!^{k-1} M^{(n-1)}_{p_1-1,\ldots,p_k-1} / \prod_{t=1}^{k} \binom{n-1}{p_t-1}$ where $M^{(n)}_{p_1,\ldots,p_k}$ is the coefficient of $t_1^{p_1} \cdots t_k^{p_k}$ in the polynomial $(\prod_{i=1}^{k}(1+t_i) - \prod_{i=1}^{k} t_i)^n$ and $\gamma^{(t)}$ for $t \in [k]$ are compositions of n with p_t parts. We interpret $M^{(n)}_{p_1,\ldots,p_k}$ in terms of certain tuples of sets.

Definition 2.2.17. *For positive integers $n, p_1, \ldots, p_k$, let $\mathcal{M}^{(n)}_{p_1,\ldots,p_k}$ be the set of of*

n-tuples $(R_1,\ldots,R_n)$ of strict subsets R_t of $[k]$ such that each integer $t\in[k]$ appears in exactly p_t of the subsets $R_1,\ldots,R_n$.

It is clear that $\#\mathcal{M}^{(n)}_{p_1,\ldots,p_k}$ is the coefficient $M^{(n)}_{p_1,\ldots,p_k}$ defined above. Note that $M^{(n)}_{p_1,p_2}=\binom{n}{p_1,p_2}$, however $M^{(n)}_{p_1,p_2,p_3}\neq\binom{n}{p_1,p_2,p_3}$ since the sets R_1,R_2 and R_3 have more than one element.

By (2.2.15), Corollary 2.1.7 will follow from the following Lemma.

Lemma 2.2.18.

$$\sum_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}\vdash n} k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}\prod_{t=1}^{k}p_{\lambda^{(t)}}(\boldsymbol{x}^{(t)})=n!^{k-1}\sum_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}\vdash n}\frac{M^{(n)}_{\ell(\lambda^{(1)}),\ldots,\ell(\lambda^{(k)})}}{\prod_{t=1}^{k}\binom{n-1}{\ell(\gamma^{(t)})-1}}\prod_{t=1}^{k}m_{\lambda^{(t)}}(\boldsymbol{x}^{(t)}). \tag{2.2.19}$$

The rest of this section is devoted to the proof of this lemma.

Proof. Let $\psi_n:=\sum_{\lambda^{(1)},\ldots,\lambda^{(k)}\vdash n}k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}\prod_{t=1}^{k}p_{\lambda^{(t)}}(\mathbf{x}^{(t)})$. We change ψ_n from the power sum basis $\{p_\lambda(\mathbf{x})\}$ to the Schur basis $\{s_\lambda(\mathbf{x})\}$, and from the Schur basis to the monomial basis $\{m_\lambda(\mathbf{x})\}$.

First we go from $\{p_\lambda(\mathbf{x})\}$ to $\{s_\lambda(\mathbf{x})\}$ in ψ_n. By (2.2.7) and the formula (2.2.13) for $k^{(n)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ we obtain

$$\psi_n(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(k)})=n^{k-1}\sum_{a=0}^{n-1}(-1)^a(a!(n-1-a)!)^{k-1}\prod_{t=1}^{k}s_{(n-a,1^a)}(\mathbf{x}^{(t)}). \tag{2.2.20}$$

Now we go from $\{s_\lambda(\mathbf{x})\}$ to $\{m_\lambda(\mathbf{x})\}$ in ψ_n. By (2.2.4) $s_{(n-a,1^a)}(\mathbf{x})=\sum_{\lambda\vdash n}K_{(n-a,1^a),\lambda}m_\lambda(\mathbf{x})$ where $K_{(n-a\,1^a),\lambda}$ is the number of SSYT of shape $(n-a,1^a)$ and type λ. This Kostka number is very easy to evaluate and only depends on a,n and $\ell(\lambda)$.

Proposition 2.2.21. *The number $K_{(n-a,1^a),\lambda}$ of SSYT of shape $(n-a,1^a)$ and type λ is $\binom{\ell(\lambda)-1}{a}$.*

Proof. A SSYT of hook shape $(n-a,1^a)$ has the entry 1 in the first cell of the first row. We claim that the SSYT is determined by the entries on the other a cells on the first column which have to be distinct, different from 1, and in strictly increasing order. Once this column is determined the entries on the other $n-a-1$ cells in the first row will be the remaining entries of the multiset $\{1^{\lambda_1},2^{\lambda_2},\ldots\}$ organized in weakly increasing order. It follows that $K_{(n-a,1^a),\lambda}=\binom{\ell(\lambda)-1}{a}$ since this is number of ways of choosing the a other entries of the first column. □

Hence by this proposition (2.2.20) becomes,

$$\psi_n(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(k)})=$$
$$=n^{k-1}\sum_{\lambda^{(1)},\ldots,\lambda^{(k)}\vdash n}\prod_{t=1}^{k}m_{\lambda^{(t)}}(\mathbf{x})\left(\sum_{a=0}^{n-1}(-1)^a(a!(n-1-a)!)^{k-1}\prod_{t=1}^{k}\binom{\ell(\lambda^{(t)})-1}{a}\right). \tag{2.2.22}$$

To complete the proof we need the following identity.

Lemma 2.2.23.

$$M^{(n)}_{p_1,\ldots,p_k} = \sum_{a=0}^{n} (-1)^a \binom{n}{a}\binom{n-a}{p_1-a}\cdots\binom{n-a}{p_k-a}. \tag{2.2.24}$$

Proof. First note that $\binom{n}{a}\binom{n-a}{p_1-a}\cdots\binom{n-a}{p_k-a}$ counts the number of n-tuples $(R_1,\ldots,R_n)$ of subsets of $[k]$ (not necessarily strict) such that: (i) each integer $t\in[k]$ appears in exactly p_t of the subsets $R_1,\ldots,R_n$ and (ii) at least a of the n subsets $R_1,\ldots,R_n$ are equal to $[k]$. Then by the principle of Inclusion-Exclusion (see [60, Thm. 2.1.1.]) the alternating sum of these terms counts precisely the number of n-tuples $(R_1,\ldots,R_n)$ of subsets of $[k]$ where (i) holds and none of the subsets $R_1,\ldots,R_n$ are equal to $[k]$. This is precisely the cardinality of $\mathcal{M}^{(n)}_{p_1,\ldots,p_k}$. $\square$

Letting $b_t = \ell(\lambda^{(t)}) - 1$ in (2.2.22), by Lemma 2.2.23 (for $n-1$ instead of n) (2.2.22) becomes

$$\sum_{a=0}^{n-1}(-1)^a(a!(n-1-a)!)^{k-1}\prod_{t=1}^{k}\binom{b_t}{a} = \frac{(n-1)!^{k-1}M^{(n-1)}_{b_1,\ldots,b_k}}{\prod_{t=1}^{k}\binom{n-1}{b_t}}.$$

In summary

$$\psi_n(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(k)}) = \sum_{\mu^{(1)},\ldots,\mu^{(k)}\vdash n} \frac{n!^{k-1}M^{(n-1)}_{p_1-1,\ldots,p_k-1}}{\prod_{t=1}^{k}\binom{n-1}{p_t-1}}\prod_{t=1}^{k} m_{\mu^{(t)}}(\mathbf{x}^{(t)}),$$

as desired. $\square$

We make a few remarks on this algebraic proof.

Remarks 2.2.25. *(i) Note that the symmetry property of $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ (Theorem 2.1.6) is evident from the formula* (2.1.8)*. Also Jackson's formula* (2.1.2) *follows from* (2.1.8) *since there are $\binom{n-1}{p_t-1}$ compositions $\gamma^{(t)}$ of n with p_t parts.*

(ii) In the algebraic proof above, the instance where the symmetry property becomes evident is when we compute the Kostka numbers $K_{(n-a,1^a),\lambda} = \binom{\ell(\lambda)-1}{a}$ when going from (2.2.20) *to* (2.2.22)*. In other words, $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ only depends on n, k and the number of parts of the compositions $\gamma^{(1)},\ldots,\gamma^{(k)}$ because the Kostka number $K_{(n-a,1^a),\lambda}$ only depends on n, a and $\ell(\lambda)$. Although the "symmetry" of colored factorizations boils down to another "symmetry" of a simple calculation of the number of SSYT of a hook shape, it remains obscure why the former objects have this property. In Section 2.3 we elucidate on why the property holds via a clear bijective argument.*

We finish the section by calculating explicit expressions for $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ for $k=2$.

Corollary 2.2.26 ([46]). *Let $\gamma^{(1)}$ and $\gamma^{(2)}$ be compositions of n with p_1 and p_2 parts. Then the number $c_{\gamma^{(1)},\gamma^{(2)}}$ of colored factorizations is*

$$c_{\gamma^{(1)},\gamma^{(2)}} = \frac{n(n-p_1)!(n-p_2)!}{(n+1-p_1-p_2)!}. \tag{2.2.27}$$

Proof. Equation (2.2.27) follows from (2.1.8) and the fact that $M^{(n-1)}_{p_1-1,p_2-1} = \binom{n-1}{p_1-1,p_2-1}$. $\square$

In Section 2.4 we use the symmetry of colored factorizations (Theorem 2.1.6) to obtain (2.2.27) and the following expression for the case $k=3$: if $\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}$ are compositions of n with p_1,p_2,p_3 parts then

$$\frac{c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}}{\prod_{t=1}^{3}(n-p_t)!} = \sum_{a\geq 0} \frac{(n-a-2)!\cdot\Theta}{a!(p_3-1-a)!(p_2-1-a)!(p_1-1)!(n-p_1-a)!(n+2-p_2-p_3+a)!}, \tag{2.2.28}$$

where $\Theta = (n+2-p_2-p_3+a)\left((n-a-1)(p_3-a)+(p_1-1)(n-p_3)\right) +$ $+(n-a_1-p_1)\left((n+1-p_2-p_3+a)(n+2-p_2-p_3+a)+(n+1-p_2)(p_2-1-a)\right)$. A more compact formula for $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$ was first computed bijectively in [48] by refining a construction by Schaeffer and Vassilieva in [54].

2.2.4 Coloring some of the permutations

Lastly, as a corollary of Theorems 2.1.4 and 2.1.6 we obtain a formula by Jackson [32] for counting factorizations where we color *some* of the permutations in the factorization. For a partition λ of n and positive integers $p_1,\ldots,p_k$ let $c_{p_1,\ldots,p_k}(\lambda)$ be the number of factorizations of $(1,2,\ldots,n)$ as a product $\pi\circ\pi_1\circ\cdots\circ\pi_k$ where π has cycle type λ and the cycles of π_t have been colored with p_t colors as in Definition 2.1.3. Note that the cycles of π are not colored. Jackson [32, Thm. 4.3] showed the following formula for $c_{p_1,\ldots,p_k}(\lambda)$.

Corollary 2.2.29 ([32, Thm.4.3]).

$$c_{p_1,\ldots,p_k}(\lambda) = \frac{n!^k}{z_\lambda}\widetilde{M}_{p_1-1,\ldots,p_k-1}(\lambda), \tag{2.2.30}$$

where $\widetilde{M}_{p_1,\ldots,p_k}(\lambda)$ is the coefficient of $z_1^{p_1}\cdots z_k^{p_k}$ in $(\prod_{i=1}^{\ell(\lambda)} U_{\lambda_i}(z_1,\ldots,z_k))/U_1(z_1,\ldots,z_k)$ for $U_m(z_1,\ldots,z_k) = \prod_{t=1}^k(1+z_t)^m - \prod_{t=1}^k z_t^m$.

Note that (2.2.30) implies (2.1.4) by setting π to be the identity (i.e., $\lambda = 1^n$). However, one can go the other way by starting with a colored factorization of $k+1$ permutations and uncoloring one of these permutations. We devote the rest of this section to following this direction to obtain (2.2.30).

Fix k and denote by U_m the polynomial $U_m(z_1,\ldots,z_k)$ and let $U_\lambda = \prod_{i=1}^{\ell(\lambda)} U_{\lambda_i}$. For a composition γ of n and positive integers $p_1,\ldots,p_k$ Let $c_{\gamma;p_1,\ldots,p_k}$ be the number of $(\ell(\gamma),p_1,\ldots,p_k)$-colored factorizations $(\pi,\pi_1,\ldots,\pi_k,\phi,\phi_1,\ldots,\phi_k)$ of $(1,2,\ldots,n)$

where π has color composition γ. For fixed $p_1, \dots, p_k$, the following equation relates $c_{p_1,\dots,p_k}(\lambda)$ with $c_{\gamma;p_1,\dots,p_k}$:

$$\sum_\lambda c_{p_1,\dots,p_k}(\lambda) p_\lambda(\mathbf{x}) = \sum_\lambda c_{\lambda;p_1,\dots,p_k} m_\lambda(\mathbf{x}). \tag{2.2.31}$$

As in Corollary 2.1.7, $c_{\gamma;p_1,\dots,p_k} = c^{(n)}_{\ell(\gamma),p_1,\dots,p_k} / \binom{n-1}{\ell(\gamma)-1}$. Then by (2.1.4)

$$\begin{aligned} c_{\gamma;p_1,\dots,p_k} &= \frac{n!^k}{\binom{n-1}{\ell(\lambda)-1}} [z^{\ell(\lambda)-1} z_1^{p_1-1} \cdots z_k^{p_k-1}]((1+z)\prod_{t=1}^k (1+z_t) - z\prod_{t=1}^k z_t)^{n-1} \\ &= n!^k [z_1^{p_1-1} \cdots z_k^{p_k-1}](\prod_{t=1}^k (1+z_t) - \prod_{t=1}^k z_t)^{\ell(\lambda)-1} (\prod_{t=1}^k (1+z_t))^{n-\ell(\lambda)}. \end{aligned}$$

Thus the RHS of (2.2.31) is equal to $[z_1^{p_1-1} \cdots z_k^{p_k-1}] \sum_\lambda |\lambda|!^{k-1} U_1^{\ell(\lambda)-1} V^{|\lambda|-\ell(\lambda)} m_\lambda(\mathbf{x})$ where $V = \prod_{t=1}^k (1+z_t)$. Since we want to show that the LHS of (2.2.31) is equal to $[z_1^{p_1-1} \cdots z_k^{p_k-1}] \sum_\lambda (|\lambda|!^{k-1}/z_\lambda)(U_\lambda/U_1) m_\lambda(\mathbf{x})$, then to prove Corollary (2.2.29) it suffices to show the following identity.

Lemma 2.2.32.

$$\sum_\lambda \frac{1}{z_\lambda} U_\lambda p_\lambda(\boldsymbol{x}) = \sum_\lambda U_1^{\ell(\lambda)} V^{|\lambda|-\ell(\lambda)} m_\lambda(\boldsymbol{x}), \tag{2.2.33}$$

where $V = \prod_{t=1}^k (1+z_t)$ *and* $U_\lambda = \prod_{i=1}^{\ell(\lambda)} U_{\lambda_i}$ *for* $U_m = \prod_{t=1}^k (1+z_t)^m - \prod_{t=1}^k z_t^m$.

Proof. To prove the Lemma we use the following identities of symmetric functions that follow easily from Proposition 2.2.3.

Proposition 2.2.34.

$$\sum_\lambda w^{\ell(\lambda)} y^{|\lambda|} m_\lambda(\boldsymbol{x}) = \prod_{i\geq 1} (1 - wyx_i(1-x_i)^{-1}). \tag{2.2.35}$$

and

$$\sum_\lambda \frac{1}{z_\lambda} u_\lambda p_\lambda(\boldsymbol{x}) = \exp(\sum_{j\geq 1} u_j p_j(\boldsymbol{x})/j), \tag{2.2.36}$$

where $u_\lambda = \prod_{i=1}^{\ell(\lambda)} u_{\lambda_i}$.

Substituting $y = \prod_{t=1}^k (1+z_t)$ and $w = U_1/V$ in (2.2.35) we get

$$\sum_\lambda U_1^{\ell(\lambda)} V^{|\lambda|-\ell(\lambda)} m_\lambda(\mathbf{x}) = \prod_{i\geq 1} (1 - U_1 x_i/(1-x_iV)^{-1}). \tag{2.2.37}$$

Since $1 - U_1 x_i/(1-x_iV)^{-1} = \frac{1-x_i\prod_{t=1}^k z_t}{1-x_i\prod_{t=1}^k (1+z_t)}$, then by Proposition 2.2.3 the RHS of

(2.2.37) becomes

$$\prod_{i\geq 1}(1-U_1x_i/(1-x_iV)^{-1}) = \frac{\exp(\sum_{j\geq 0}\frac{1}{j}p_j(\mathbf{x})\prod_{t=1}^{k}(1+z_t)^j)}{\exp(\sum_{j\geq 0}\frac{1}{j}p_j(\mathbf{x})\prod_{t=1}^{k}z_t^j)}$$
$$= \exp(\sum_{j\geq 0}\frac{1}{j}U_jp_j(\mathbf{x})). \quad (2.2.38)$$

By Proposition 2.2.3, $\exp(\sum_{j\geq 0}\frac{1}{j}U_jp_j(\mathbf{x})) = \sum_\lambda \frac{1}{z_\lambda}U_\lambda p_\lambda(\mathbf{x})$. Combining (2.2.37) and (2.2.38) we finish the proof of the Lemma. □

Having proved Lemma 2.2.32 we conclude the proof of Corollary 2.2.29.

2.3 Symmetry colored factorizations all k

This section is from Sections 3 and 4 of [10], joint work with O. Bernardi.

2.3.1 Background on maps

Graphs and maps. Our **graphs** are undirected and can have loops and multiple edges. A **digraph**, or **directed graph**, is a graph where every edge is oriented; oriented edges are called **arcs**. An **Eulerian tour** of a directed graph is a directed path starting and ending at the same vertex and taking every arc exactly once. An edge e of a graph defines two **half-edges** each of them incident to an endpoint of e. A **rotation system** for a graph G is an assignment for each vertex v of G of a cyclic ordering for the half-edges incident to v.

We now review the connection between rotation systems and embeddings of graphs in surfaces. We call **surface** a compact, connected, orientable, 2-dimensional manifold without boundary (such a surface is characterized by its **genus** $g \geq 0$). A **map** is a cellular embedding of a connected graph in an oriented surface considered up to orientation preserving homeomorphism[1]. By **cellular** we mean that the **faces** (connected components of the complement of the graph) are simply connected. For a map, the angular section between two consecutive half-edges around a vertex is called a **corner**. The **degree** of a vertex or a face is the number of incident corners. A map M naturally defines a rotation system $\rho(M)$ of the underlying graph G by taking the cyclic order of the half-edges incident to a vertex v to be the clockwise order of these half-edges around v. The following classical result (see e.g. [45]) states the relation between maps and graphs with rotation systems.

Lemma 2.3.1. *For any connected graph G, the function ρ is a bijection between the set of maps having underlying graph G and the set of rotation systems of G.*

Constellations and cacti. A k**-constellation**, or **constellation** for short, is a map with two types of faces **black** and **white**, and k types of vertices $1, 2, \ldots, k$, such that:

(i) each edge separates a black face and a white face,
(ii) each black face has degree k and is incident to vertices of type $1, 2, \ldots, k$ in this order clockwise around the face.

Two constellations are shown in Figure 2-4. The black faces are also called **hyperedges**. The **size** of a constellation is the number of hyperedges. A constellation of size n is **labelled** if its hyperedges receive distinct labels in $[n]$.

We now recall the link between constellations and products of permutations. We call k-hypergraph a pair $G = (V, E)$ where V is a set of vertices, each of them having a **type** in $[k]$, and E is a set of **hyperedges** which are subsets of V containing exactly one vertex of each type. A **rotation-system** for the hypergraph G is an assignment for each vertex v of a cyclic order of the hyperedges *incident to* v (i.e., containing

[1]Maps can be considered on non-orientable surfaces [51, 23, 35] where coloring arguments are also useful for enumeration [6, 47, 63].

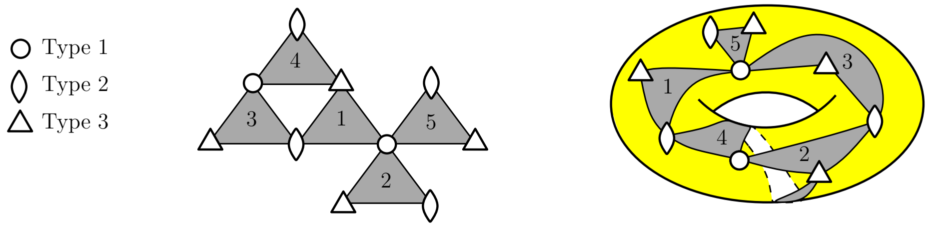


Figure 2-4: Two hyperedge-labelled 3-constellations of size 5 (the shaded triangles represent the hyperedges). The 3-constellation on the left (which is embedded in the sphere) encodes the triple (π_1, π_2, π_3), where $\pi_1 = (1, 2, 5)(3, 4)$, $\pi_2 = (1, 3)(2)(4)(5)$, $\pi_3 = (1, 4)(2)(3)(5)$, so that $\pi_1\pi_2\pi_3 = (1, 3, 2, 5)(4)$. The 3-cactus on the right (which is embedded in the torus) encodes the triple (π_1, π_2, π_3), where $\pi_1 = (1, 3, 5)(2, 4)$, $\pi_2 = (1, 4)(2, 3)(5)$, $\pi_3 = (1)(2, 4)(3)(5)$, so that $\pi_1\pi_2\pi_3 = (1, 2, 3, 4, 5)$.

v). Clearly each k-constellation defines a connected k-hypergraph together with a rotation system (the clockwise order of the hyperedges around each vertex). In fact Lemma 2.3.1 readily implies the following result.

Lemma 2.3.2. *For any connected k-hypergraph G, there is a bijection between k-constellations of underlying k-hypergraph G and the rotation systems of G.*

Now given a hyperedge-labelled k-constellation C of size n, we define some permutations $\pi_1, \ldots, \pi_k$ as follows: for each $t \in [k]$ we define the cycles of the permutation π_t to be the counterclockwise order of the hyperedges around the vertices of type t. Examples are given in Figure 2-4. We then say that the hyperedge-labelled k-constellation C **represents** the tuple $\varrho(C) = (\pi_1, \ldots, \pi_k)$. From Lemma 2.3.2 it is easy to establish the following classical result (see e.g. [36]).

Lemma 2.3.3. *The representation mapping ϱ is a bijection between hyperedge-labelled k-constellations of size n and tuples of permutations $(\pi_1, \ldots, \pi_k)$ of $[n]$ acting transitively on $[n]$. Moreover the number of white faces of the constellation is equal to the number of cycles of the product $\pi_1\pi_2 \cdots \pi_k$.*

An edge of a constellation has **type** $t \in [k]$ if its endpoints have types t and $t+1$ (the types of the vertices and edges are considered modulo k). A k-constellation has **type** $(p_1, \ldots, p_k)$ if it has p_t vertices of type t for all $t \in [k]$. The **hyperdegree** of a vertex is the number of incident hyperedges. A constellation of type $(p_1, \ldots, p_k)$ is **vertex-labelled** if for each $t \in [k]$ the p_t vertices of type t have distinct labels in $[p_t]$. We say that such a constellation has **vertex-compositions** $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ if for all $t \in [k]$, $\gamma^{(t)}$ is a composition of size n and length p_t whose ith part is the hyperdegree of the vertex of type t labelled i.

A k-constellation is **rooted** if one of its hyperedges is distinguished as the **root hyperedge**. The vertex of type k incident to the root hyperedge is called **root vertex**. There are $n!$ distinct ways of labelling a rooted constellation of size n (because a rooted constellation has no symmetry preserving the root hyperedge). Hence, there is a 1-to-$(n-1)!$ correspondence between rooted constellations of size n and hyperedge-labelled constellations of size n.

A k-**cactus** is a k-constellation with a single white face. By Lemma 2.3.3 the hyperedge-labelled k-cacti correspond bijectively to the factorizations of one of the $(n-1)!$ long cycles into k factors (transitivity is redundant in this case), while rooted cacti correspond bijectively to the factorizations of the permutation $(1, 2, \ldots, n)$. Since Jackson's counting formula is about **colored factorizations** of $(1, 2, \ldots, n)$ (see Definition 2.1.3), we now consider **vertex-colored cacti**. Given some positive integers $q_1, \ldots, q_k$, a $(q_1, \ldots, q_k)$**-colored cacti** is a k-cacti together with an assignment of **colors** to vertices, such that for every $t \in [k]$ the vertices of type t are colored using every color in $[q_t]$. A $(2, 1, 3)$-colored cacti is represented in Figure 2-5. The **color-compositions** of a $(q_1, \ldots, q_k)$-colored cacti of size n is the tuple $(\gamma^{(1)}, \ldots, \gamma^{(k)})$, where for all $t \in [k]$, $\gamma^{(t)}$ is a composition of size n and length q_t whose ith part is the number of hyperedges incident to vertices of type t colored i. It is clear from the representation mapping ϱ, that $(q_1, \ldots, q_k)$-colored cacti of color-compositions $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ are in bijection with the $(q_1, \ldots, q_k)$-colored factorizations of $(1, 2, \ldots, n)$ with color-compositions $(\gamma^{(1)}, \ldots, \gamma^{(k)})$.

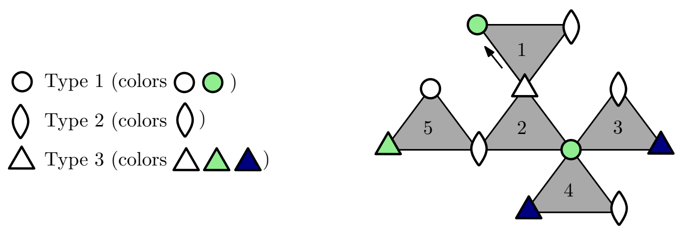


Figure 2-5: A $(2, 1, 3)$-colored cacti (embedded in the sphere) with color-compositions $(\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})$, where $\gamma^{(1)} = (1, 4)$, $\gamma^{(2)} = (5)$ and $\gamma^{(3)} = (2, 1, 2)$.

From now on, all our results and proofs are stated in terms of constellations and cacti.

2.3.2 From cacti to tree-rooted constellations

In this section we establish a bijection between vertex-colored cacti and certain constellations with a distinguished spanning tree. Let C be a k-constellation and let v_0 be a vertex. We call v_0**-arborescence** of C a spanning tree A such that every vertex $v \neq v_0$ of type t is incident to exactly one edge of type t in A (equivalently, the spanning tree A is oriented from the leaves toward v_0 by orienting every edge of A of type $t \in [k]$ from its endpoint of type t toward its endpoint of type $t + 1$). A **tree-rooted constellation** is a pair (C, A) made of a rooted constellation C together with a v_0-arborescence A, where v_0 is the root vertex of C. An example of tree-rooted constellation is given in Figure 2-6 (bottom right).

Theorem 2.3.4. *Let $p_1, \ldots, p_k$ be positive integers. There is a bijection Φ between the set $\mathcal{C}^{(n)}_{p_1,\ldots,p_k}$ of $(p_1, \ldots, p_k)$-colored rooted k-cacti of size n (these encode the $(p_1, \ldots, p_k)$-colored factorizations of $(1, 2, \ldots, n)$), and the set $\mathcal{T}^n_{p_1,\ldots,p_k}$ of vertex-labelled tree-rooted k-constellations of size n and type $(p_1, \ldots, p_k)$.*

Moreover, the bijection has the following **degree preserving property***: for any vertex-colored cactus C, the number of edges joining vertices of type t and color i to vertices of type $t+1$ and color j in C is equal to the number of edges joining the vertex of type t labelled i to the vertex of type $t+1$ labelled j in the tree-rooted constellation $\Phi(C)$.*

Remark. The degree preserving property of Theorem 2.3.4 implies that for any tuple of compositions $(\gamma^{(1)}, \ldots, \gamma^{(k)})$, the mapping Φ establishes a bijection between cacti of color-compositions $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ and tree-rooted constellations of vertex-compositions $(\gamma^{(1)}, \ldots, \gamma^{(k)})$.

Remark. In the case $k = 2$ the tree-rooted k-constellations can be identified with rooted bipartite maps with a distinguished spanning tree (simply by considering the hyperedges as edges). These objects are easy to count (see [6]), so that the case $k = 2$ of Theorem 2.1.4 follows easily from Theorem 2.3.4 in this case.

The remaining of this section is devoted to the proof of Theorem 2.3.4. Our strategy parallels the one developed in [6] (building on some ideas of Lass [38]) in order to prove extensions of the Harer-Zagier formula. This proof is illustrated in Figure 2-6. We shall recombine the information given by a vertex-colored cactus into the information given by a tree-rooted constellation through the *BEST Theorem* (see Lemma 2.3.6 below).

We call **k-digraph** a directed graph with k types of vertices $1, \ldots, k$, such that every vertex has as many ingoing and outgoing arcs, and every arc goes from a vertex of type t to a vertex of type $t+1$ for some $t \in [k]$ (as usual the types of vertices are considered modulo k). An arc going from a vertex of type t to a vertex of type $t+1$ is said to have **type** t. Note that a k-digraph has as many arcs of each type, and we say that it has **size** n if it has n arcs of each type. An **arc-labelling** of a k-digraph of size n is an assignment of distinct labels in $[n]$ to the n arcs of type t, in such a way that for any $(t, i) \in [k] \times [n]$ the end of the arc of type t and label i is the origin of the arc of type $t+1$ and label i. Observe that arc-labelled k-digraphs easily identify with hyperedge-labelled k-hypergraphs. A k-digraph has **type** $(p_1, \ldots, p_k)$ if for each $t \in [k]$ there are p_t vertices of type t. It is **vertex-labelled** by assigning distinct labels in $[p_t]$ to its p_t vertices of type t for all $t \in [k]$.

Lemma 2.3.5. *There is a bijection Ξ between the set of hyperedge-labelled rooted $(p_1, \ldots, p_k)$-colored cacti of size n, and the set of pairs (G, η) where G is a arc-labelled vertex-labelled k-digraph of type $(p_1, \ldots, p_k)$ and η is an Eulerian tour of G starting and ending at a vertex of type k.*

Lemma 2.3.5 is illustrated in the top part of Figure 2-6.

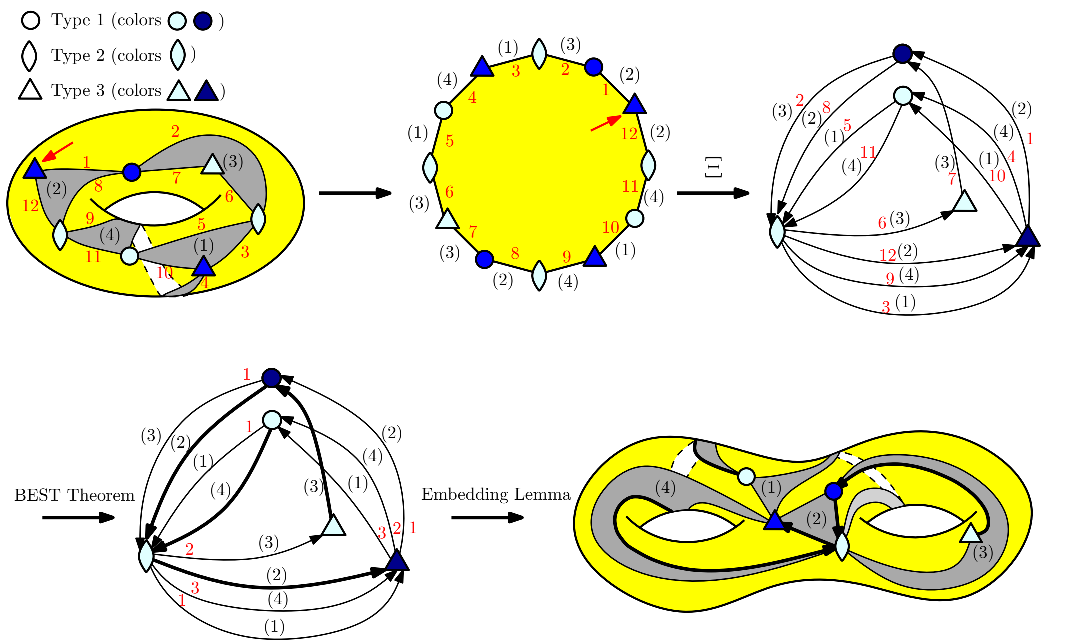


Figure 2-6: From a vertex-colored cactus to a tree-rooted constellation via the BEST Theorem.

Proof. We call **black k-gon** a polygon with k vertices of type $1, 2, \ldots, k$ in clockwise order, and **white kn-gon** a polygon with kn vertices, such that the type of vertices increases by one (modulo k) along each edge in counterclockwise order (modulo k). A white kn-gon is **rooted** if a corner incident to a vertex of type k is distinguished as the **root-corner**; it is $(p_1, \ldots, p_k)$-colored if for all $t \in [k]$ the vertices of type t are colored using every color in $[p_t]$.

Observe that the n hyperedges of a k-cactus of size n are black k-gons, while its white face is a white kn-gon (since faces of cactus are simply connected). Moreover the k-cactus is completely determined (up to homeomorphism) by specifying the gluing of the black k-gons with the white kn-gon (that is specifying the pair of edges to be identified). Thus, a rooted hyperedge-labelled $(p_1, \ldots, p_k)$-colored cactus is obtained by taking a rooted $(p_1, \ldots, p_k)$-colored white kn-gon, and gluing its edges to the edges of n labelled black k-gon *so as to respect the color and type of the vertices* (certain vertices of the white kn-gon are identified by the gluing). Now, a rooted $(p_1, \ldots, p_k)$-colored white kn-gon is bijectively encoded by a pair $(\tilde{G}, \eta)$, where $\tilde{G}$ is a vertex-labelled k-digraph of type $(p_1, \ldots, p_k)$ and η is an Eulerian tour of $\tilde{G}$ (the Eulerian tour gives the order of the colors around the white kn-gon in counterclockwise direction starting from the root-corner). Moreover, the gluings of the n labelled black k-gons (respecting the type and coloring) are in bijection with the arc-labellings of $\tilde{G}$. This establishes the claimed bijection. □

We now recall the BEST Theorem for Eulerian tours[2]. Let G be a directed graph

[2]This Theorem is due to de Bruijn, van Aardenne-Ehrenfest, Smith and Tutte. See [57, Theorem

and let v_0 be a vertex. We call v_0-**Eulerian tour** an Eulerian-tour starting and ending at vertex v_0. Observe that a v_0-Eulerian tour is completely characterized by its **local-order**, that is, the assignment for each vertex v of the order in which the outgoing edges incident to v are used. Note however that not every local order corresponds to an Eulerian tour. We call v_0-**arborescence** a spanning tree A of G oriented from the leaves toward v_0 (i.e., every vertex $v \neq v_0$ has exactly one outgoing arc in A).

Lemma 2.3.6 (BEST Theorem). *Let G be an arc-labelled directed graph where every vertex has as many ingoing arcs as outgoing ones, and let v_0 be a vertex of G. A local order corresponds to a v_0-Eulerian tour if and only if the set of last outgoing arcs out of the vertices $v \neq v_0$ form a v_0-arborescence. Consequently, there is a bijection between the set of v_0-Eulerian tours of G and the set of pairs (A, τ), where A is a v_0-arborescence, and τ is an assignment for each vertex v of a total order of the incident outgoing arcs not in A.*

We now complete the proof of Theorem 2.3.4. By combining Lemma 2.3.5 and the BEST Theorem, one gets a bijection between rooted hyperedge-labelled $(p_1, \dots, p_k)$-colored cacti and triples (G, A, θ) where G is an arc-labelled vertex-labelled k-digraph of type $(p_1, \dots, p_k)$, A is a v_0-arborescence of G for a vertex v_0 of type k, and τ is an assignment for each vertex v of a total order of the arcs not in A going out of v. Observe that τ encodes the same information as a pair (a_0, τ'), where a_0 is an arc going out of v_0 and τ' is an assignment for each vertex v of a cyclic order of the arcs going out of v. Now the arc-labelled vertex-labelled k-digraph G encodes the same information as a hyperedge-labelled vertex-labelled k-hypergraph G', and τ' can be seen as a rotation system for G'. Thus, by Lemma 2.3.2 the pair (G, τ) encodes the same information as a rooted hyperedge-labelled vertex-labelled k-constellation C of type $(p_1, \dots, p_k)$ (note that the hypergraph G' is clearly connected since it has an arborescence A). Lastly, the v_0-arborescence A of G clearly encodes a v_0-arborescence of the constellation C, where v_0 is the root vertex of C.

We thus have obtained a bijection between rooted hyperedge-labelled $(p_1, \dots, p_k)$-colored cacti and the hyperedge-labelled vertex-labelled tree-rooted constellations. The labelling of the hyperedges can actually be disregarded since there are $n!$ distinct ways of labelling the hyperedges of a rooted constellation of size n. This gives the bijection announced in Theorem 2.3.4. Moreover it is easy to check that it has the claimed **degree preserving property**. $\square$

2.3.3 Symmetries for tree-rooted constellations

In this section we prove that for vertex-labelled tree-rooted constellations of a given type $(p_1, \dots, p_k)$, every vertex-compositions is equally likely. This together with Theorem 2.3.4 proves the symmetry property stated in Theorem 2.1.6.

We denote by $\mathcal{T}_{\gamma^{(1)}, \dots, \gamma^{(k)}}$ the set of vertex-labelled tree-rooted constellations of vertex-compositions $(\gamma^{(1)}, \dots, \gamma^{(k)})$.

5.6.2] for a proof.

Theorem 2.3.7. *If $\gamma^{(1)}, \ldots, \gamma^{(k)}, \delta^{(1)}, \ldots, \delta^{(k)}$ are compositions of n such that $\ell(\gamma^{(t)}) = \ell(\delta^{(t)})$ for all $t \in [k]$, then the sets $\mathcal{T}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ and $\mathcal{T}_{\delta^{(1)},\ldots,\delta^{(k)}}$ are in bijection.*

Remark. Theorem 2.3.7 gives the hope of counting tree-rooted constellations of given type, by looking at the simplest possible vertex-compositions. For instance, one can try to enumerate the set $\mathcal{T}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ where $\gamma^{(t)} = (n - p_t + 1, 1, 1, \ldots, 1)$ for all $t \in [k]$ (similar ideas lead to a very easy way of counting k-cacti embedded in the sphere [8]). However, our efforts in this direction only led to a restatement of Jackson counting formula as a *probabilistic puzzle* similar to [10][Thm. 1.6.] which we could not easily solve for $k \geq 3$.

Proof. Let $t \in [k]$ and $i, j \in [p_t]$. In order to prove Theorem 2.3.7 it suffices to exhibit a bijection $\varphi_{t,i,j}$ between $\mathcal{T}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ and $\mathcal{T}_{\delta^{(1)},\ldots,\delta^{(k)}}$ when $\gamma^{(s)} = \delta^{(s)}$ for all $s \neq t$, $\gamma_x^{(t)} = \delta_x^{(t)}$ for all $x \neq i, j$, $\gamma_i^{(t)} - 1 = \delta_i^{(t)}$ and $\gamma_j^{(t)} + 1 = \delta_j^{(t)}$. In other words, we want to construct a bijection $\varphi_{t,i,j}$ which decreases by one the hyperdegree of the vertex of type t labelled i and increases by one the hyperdegree of the vertex of type t labelled j. Recall from Lemma 2.3.2 that a k-constellation is defined by a (connected) k-hypergraph together with a rotation system (clockwise order of hyperedges around the vertices); therefore it is well defined to *unglue* a hyperedge from a vertex of type t and *reglue* it in a specified corner of another vertex of type t. We will use these operations to define the mapping $\varphi_{t,i,j}$ below; see Figure 2-7.

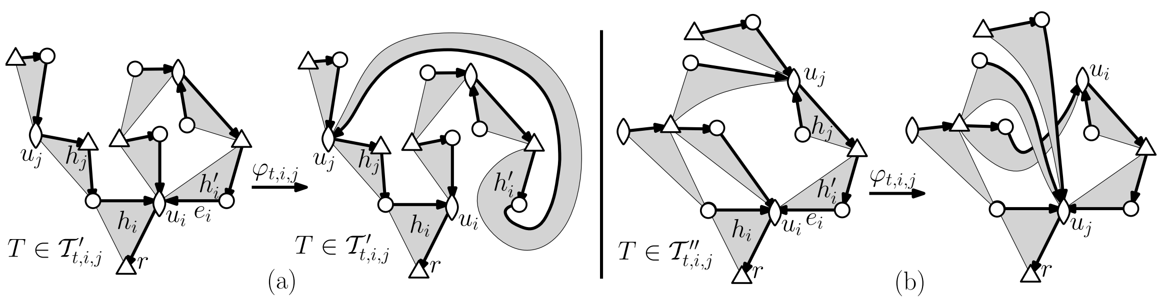


Figure 2-7: The bijection $\varphi_{t,i,j}$ applied to a tree-rooted constellation in $\mathcal{T}'_{t,i,j}$ (left), or in $\mathcal{T}''_{t,i,j}$ (right). The tree-rooted k-constellations are represented as k-hypergraphs together with a rotation system (so the overlappings of the hyperedges in this figure are irrelevant).

Let $\mathcal{T}_{t,i}$ be the set of vertex-labelled tree-rooted constellations of type $(p_1, \ldots, p_k)$ such that the vertex of type t labelled i has hyperdegree at least two. Let T be a tree-rooted constellation in $\mathcal{T}_{t,i}$, let u_i and u_j be the vertices of type t labelled i and j respectively, let r be the root vertex, and let A be the marked r-arborescence. If $u_i \neq r$ we denote by h_i be hyperedge incident to the edge joining u_i to its parent in A, while if $u_i = r$ we denote by h_i the the root hyperedge. We define h_j similarly. Let h_i' be the hyperedge preceding h_i in clockwise order around u_i and let e_i be the edge of type $t-1$ incident to h_i'. Observe that $h_i \neq h_i'$ since the hyperdegree of u_i is at least two.

In order to define the mapping $\varphi_{t,i,j}$ we need to consider two cases which are illustrated in Figure 2-7. We first define a partition $\mathcal{T}_{t,i} = \mathcal{T}'_{t,i,j} \cup \mathcal{T}''_{t,i,j}$ by declaring that T is in $\mathcal{T}'_{t,i,j}$ if the edge e_i is not on the path from u_j to the root vertex r in the arborescence A, and that T is in $\mathcal{T}''_{t,i,j}$ otherwise. Suppose first that T is in $\mathcal{T}'_{t,i,j}$. In this case we define $\varphi_{t,i,j}(T)$ as the constellation (with marked edges) obtained from the tree-rooted constellation T (with marked edges corresponding to the arborescence A) by ungluing the hyperedge h'_i from u_i and gluing it to u_j in the corner preceding the hyperedge h_j in clockwise order around u_j; see Figure 2-7(a). Observe that $\varphi_{t,i,j}(T)$ is a tree-rooted constellation (in particular the marked edges form an r-arborescence A' of $\varphi_{t,i,j}(T)$). Moreover $\varphi_{t,i,j}(T)$ is in $\mathcal{T}_{t,j}$ and more precisely in $\mathcal{T}'_{t,j,i}$. It is also easy to see that $\varphi_{t,j,i}(\varphi_{t,i,j}(T)) = T$. Suppose now that T is in $\mathcal{T}''_{t,i,j}$. In this case we define $\varphi_{t,i,j}(T)$ as the constellation (with marked edges) obtained from T (with marked edges corresponding to the arborescence A) as follows: we unglue all the hyperedges incident to u_i except h_i and h'_i, we unglue all the hyperedges incident to u_j except h_j, we reglue the hyperedges unglued from u_j to u_i in the corner preceding h'_i in clockwise order around u_i (without changing their clockwise order), we reglue the hyperedges unglued from u_i to u_j (in the unique possible corner), and lastly we exchange the labels i and j of the vertices u_i and u_j; see Figure 2-7(b). It is easy to see that $\varphi_{t,i,j}(T)$ is a tree-rooted constellation (in particular the marked edges form an r-arborescence of $\varphi_{t,i,j}(T)$). Moreover $\varphi_{t,i,j}(T)$ is in $\mathcal{T}_{t,j}$ and more precisely in $\mathcal{T}''_{t,j,i}$. It is also easy to see that $\varphi_{t,j,i}(\varphi_{t,i,j}(T)) = T$.

We have shown that $\varphi_{t,i,j}$ is a mapping from $\mathcal{T}_{t,i}$ to $\mathcal{T}_{t,j}$. Moreover $\varphi_{t,j,i} \circ \varphi_{t,i,j} = Id$ for all i, j, thus $\varphi_{t,i,j} = \varphi_{t,j,i}^{-1}$ is a bijection. Lastly, the bijection $\varphi_{t,i,j}$ decreases by one the hyperdegree of the vertex of type t labelled i and increases by one the hyperdegree of the vertex of type t labelled j. Thus $\varphi_{t,i,j}$ has all the claimed properties. $\square$

A natural question is whether there is also symmetry for colored factorizations of a permutation with more than one cycle. Let $\mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}(f)$ be the set of colored factorizations of a fixed permutation with f cycles of color compositions $(\gamma^{(1)}, \ldots, \gamma^{(k)})$. Thus $\mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}(1) \cong \mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$. Computer experiments for $n \leq 7$ give evidence for symmetry of colored factorizations for two or three cycles but not for four cycles.

Conjecture 2.3.8. *Let $\gamma^{(1)}, \delta^{(1)}, \ldots, \gamma^{(k)}, \delta^{(k)}$ be compositions of n such that for every $t \in [k]$ $\ell(\gamma^{(t)}) = \ell(\delta^{(t)})$. Then for $f = 2$ and 3 (but not for $f = 4$) $\#\mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}(f) = \#\mathcal{C}_{\delta^{(1)},\ldots,\delta^{(k)}}(f)$.*

2.4 Applications of symmetry I: enumerating colored factorizations of two and three factors

In this section we use the symmetry of colored factorizations (Theorem 2.1.6) to compute explicit formulae for $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ when $k=2,3$ (see (2.2.27) and (2.2.28)). The approach consists of working with vertex-labelled tree-rooted k-constellations which are in bijection with colored factorizations (by Theorem 2.3.4). Then applying the symmetry to these tree-rooted constellations we choose a convenient composition $\gamma^{(t)}$ of n with a given length like a hook. This greatly simplifies the enumeration and one can obtain explicit expressions for $c_{\gamma^{(1)},\gamma^{(2)}}$ and $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$. These formulae were first obtained in [46, 48] by Morales and Vassilieva via a bijective argument and multivariate Lagrange inversion.

The proof below of the formula for $c_{\gamma^{(1)},\gamma^{(2)}}$ is from [7, Sec. 5.2.], joint work with O. Bernardi, R.X. Du and R.P. Stanley.

Case $k=2$:

Corollary 2.4.1 ([46])**.** *If $\gamma^{(1)},\gamma^{(2)}$ are compositions of n with p_1,p_2 parts then the number $c_{\gamma^{(1)},\gamma^{(2)}}$ of colored factorizations of two factors is*

$$c_{\gamma^{(1)},\gamma^{(2)}} = \frac{n(n-p_1)!(n-p_2)!}{(n+1-p_1-p_2)!}.$$

Proof. If $\gamma^{(1)},\delta^{(1)},\gamma^{(2)},\delta^{(2)}$ are compositions of n such that $\ell(\gamma^{(1)})=\ell(\delta^{(1)})=p_1$ and $\ell(\gamma^{(2)})=\ell(\delta^{(2)})=p_2$ then by Theorem 2.1.6 $c_{\gamma^{(1)},\gamma^{(2)}}=c_{\delta^{(1)},\delta^{(2)}}$. From this property one can compute the cardinality of $\#\mathcal{T}^n_{\gamma^{(1)},\gamma^{(2)}}=c_{\gamma^{(1)},\gamma^{(2)}}$ by choosing the most convenient compositions $\gamma^{(1)}$, $\gamma^{(2)}$ of length p_1 and p_2. We take the hooks $\gamma^{(1)}=(n-p_1+1,1,1,\ldots,1)$ and $\gamma^{(2)}=(n-p_2+1,1,1,\ldots,1)$, so that $c_{\gamma^{(1)},\gamma^{(2)}}$ is the number of (p_1,p_2)-labelled bipartite tree-rooted maps with the type 1 and type 2 vertices labelled 1 of degrees $n-p_1+1$ and $n-p_2+1$ respectively, and all the other vertices of degree 1. In order to construct such an object (see Figure 2-8), one must choose the unrooted plane tree (1 choice), the labelling of the vertices ($(p_1-1)!(p_2-1)!$ choices), the $n-p_1-p_2+1$ edges not in the tree ($\binom{n-p_1}{n-p_1-p_2+1}\binom{n-p_2}{n-p_1-p_2+1}(n-p_1-p_2+1)!$ choices), and lastly the root (n choices). This gives (2.2.27): that $c_{\gamma^{(1)},\gamma^{(2)}}=\frac{n(n-p_1)!(n-p_2)!}{(n+1-p_1-p_2)!}$. $\square$

Case $k=3$**:** If $\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}$, the number $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$ was computed bijectively by Morales and Vassilieva [48]. That calculation used the the multivariate Lagrange's Implicit Function Theorem [25, Sec. 1.2.9.]. Instead of using this theorem, we use the symmetry to obtain an equivalent formula.

Corollary 2.4.2. *If $\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}$ are compositions of n with p_1,p_2,p_3 parts then the number $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$ of colored factorizations of three factors is*

$$c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}} = (n-p_1)!(n-p_2)!(n-p_3)!\times$$
$$\sum_{a\geq 0}\frac{(n-a-2)!\cdot\Theta}{a!(p_3-1-a)!(p_2-1-a)!(p_1-1)!(n-p_1-a)!(n+2-p_2-p_3+a)!}, \quad (2.4.3)$$

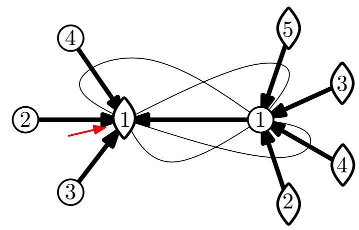


Figure 2-8: A tree-rooted map in $\mathcal{T}_{\gamma^{(1)},\gamma^{(2)}}$, where $\gamma^{(1)} = (8,1,1,1,1)$, $\gamma^{(2)} = (9,1,1,1)$. Here the map is represented using the "rotation system interpretation", so that the edge-crossings are irrelevant.

where

$$\Theta = (n+2-p_2-p_3+a)\left((n-a-1)(p_3-a)+(p_1-1)(n-p_3)\right)+$$
$$+(n-a_1-p_1)\left((n+1-p_2-p_3+a)(n+2-p_2-p_3+a)+(n+1-p_2)(p_2-1-a)\right).$$

Remark 2.4.4. *The expression in [48] for $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$ is more compact:*

$$c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}} = n^2(n-p_1)!(n-p_2)!(n-p_3)!\sum_{r\geq 0}\frac{r!(n-1-r)!\binom{p_1-1}{r}\binom{p_2-1}{r}}{(n+1-p_1-p_2-r)!(n-p_3-r)!}.$$

■

Proof. If $\gamma^{(1)},\delta^{(1)},\gamma^{(2)},\delta^{(2)},\gamma^{(3)},\delta^{(3)}$ are compositions of n such that $\ell(\gamma^{(t)}) = \ell(\delta^{(t)}) = p_t$ for $t = 1,2,3$ then by Theorem 2.1.6 $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}} = c_{\delta^{(1)},\delta^{(2)},\delta^{(3)}}$. From this property one can compute the cardinality of $\#\mathcal{T}^n_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}} = c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$ by choosing the most convenient compositions $\gamma^{(t)}$ of length p_t. We take the hooks $\gamma^{(t)} = (n-p_t+1,1,1,\ldots,1)$, so that $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$ is the number of (p_1,p_2,p_3)-labelled tree-rooted constellations with the type t vertex labelled 1 of hyperdegrees $n-p_t+1$, and all the other vertices of hyperdegree 1.

We do a refined counting of such objects. For nonnegative integers a_1,a_2,a_3, let $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)})}(a_1,a_2,a_3)$ be the number of tree-rooted constellations of vertex-compositions $(\gamma^{(1)},\gamma^{(2)},\gamma^{(3)})$ where $\gamma^{(t)} = (n+1-p_t,1,1,\ldots,1)$ for $t = 1,2,3$ where the type t vertex labelled 1 is incident to a_t 3-gons whose other two vertices are of hyperdegree 1. The following lemma gives a formula for $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1,a_2,a_3)$.

Lemma 2.4.5. *For positive integers n,p_1,p_2,p_3 and nonnegative integers a_1,a_2,a_3 if $\gamma^{(t)} = (n+1-p_t,1,1,\ldots,1)$ is a composition of n for $t = 1,2,3$ then*

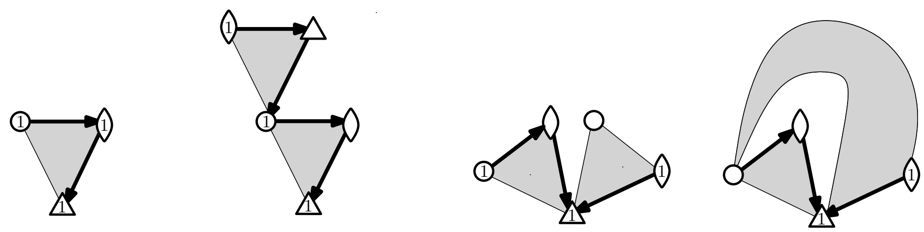
$$c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1,a_2,a_3) = \frac{\prod_{t=1}^{3}(p_t-1)!(n-p_t)!}{(n-1)!}\times$$
$$\times\binom{n}{a_1,a_2,a_3,p_1-1-a_2-a_3,p_2-1-a_1-a_3,p_3-1-a_1-a_2}\cdot\Delta,$$

where

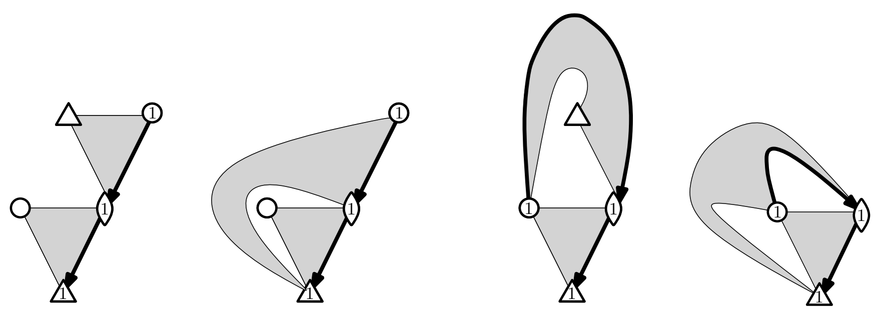
$$\Delta = (n - \sum_{t=1}^{3} a_t)(n+3-\sum_{t=1}^{3} p_t + a_t) + \sum_{t=1}^{3}(p_t - 1 - a_{t+1} - a_{t+2})(p_{t+1} - 1 - a_{t+2} - a_t),$$

and the last sum is modulo 3.

Proof. A tree-rooted constellation (C, A) counted in $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1, a_2, a_3)$ is built in the following way: we first choose the part of the arborescence A incident to the vertices labelled 1 (of hyperdgree $n + 1 - p_t$) and the 3-gons of C with the edges of the part of the arborescence. There are eight possible cases. See Figure 2-9 for an illustration of these cases. Note that Case 4. can be obtained from Case 3. by identifying the type 1 vertices. Similarly, Cases 6.,7., and 8. can be obtained from Case 5. by identifying the type 3 vertices, or the type 1 vertices, or both type 3 and type 1 vertices respectively.



	Case 1.	Case 2.	Case 3.	Case 4.
$a_{23}:$	$p_1 - 1 - a_2 - a_3$	$p_1 - 1 - a_2 - a_3$	$p_1 - 2 - a_2 - a_3$	$p_1 - 1 - a_2 - a_3$
$a_{13}:$	$p_2 - 1 - a_1 - a_3$	$p_2 - 2 - a_1 - a_3$	$p_2 - 2 - a_1 - a_3$	$p_2 - 2 - a_1 - a_3$
$a_{12}:$	$p_3 - 1 - a_1 - a_2$	$p_3 - 2 - a_1 - a_2$	$p_3 - 1 - a_1 - a_2$	$p_3 - 1 - a_1 - a_2$
$a_{123}:$	$n + 2 - \sum_{t=1}^{3} p_t + a_t$	$n + 3 - \sum_{t=1}^{3} p_t + a_t$	$n + 3 - \sum_{t=1}^{3} p_t + a_t$	$n + 2 - \sum_{t=1}^{3} p_t + a_t$



	Case 5.	Case 6.	Case 7.	Case 8.
$a_{23}:$	$p_1 - 2 - a_2 - a_3$	$p_1 - 2 - a_2 - a_3$	$p_1 - 1 - a_2 - a_3$	$p_1 - 1 - a_2 - a_3$
$a_{13}:$	$p_2 - 1 - a_1 - a_3$	$p_2 - 1 - a_1 - a_3$	$p_2 - 1 - a_1 - a_3$	$p_2 - 1 - a_1 - a_3$
$a_{12}:$	$p_3 - 2 - a_1 - a_2$	$p_3 - 1 - a_1 - a_2$	$p_3 - 2 - a_1 - a_2$	$p_3 - 1 - a_1 - a_2$
$a_{123}:$	$n + 3 - \sum_{t=1}^{3} p_t + a_t$	$n + 2 - \sum_{t=1}^{3} p_t + a_t$	$n + 2 - \sum_{t=1}^{3} p_t + a_t$	$n + 1 - \sum_{t=1}^{3} p_t + a_t$

Figure 2-9: Possible choices for the part of the tree-rooted 3-constellation with hook type $\gamma^{(t)} = (n - p_t + 1, 1^{p_t - 1})$ incident to the vertices labelled 1 (thus of hyperdegree $n - p_t + 1$). For each case we indicate the values of $a_{tt'}$: the number of additional 3-gons whose vertices of types t and t' are labelled 1, and of $a_{tt't''}$: the number of additional 3-gons whose all vertices are labelled 1.

For $t \neq t'$, let $a_{tt'}$ be the number of 3-gons whose vertices of types t and t' are labelled 1, and thus have hyperdegree $n+1-p_t$ and $n+1-p_{t'}$ respectively, but the other vertex is of hyperdegree 1, and a_{123} is the number of 3-gons with all vertices labelled 1.

The contribution for each case $i = 1, 2, \ldots, 8$ will be $\prod_{t=1}^{3}(p_t-1)! \cdot N_i \cdot n$, for the choices of the labels of the vertices of each type of hyperdegree 1 ($\prod_{t=1}^{3}(p_t-1)!$ choices), the choices of the $a_t, a_{tt'}, a_{123}$ 3-gons for case i (N_i choices that will be computed below), and lastly choosing the root (n choices). Then

$$c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1, a_2, a_3) = n \cdot \prod_{t=1}^{3}(p_t-1)! \cdot (N_1 + N_2 + \cdots + N_8). \tag{2.4.6}$$

Next we find N_i for $i = 1, 2, \ldots, 8$.

Proposition 2.4.7.

$$N_1 = \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\binom{n-1}{a_1, a_2, a_3, a_{12}, a_{23}, a_{13}, a_{123}}, \tag{2.4.8}$$

and for $i = 2, 3, \ldots, 8$,

$$N_i = \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-2)!}\binom{n-2}{a_1, a_2, a_3, a_{12}, a_{23}, a_{13}, a_{123}}, \tag{2.4.9}$$

where a_{23}, a_{13}, a_{12}, *and* a_{123} *for each case are given in Figure 2-9.*

Proof. For the first case note that $p_1 - 1 = a_2 + a_3 + a_{23}$, $p_2 - 1 = a_1 + a_3 + a_{13}$ and $p_3 - 1 = a_1 + a_2 + a_{13}$. There are $\binom{n-p_1}{a_1, a_{12}, a_{13}}\binom{n-p_2}{a_2, a_{23}, a_{12}}\binom{n-p_3}{a_3, a_{23}, a_{13}}$ choices for the kind of 3-gons incident to type t vertex labelled 1. Then there are $a_{12}!a_{13}!a_{23}!(a_{123}!)^2$ ways of identifying the 3-gons (see Figure 2-10). Recombining the binomials we obtain $\frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\binom{n-1}{a_1, a_2, a_3, a_{12}, a_{23}, a_{13}, a_{123}}$.

By a similar argument to that of Case 1. one can show the formula in (A.0.2) for N_i, $i = 2, 3, \ldots, 8$. ☐

By using the formulas in (A.0.1), (A.0.2) for N_i for $i = 1, 2, \ldots, 8$ in (2.4.6) we obtain the desired expression for $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1, a_2, a_3)$. This finishes the proof of Lemma 2.4.5.

☐

Since $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}} = \sum_{a_1,a_2,a_3 \geq 0} c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1, a_2, a_3)$, from Lemma 2.4.5 and (2.4.6) we can obtain (2.4.3) via manipulations with binomials and using the Chu-Vandermonde identity repeatedly. This is done in the Appendix A to complete the proof of Corollary 2.4.2. ☐

Remark 2.4.10. *In this section we have seen that using the symmetry of colored factorizations reduces the calculation of* $c_{\gamma^{(1)},\gamma^{(2)}}$ *to one case (see Figure 2-8) but reduces the calculation of* $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$ *to eight cases (see Figure 2-9) and several manipulations*

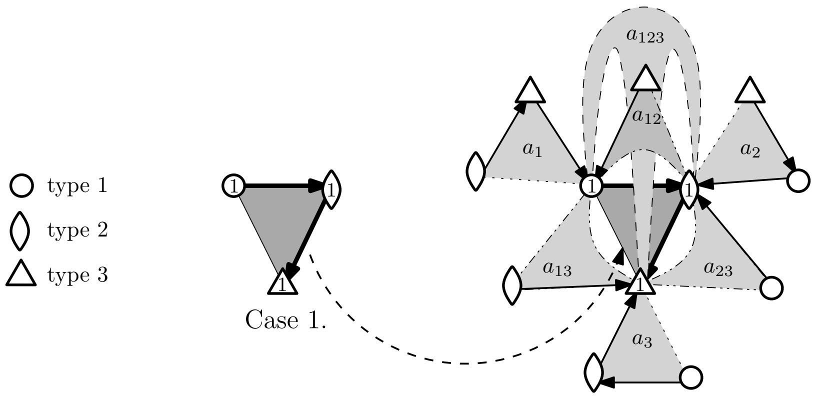


Figure 2-10: Illustration for Case 1. of the 3-gons counted by a_1, a_2, a_3 (one vertex of hyperdegree $n+1-p_t$, the other two of hyperdegree 1) , by a_{12}, a_{13}, a_{23} (two vertices of hyperdegree $n+1-p_t$, the other of hyperdegree 1), and by a_{123} (all vertices of hyperdegree $n+1-p_t$). Here the 3-constellation is represented using the "rotation system interpretation", so that the crossings of the 3-gons are irrelevant.

with binomials. Thus, the symmetry alone does not seem to be sufficient to tractably compute $c_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ *for all* k *and thus to prove Jackson's formula (Corollary* (2.1.7)*). In [10] there are two more combinatorial constructions starting from tree-rooted* k*-constellations aimed at fully proving this formula.* ■

2.5 Applications of symmetry II: enumerating planar cacti

This section is from [8], joint work with O. Bernardi.

2.5.1 Introduction

We study the enumeration of rooted planar k-cacti by given vertex degree distributions. We carry out this enumeration by showing, via simple bijective arguments, two symmetry properties that these objects satisfy. These symmetry properties reduce the complexity of the enumeration to a calculation of a particular simple case. We recover the classical formulas of Goulden-Jackson [24] that were obtained using the multivariate Lagrange's Implicit Function Theorem [25, Sec. 1.2.9.].

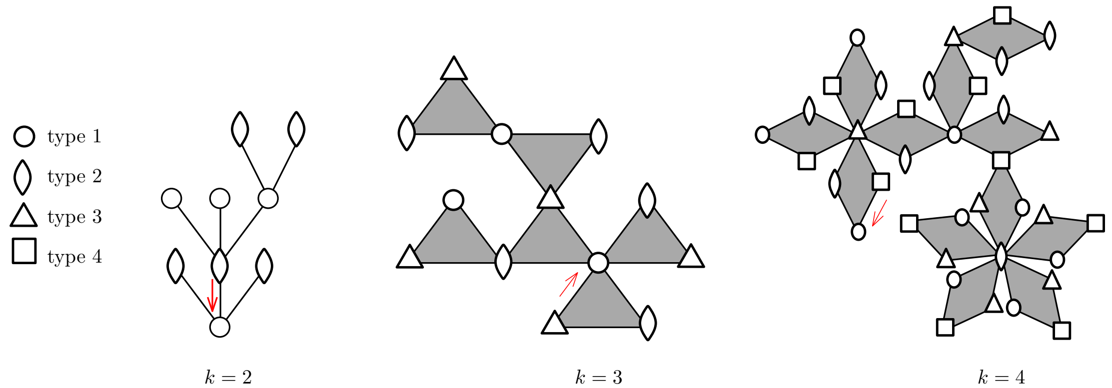


Figure 2-11: Examples of rooted planar k-cacti for $k = 2, 3$ and 4.

2.5.2 Enumeration of planar rooted k-cacti using symmetry

A k-**cactus** is a connected simple graph such that every edge lies on exactly one k-cycle which we call a k-gon. We say a k-cactus is **planar** if it is embedded in the plane such that every edge is part of the unbounded region. Thus the planar cacti consists of an unbounded face and the bounded connected regions which are precisely the k-gons. A planar k-cacti has **size** n if it has n k-gons and it is rooted if it has a distinguished edge. The **degree** of a vertex is the number of k-gons incident to the vertex.

We associate to a rooted planar k-cacti the following canonical coloring or assignment of types to the vertices. Starting at the root edge, we traverse the boundary of the unbounded region always keeping it to the left. As we traverse the region we assign to the vertices types $1, 2, \ldots, k$ cyclically starting by assigning the vertices of the root edge types k and 1 respectively. It is clear that the assignment is well-defined, meaning that if a vertex has degree j it will be visited j times by the traversal of the unbounded face but it will be assigned the same type each time. We call the type 1 vertex of the root edge the **root vertex**. Also note that between any two vertices u and v of a planar k-cacti there is a unique shortest **path of k-gons** between two k-gons with vertices u and v.

We say the cacti is **vertex-labelled** if for each $t \in [k]$ the p_t type t vertices have distinct labels in $[p_t]$. The **degree distribution** of the type t vertices is the composition $\gamma^{(t)}$ of n with p_t parts whose ith part gives the degree of the type t vertex labelled i. If $\gamma^{(1)}, \ldots, \gamma^{(k)}$ are compositions of n, a planar cacti has vertex-degree distribution $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ if for all $t \in [k]$, the type t vertices have degree distribution $\gamma^{(t)}$. Let $\mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ be the set of such vertex-labelled rooted planar cacti, and $C_{\gamma^{(1)},\ldots,\gamma^{(k)}} = \#\mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$. See Figure 2-11 for examples of vertex-labelled rooted planar k-cacti for $k = 2, 3$ and 4.

If $p_t = \ell(\gamma^{(t)})$ then $1 \le p_t \le n$ and by the Euler characteristic of the cacti we have that $\sum_{t=1}^{k} p_t = (k-1)n + 1$. In [24, Thm. 3.2] Goulden and Jackson counted these rooted planar k-cacti (but with unlabelled vertices) using the multivariate Lagrange Inversion Theorem [25, Sec. 1.2.9.].

Theorem 2.5.1 ([24, Theorem 3.2])**.** *Let $\gamma^{(1)}, \ldots, \gamma^{(k)}$ be k compositions of n with $\ell(\gamma^{(t)}) = p_t$ parts such that $\sum_{t=1}^{k} p_t = (k-1)n + 1$ then the number of vertex-labelled rooted planar k-cacti where the type t vertices have degree distribution given by $\gamma^{(t)}$ is*

$$C_{\gamma^{(1)},\ldots,\gamma^{(k)}} = n^{k-1} \cdot \prod_{t=1}^{k} (p_t - 1)!. \tag{2.5.2}$$

Remark 2.5.3. *The rooted planar k-cacti counted in [24] have unlabelled vertices. Thus the number of k-cacti they consider is $C_{\gamma^{(1)},\ldots,\gamma^{(k)}} / \prod_{t=1}^{k} \prod_j n_j(\gamma^{(t)})!$ where $n_j(\gamma^{(t)})$ is the number of parts of $\gamma^{(t)}$ equal to j.* ■

If instead we count rooted planar k-cacti with unlabelled vertices not by degree distribution but by the number of vertices of each type we get the **Narayana numbers** $\frac{1}{n}\binom{n}{p_1}\binom{n}{p_1-1}$ for $k = 2$ and for general k we obtain the following formula for these numbers.

Corollary 2.5.4. *For positive integers $p_1, \ldots, p_k$ such that $1 \le p_t \le n$ and $\sum_{t=1}^{k} p_t = (k-1)n + 1$, let $C(n; p_1, \ldots, p_k)$ be the number of rooted planar k-cacti of size n with unlabelled vertices and p_t vertices of type t for $t = 1, \ldots, k$. Then*

$$C(n; p_1, \ldots, p_t) = n^{k-1} \cdot \prod_{t=1}^{k} \frac{1}{p_t} \binom{n-1}{p_t}. \tag{2.5.5}$$

Proof. The set of rooted planar k-cacti of size n with p_t labelled vertices of type t has cardinality $(\prod_{t=1}^{k} p_t!) \cdot C(n; p_1, \ldots, p_t)$. Also this set is the disjoint union $\bigcup_{\gamma^{(t)} \models n,\ \ell(\gamma^{(t)}) = p_t} \mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$. But since $C_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ only depends on n, k and p_t and there are $\binom{n-1}{p_t-1}$ compositions of n with p_t parts we have that

$$\sum_{\gamma^{(t)} \models n, \ell(\gamma^{(t)}) = p_t} C_{\gamma^{(1)},\ldots,\gamma^{(k)}} = \prod_{t=1}^{k} \binom{n-1}{p_t - 1} \cdot n^{k-1} \cdot \prod_{t=1}^{k} (p_t - 1)!.$$

Using this equation and doing some straightforward cancellations we obtain the desired formula for $C(n; p_1, \ldots, p_k)$. $\square$

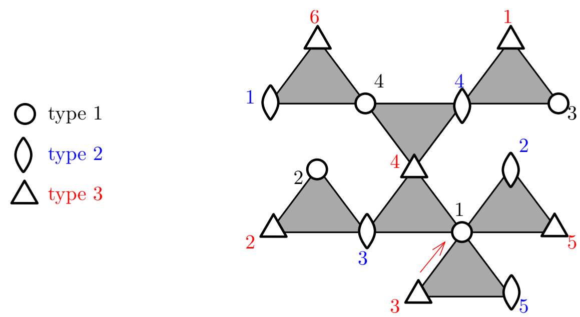


Figure 2-12: A rooted planar 3-cacti of size 7 with labelled vertices and vertex degree distribution $(\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})$ where $\gamma^{(1)} = (3, 1, 1, 2)$, $\gamma^{(2)} = (1, 1, 2, 2, 1)$ and $\gamma^{(3)} = (1, 1, 1, 2, 1, 1)$.

Remark 2.5.6. *Rooted planar k-cacti are related to* **minimal factorizations** *of the cycle $(1, 2, \ldots, n)$ into k permutations (see [24][Thm. 2.1] and [36][Sec. 1.3]).* $\blacksquare$

Symmetry of degree distribution by type for planar cacti

The first symmetry result states that $C_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ only depends on n, k and on the number of parts of $\gamma^{(t)}$.

Theorem 2.5.7 (Symmetry of degree distribution of cacti)**.** *Let $\gamma^{(1)}, \ldots, \gamma^{(k)}$ and $\delta^{(1)}, \ldots, \delta^{(k)}$ be compositions of n such that $\ell(\gamma^{(t)}) = \ell(\delta^{(t)})$ for all $t \in [k]$ then*

$$C_{\gamma^{(1)},\ldots,\gamma^{(k)}} = C_{\delta^{(1)},\ldots,\delta^{(k)}}. \tag{2.5.8}$$

This means that we can compute $C_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ where $p_t = \ell(\gamma^{(t)})$ by choosing simple compositions of n with p_t parts like **hooks** $\gamma^{(t)} = (n + 1 - p_t, 1^{p_t - 1})$.

Corollary 2.5.9. *Theorem 2.5.1 is equivalent to showing that the number of rooted planar k-cacti of vertex-degree distribution $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ where $\gamma^{(t)}$ is the hook $(n + 1 - p_t, 1^{p_t - 1})$ is*

$$C_{(n+1-p_1,1^{p_1-1}),\ldots,(n+1-p_k,1^{p_k-1})} = n^{k-1} \cdot \prod_{t=1}^{k} (p_t - 1)!.$$

In Section 2.5.2 we finish the proof of Theorem 2.5.1 by computing the number of such particular planar k-cacti using another "symmetry". The rest of this section is devoted to the proof of Theorem 2.5.7.

Proof. Let $t \in [k]$ and $i, j \in [p_t]$. In order to prove Theorem 2.5.7 it suffices to give a bijection $\Phi_{t;i,j}$ between $\mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ and $\mathcal{C}_{\delta^{(1)},\ldots,\delta^{(k)}}$ where $\gamma^{(s)} = \delta^{(s)}$ for $s \neq t$, but $\gamma_x^{(t)} = \delta_x^{(t)}$ for all $x \neq i, j$ and $\gamma_i^{(t)} - 1 = \delta_i^{(t)}$ and $\gamma_j^{(t)} + 1 = \delta_j^{(t)}$. In other words, we want to give a bijection $\Phi_{t;i,j}$ which decreases by one the degree of the type t vertex labelled i and increases by one the degree of the type t vertex labelled j and leaves unchanged the degrees of all the other vertices. Note that since the k-cacti are planar it is well defined to *unglue* a k-gon from a vertex of type t and *reglue* it in a specified corner of another vertex of type t. We will use these operations to define the mapping $\Phi_{t;i,j}$ below; see Figure 2-13.

Let $\mathcal{C}^{(t;i)}$ be the set of vertex-labelled rooted planar k-cacti with p_t type t vertices for $t = 1, \ldots, k$ such that the vertex of type t labelled i has degree at least two. Let C be a cacti in $\mathcal{C}^{(t;i)}$, let u_i and u_j be the vertices of type t labelled i and j respectively and let P be the path of k-gons from u_i to u_j.

We denote by g_i be the k-gon in P with vertex u_i. We define g_j to be the k-gon in P with vertex u_j. Let g_i' be the k-gon preceding g_i in clockwise order around u_i. Observe that $g_i \neq g_i'$ since the degree of u_i is at least two. We define $\Phi_{t;i,j}(C)$ as the planar cacti obtained from C by ungluing the k-gon g_i' from u_i and gluing it to u_j in the corner preceding the k-gon g_j in clockwise order around u_j; see Figure 2-13. Observe that $\Phi_{t;i,j}(C)$ is in $\mathcal{C}^{(t;j)}$. It is also easy to see that $\Phi_{t;j,i}(\Phi_{t;i,j}(C)) = C$.

We have shown that $\Phi_{t;i,j}$ is a mapping from $\mathcal{C}^{(t;i)}$ to $\mathcal{C}^{(t;j)}$. Moreover $\Phi_{t;j,i} \circ \Phi_{t;i,j} = Id$ for all i, j, thus $\Phi_{t;i,j} = \Phi_{t;j,i}^{-1}$ is a bijection. Lastly, the bijection $\Phi_{t;i,j}$ decreases by one the degree of the vertex of type t labelled i and increases by one the degree of the vertex of type t labelled j. Thus $\Phi_{t;i,j}$ has all the claimed properties. $\square$

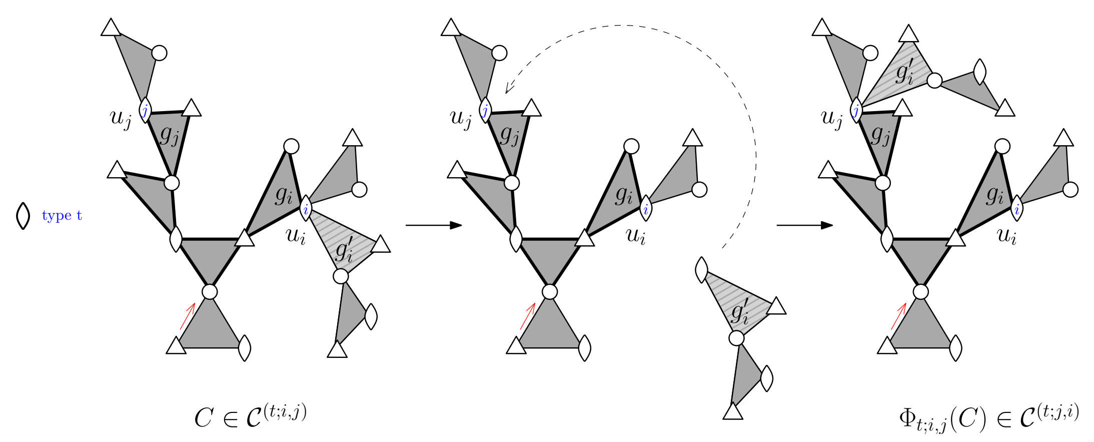


Figure 2-13: The bijection $\Phi_{t;i,j}$ applied to a planar cacti in $\mathcal{C}^{(t;i)}$. The k-gons with a **bold** border are part of the path P of k-gons from u_i to u_j.

Symmetry of number of vertices by type for planar hook cacti

From Corollary 2.5.9 the number of rooted planar k-cacti of degree distribution $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ is the same as the number of rooted planar k-cacti of degree distribution given by hooks $\gamma^{(t)} = (n+1-p_t, 1^{p_t-1})$ where $p_t = \ell(\gamma^{(t)})$. Thus, we now focus on this hook case and so given positive integers $p_1, \ldots, p_k$ such that $1 \leq p_t \leq n$ and $\sum_{t=1}^k p_i = (k-1)n+1$, let $\mathcal{H}^{(n)}_{p_1,\ldots,p_k}$ be the set $\mathcal{C}_{\gamma^{(1)},\ldots,\gamma^{(k)}}$ of rooted planar k-cacti with vertex degree distribution $(\gamma^{(1)}, \ldots, \gamma^{(k)})$ where $\gamma^{(t)}$ is the hook $(n+1-p_t, 1^{p_t-1})$. Note that the type t vertex labelled 1 is the vertex of degree $n+1-p_t$. We call such cacti: rooted **planar hook k-cacti** and we let $H^{(n)}_{p_1,\ldots,p_k} = \#\mathcal{H}^{(n)}_{p_1,\ldots,p_k}$. See Figure 2-14 (a) for an example.

Counting $H^{(n)}_{p_1,\ldots,p_k}$ directly for small k like $k = 2, 3$ is tractable but for bigger k the computations and case analysis gets trickier. So instead of counting planar hook k-cacti directly we use another symmetry that these hook k-cacti satisfy. This symmetry states that $H^{(n)}_{p_1,\ldots,p_k}$, up to some easy factors from vertex labelling, depends only on n and k.

Theorem 2.5.10 (Symmetry of number of vertices of hook cacti)**.** *Let $p_1, \ldots, p_k$ be positive integers such that $1 \leq p_i \leq n$ and $\sum_{t=1}^k p_t = (k-1)n+1$. If there is an index $s \in \{1, \ldots, k\}$ such that $p_a < n$ then $p_b > 1$ for all other b in $[k]\backslash\{s\}$. For each such b there is a (p_b-1)-to-p_a correspondence between the sets of planar hook k-cacti $\mathcal{H}^{(n)}_{p_1,\ldots,p_k}$ and $\mathcal{H}^{(n)}_{p_1,\ldots,p_b-1,\ldots,p_a+1,\ldots,p_k}$.*

The proof of this theorem is at the end of this section. By this symmetry we can compute $H^{(n)}_{p_1,\ldots,p_k}$ by repeatedly increasing the number of type t-vertices for $t = 2, \ldots, k$ and decreasing the number of say type 1-vertices. Therefore we can relate $H^{(n)}_{p_1,\ldots,p_k}$ with the extreme case when $p_1 = 1$ and $p_2 = p_3 = \cdots = p_k = n$.

Corollary 2.5.11. *Let $p_1, \ldots, p_k$ be positive integers such that $1 \leq p_t \leq n$ and $\sum_{t=1}^k p_t = (k-1)n+1$. Then*

$$H^{(n)}_{p_1,\ldots,p_k} = \frac{(p_1-1)!}{\prod_{t=2}^k (n-1)_{n+1-p_t}} \cdot H^{(n)}_{1,n,\ldots,n},$$

where $(x)_m = x(x-1)\cdots(x-m+1)$.

Proof. If there is an index a in $\{2, \ldots, k\}$ such that $p_a < n$ then by Theorem 2.5.10 for $b = 1$ we have that

$$H^{(n)}_{p_1,\ldots,p_k} = \frac{p_1-1}{p_a} \cdot H^{(n)}_{p_1-1,p_2,\ldots,p_{s-1},p_s+1,p_{s+1},\ldots,p_k}.$$

Then by repeated application of this result we reach to $p_1 = 1$ and $p_2 = p_3 = \cdots = p_t = n$ and obtain the desired formula. □

Thus by Corollaries 2.5.9 and 2.5.11, proving Theorem 2.5.1 is equivalent to showing that $H^{(n)}_{1,n,\ldots,n} = (n!)^{k-1}$. Computing this number $H^{(n)}_{1,n,\ldots,n}$ of rooted planar hook k-cacti is very easy.

Proposition 2.5.12. *The number of rooted planar k-cacti of size n with one type 1 vertex and n type t vertices for $t = 2, \ldots, k$ is*

$$H^{(n)}_{1,n,\ldots,n} = (n!)^{k-1}.$$

Proof. A rooted planar hook k-cacti with $p_1 = 1$ and $p_2 = p_3 = \cdots = p_k = n$ has only one type 1 vertex of degree n which is the root labelled 1. For $t = 2, \ldots, k$, the k-cacti has n type t vertices of degree 1 with distinct labels in $[n]$. See Figure 2-14 (b) for an illustration of such a planar hook k-cacti with the restriction on ℓ'_1.

Thus we calculate $H^{(n)}_{1,n,\ldots,n}$ in the following way: there is one k-cacti consisting of a type 1 vertex which is the root incident to n k-gons whose vertices (except for the root) have all degree 1. And for each type $t = 2, \ldots, k$, there are $n!$ ways of labelling the n vertices with the distinct labels $[n]$. This gives $(n!)^{k-1}$. $\square$

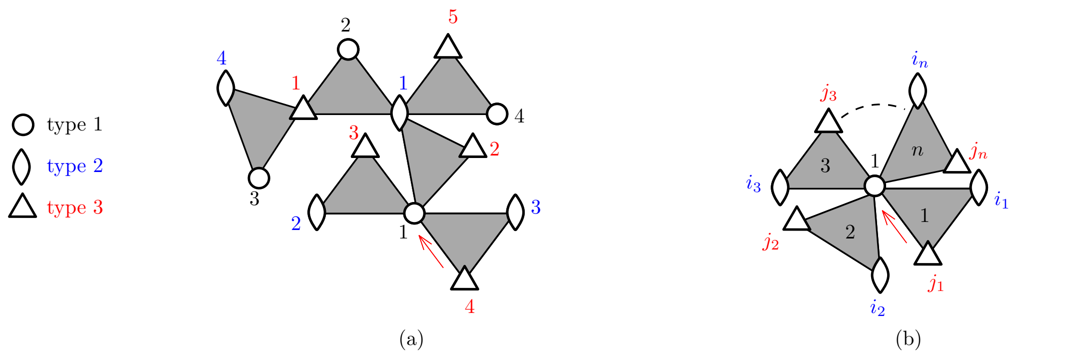


Figure 2-14: (a) Example of a hook cacti of size 6 with $p_1 = 4$, $p_2 = 4$, $p_3 = 5$. (b) Illustration of hook cacti of size n with $p_1 = 1$, $p_2 = p_3 = n$. There are $(n!)^2$ such hook cacti since this is the number of ways to label the n vertices of type 2 and type 3.

Now to complete the combinatorial proof of Theorem 2.5.1 we need to prove bijectively Theorem 2.5.10. We devote the rest of this section on this proof. We exhibit a map in the same spirit as the bijection $\Phi_{t;i,j}$ in Theorem 2.5.7.

Proof. Since the positive integers $p_1, \ldots, p_k$ satisfy $1 \leq p_t \leq n$ and $\sum_{t=1}^{k} p_t = (k-1)n+1$ if $p_a < n$ for some $a \in \{2, \ldots, k\}$ then $p_b > 1$ for all other b in $[k]\backslash\{a\}$. Fix one such b. To prove Theorem 2.5.10 it suffices to exhibit a $(p_b - 1)$-to-p_a correspondence between the sets $\mathcal{H}^{(n)}_{p_1,\ldots,p_k}$ and $\mathcal{H}^{(n)}_{q_1,\ldots,q_k}$ where $q_a = p_a + 1$, $q_b = p_b - 1$ and $q_t = p_t$ for $t \in [k]\backslash\{a, b\}$.

Given a planar hook k-cacti H in $\mathcal{H}^{(n)}_{p_1,\ldots,p_k}$ its type a vertex v_a labelled 1 has degree $n + 1 - p_a > 1$. Let g_a be the k-gon on Q with vertex v_a and let g'_a be the k-gon preceding g_a in clockwise order around v_a. Note that $g_a \neq g'_a$ since v_a has degree $n + 1 - p_a > 1$. Let v'_b be the type b vertex of g'_a. Since H is a planar hook cacti and $g_a \neq g'_a$ then $v'_a \neq v_a$ and so v'_a has degree one and label $\ell'_b \in \{2, \ldots, p_b\}$. For simplicity we will consider those hook cacti where $\ell'_b = p_b$. That is, let $\mathcal{H}^{(n;a,b)}_{p_1,\ldots,p_k}$ be the set of planar hook k-cacti in $\mathcal{H}^{(n)}_{p_1,\ldots,p_k}$ where the type b vertex v'_b of the k-gon g'_a is labelled $\ell'_b = p_b$. Note that there is a $(p_b - 1)$-to-1 correspondence between $\mathcal{H}^{(n)}_{p_1,\ldots,p_k}$ and $\mathcal{H}^{(n;a,b)}_{p_1,\ldots,p_k}$.

Next we build a map $\Psi_{a,b}$ between $\mathcal{H}^{(n;a,b)}_{p_1,\ldots,p_k}$ and $\mathcal{H}^{(n;b,a)}_{q_1,\ldots,q_k}$. Given a planar hook k-cacti H in $\mathcal{H}^{(n;a,b)}_{p_1,\ldots,p_k}$, let the vertices v_a, v'_b and the k-gons g_a, g'_a be as defined in the previous paragraph. Let v_b be the type b vertex in H labelled 1 which has degree $n+1-p_b$. Let Q be the path of k-gons from v_a to v_b. Let g_b be the k-gon on Q with vertex v_b. We define $\Psi_{a,b}(H)$ to be the planar k-cacti obtained from H by (i) ungluing the k-gon g'_a from v_a, (ii) unlabelling v'_b and the unglued type a vertex v'_a of g'_a and (iii) gluing the k-gon via v'_b to the vertex v_b of type b in the corner preceding g_b in clockwise order around v_b. Finally: (iv) we relabel v'_a with the label p_a+1; see Figure 2-15. The resulting planar k-cacti $\Psi_{a,b}(H)$ is in $\mathcal{H}^{(n;b,a)}_{q_1,\ldots,q_k}$. That is, it is a hook cacti with $q_t = p_t$ type t vertices for $t \in [k]\backslash\{a,b\}$, $q_b = p_b - 1$ type b vertices, and $q_a = p_a + 1$ type a vertices where the type a vertex v'_a of degree one in the k-gon preceding g_b in clockwise order around the root r has label $q_a = p_a + 1$.

The map $\Psi_{a,b} \circ \Psi_{b,a} = Id$ for all $a, b \in [k]$, thus $\Psi_{a,b} = \Psi^{-1}_{b,a}$ is a bijection. See Figure 2-15 for an illustration of this bijection.

Finally, since $q_q = p_q + 1$, then there is a 1-to-p_a correspondence between $\mathcal{H}^{(n;b,a)}_{q_1,\ldots,q_k}$ and $\mathcal{H}^{(n)}_{q_1,\ldots,q_k}$.

Putting it all together we obtain a (p_b-1)-to-p_a correspondence between $\mathcal{H}^{(n)}_{p_1,\ldots,p_k}$ and $\mathcal{H}^{(n)}_{q_1,\ldots,q_k}$ as desired. □

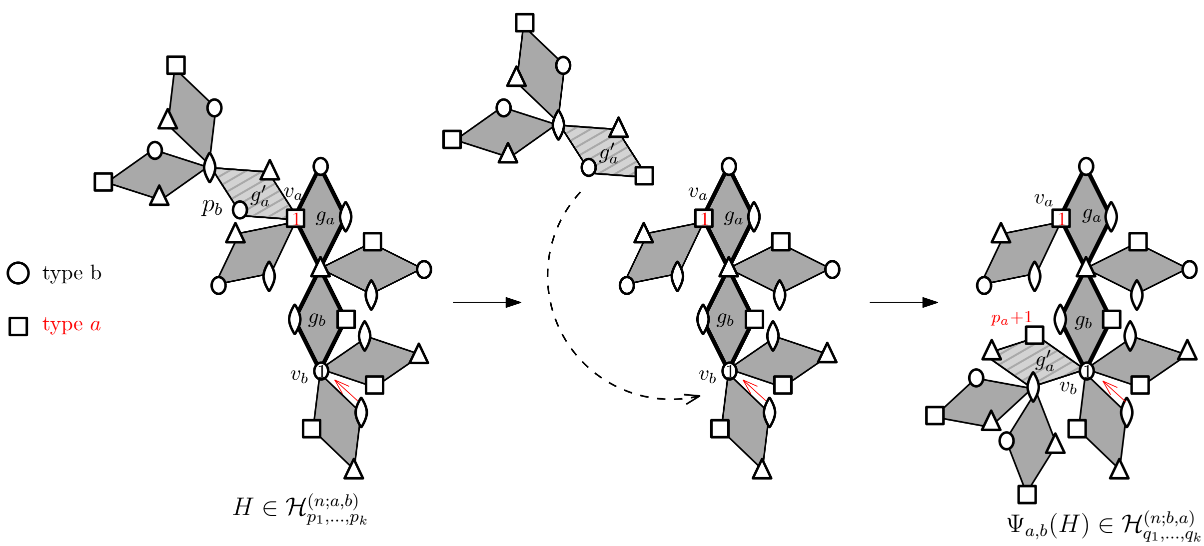


Figure 2-15: The bijection $\Psi_{a,b}$. The k-gons with a **bold** border are part of the path Q of k-gons from v_a to v_b.

With the proof of Theorem 2.5.10 we complete the combinatorial proof of Theorem 2.5.1.

Remark 2.5.13. *Theorem 2.5.7, the symmetry of degree distribution of planar k-cacti, is a special case of a symmetry property of* **vertex colored one-face k-constellations** *proved in [10][Thm. 1.3]. However, Theorem 2.5.10, the symmetry* *of number of vertices of planar hook k-cacti, does not hold is this more general setting of colored constellations.* ■

2.6 Applications of symmetry III: separation probabilities

This section is from [7], joint work with O. Bernardi, R.X. Du and R.P. Stanley.

2.6.1 Background on separation

Now we apply the symmetry of colored factorizations of the long cycle $(1, 2, \dots, n)$ (Theorem 2.1.6) to study *separation probabilities* for products of permutations. The archetypal question can be stated as follows: *In the symmetric group $\mathfrak{S}_n$, what is the probability that the elements $1, 2, \dots, k$ are in distinct cycles of the product of two n-cycles chosen uniformly randomly?*

The answer is very elegant: the probability is $\frac{1}{k!}$ if $n-k$ is odd and $\frac{1}{k!}+\frac{2}{(k-2)!(n-k+1)(n+k)}$ if $n-k$ is even. This result was originally conjectured by Bóna [13] for $k = 2$ and proved for all k by Du and Stanley [59]. Du and Stanley also proposed additional conjectures related to this question. In this section we prove such conjecture and study further generalizations. The approach here is different from the one used in [59].

Let us define a larger class of problems. Given a tuple $A = (A_1, \dots, A_k)$ of k disjoint non-empty subsets of $\{1, \dots, n\}$, we say that a permutation π is **A-separated** if no cycle of π contains elements of more than one of the subsets A_i. Now, given two integer partitions λ, μ of n, one can wonder about the probability $P_{\lambda,\mu}(A)$ that the product of two uniformly random permutations of cycle type λ and μ is A-separated. The example presented above corresponds to $A = (\{1\}, \dots, \{k\})$ and $\lambda = \mu = (n)$. Clearly, the separation probabilities $P_{\lambda,\mu}(A)$ only depend on A through the size of the subsets $\#A_1, \dots, \#A_k$, and we shall denote $\sigma^{\alpha}_{\lambda,\mu} := P_{\lambda,\mu}(A)$, where $\alpha = (\#A_1, \dots, \#A_k)$ is a composition (of size $m \leq n$). Note also that $\sigma^{\alpha}_{\lambda,\mu} = \sigma^{\alpha'}_{\lambda,\mu}$ whenever the composition α' is a permutation of the composition α. Below, we focus on the case $\mu = (n)$ and we further denote $\sigma^{\alpha}_{\lambda} := \sigma^{\alpha}_{\lambda,(n)}$.

In this section, we first express the separation probabilities $\sigma^{\alpha}_{\lambda}$ as some coefficients in an explicit generating function. Using this expression we then prove the following symmetry property: if $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ are compositions of the same size $m \leq n$ and of the same length k, then

$$\frac{\sigma^{\alpha}_{\lambda}}{\prod_{i=1}^{k} \alpha_i!} = \frac{\sigma^{\beta}_{\lambda}}{\prod_{i=1}^{k} \beta_i!}. \tag{2.6.1}$$

Moreover, for certain partitions λ (including the cases $\lambda = (n)$ and $\lambda = 2^N$) we obtain explicit expressions for the probabilities $\sigma^{\alpha}_{\lambda}$ for certain partitions λ. For instance, the separation probability $\sigma^\alpha_{(n)}$ for the product of two n-cycles is found to be

$$\sigma^\alpha_{(n)} = \frac{(n-m)!\prod_{i=1}^k \alpha_i!}{(n+k)(n-1)!}\left(\frac{(-1)^{n-m}\binom{n-1}{k-2}}{\binom{n+m}{m-k}} + \sum_{r=0}^{m-k}\frac{(-1)^r\binom{m-k}{r}\binom{n+r+1}{m}}{\binom{n+k+r}{r}}\right). \qquad (2.6.2)$$

This includes the case $\alpha = 1^k$ proved by Du and Stanley [59].

Our general expression for the separation probabilities σ^α_λ is derived using the formula for $c^{(n)}_{\gamma^{(1)},\gamma^{(2)}}$ in Equation (2.2.27). This formula, proved bijectively in [46] and later in [6], displays the symmetry mentioned in Theorem 2.1.6 which turns out to be of crucial importance for our method.

Outline. In Section 2.6.2 we present our strategy for computing the separation probabilities. This involves counting certain colored factorizations of the n-cycle. We then gather our main results in Section 2.6.3. In particular we prove the symmetry property (2.6.1) and obtain formulas for the separation probabilities σ^α_λ for certain partitions λ including $\lambda = (n)$ or when $\lambda = 2^N$.

2.6.2 How to go from separation probabilities to colored factorizations

In this section, we first translate the problem of determining the separation probabilities σ^α_λ into the problem of enumerating certain sets $\mathcal{S}^\alpha_\lambda$. Then, we introduce a symmetric function $G^\alpha_n(\mathbf{x}, t)$ whose coefficients in one basis are the cardinalities $\#\mathcal{S}^\alpha_\lambda$, while the coefficients in another basis count certain "colored" separated factorizations of the permutation $(1, \ldots, n)$. Lastly, we give exact counting formulas for these colored separated factorizations. Our main results will follow as corollaries in Section 2.6.3.

For a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of size $m \leq n$, we denote by $\mathcal{A}^\alpha_n$ the set of tuples $A = (A_1, \ldots, A_k)$ of pairwise disjoint subsets of $[n]$ with $\#A_i = \alpha_i$ for all i in $[k]$. Observe that $\#\mathcal{A}^\alpha_n = \binom{n}{\alpha_1,\alpha_2,\ldots,\alpha_k,n-m}$.

Now, recall from Section 2.6.1 that σ^α_λ is the probability for the product of a uniformly random permutation of cycle type λ with a *uniformly random* n-cycle to be A-separated for a *fixed* tuple A in $\mathcal{A}^\alpha_n$. Alternatively, it can be defined as the probability for the product of a uniformly random permutation of cycle type λ with a *fixed* n-cycle to be A-separated for a *uniformly random* tuple A in $\mathcal{A}^\alpha_n$ (since the only property that matters is that the elements in A are randomly distributed in the n-cycle).

Definition 2.6.3. *For an integer partition λ of n, and a composition α of $m \leq n$, we denote by $\mathcal{S}^\alpha_\lambda$ the set of pairs (π, A), where π is a permutation in $\mathcal{C}_\lambda$ and A is a tuple in $\mathcal{A}^\alpha_n$ such that the product $\pi \circ (1, 2, ..., n)$ is A-separated.*

From the above discussion we obtain for any composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of size m,

$$\sigma_\lambda^\alpha = \frac{\#\mathcal{S}_\lambda^\alpha}{\binom{n}{\alpha_1, \alpha_2, \ldots, \alpha_k, n-m} \#\mathcal{C}_\lambda}. \tag{2.6.4}$$

Enumerating the sets $\mathcal{S}_\lambda^\alpha$ directly seems rather challenging. However, we will show below how to enumerate a related class of "colored" separated permutations denoted by $\mathcal{T}_\gamma^\alpha(r)$. Similar to Definition 2.1.3, we define a **cycle coloring** of a permutation $\pi \in \mathfrak{S}_n$ in $[q]$ to be a mapping c from $[n]$ to $[q]$ such that if $i, j \in [n]$ belong to the same cycle of π then $c(i) = c(j)$. We think of $[q]$ as the set of *colors*, and $c^{-1}(i)$ as set of *elements colored* i.

Definition 2.6.5. *Let $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ be a composition of size n and length ℓ, and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of size $m \leq n$ and length k. For a nonnegative integer r we define $\mathcal{T}_\gamma^\alpha(r)$ as the set of quadruples (π, A, c_1, c_2), where π is a permutation of $[n]$, $A = (A_1, \ldots, A_k)$ is in $\mathcal{A}_n^\alpha$, and*

- *(i) c_1 is a cycle coloring of π in $[\ell]$ such that there are γ_i element colored i for all i in $[\ell]$,*
- *(ii) c_2 is a cycle coloring of the product $\pi \circ (1, 2, \ldots, n)$ in $[k + r]$ such that every color in $[k + r]$ is used and for all i in $[k]$ the elements in the subset A_i are colored i.*

Note that condition (ii) in Definition 2.6.5 and the definition of cycle coloring implies that the product $\pi \circ (1, 2, \ldots, n)$ is A-separated.

In order to relate the cardinalities of the sets $\mathcal{S}_\lambda^\alpha$ and $\mathcal{T}_\gamma^\alpha(r)$, it is convenient to use symmetric functions (in the variables $\mathbf{x} = \{x_1, x_2, x_3, \ldots\}$). Namely, given a composition α of $m \leq n$, we define

$$G_n^\alpha(\mathbf{x}, t) := \sum_{\lambda \vdash n} p_\lambda(\mathbf{x}) \sum_{(\pi, A) \in \mathcal{S}_\lambda^\alpha} t^{\text{excess}(\pi, A)},$$

where the outer sum runs over all the integer partitions of n, and $\text{excess}(\pi, A)$ is the number of cycles of the product $\pi \circ (1, 2, \ldots, n)$ containing none of the elements in A. Recall that the power symmetric functions $p_\lambda(\mathbf{x})$ form a basis of the ring of symmetric functions, so that the contribution of a partition λ to $G_n^\alpha(\mathbf{x}, t)$ can be recovered by extracting the coefficient of $p_\lambda(\mathbf{x})$ in this basis:

$$\#\mathcal{S}_\lambda^\alpha = [p_\lambda(\mathbf{x})] \; G_n^\alpha(\mathbf{x}, 1). \tag{2.6.6}$$

As we prove now, the sets $\mathcal{T}_\gamma^\alpha(r)$ are related to the coefficients of $G_n^\alpha(\mathbf{x}, t)$ in the basis of monomial symmetric functions.

Proposition 2.6.7. *If α is a composition of length k, then*

$$G_n^\alpha(\boldsymbol{x}, t+k) = \sum_{\gamma \vdash n} m_\gamma(\boldsymbol{x}) \sum_{r \geq 0} \binom{t}{r} \#\mathcal{T}_\gamma^\alpha(r), \tag{2.6.8}$$

where the outer sum is over all integer partitions of n, and $\displaystyle\binom{t}{r} := \frac{t(t-1)\cdots(t-r+1)}{r!}$.

Proof. Since both sides of (2.6.8) are polynomial in t and symmetric function in $\mathbf{x}$ it suffices to show that for any nonnegative integer t and any partition γ the coefficient of $\mathbf{x}^\gamma$ is the same on both sides of (2.6.8). We first determine the coefficient $[\mathbf{x}^\gamma]G_n^\alpha(\mathbf{x}, t+k)$ when t is a nonnegative integer. Let λ be a partition, and π be a permutation of cycle type λ. Then the symmetric function $p_\lambda(\mathbf{x})$ can be interpreted as the generating function of the cycle colorings of π, that is, for any sequence $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ of nonnegative integers, the coefficient $[\mathbf{x}^\gamma]p_\lambda(\mathbf{x})$ is the number of cycle colorings of π such that γ_i elements are colored i, for all $i > 0$. Moreover, if π is A-separated for a certain tuple $A = (A_1, \ldots, A_k)$ in $\mathcal{A}_n^\alpha$, then $(t+k)^{\mathrm{excess}(S,\pi)}$ represents the number of cycle colorings of the permutation $\pi \circ (1, 2, \ldots, n)$ in $[k+t]$ (not necessarily using every color) such that for all $i \in [k]$ the elements in the subset A_i are colored i. Therefore, for a partition γ and a nonnegative integer t, the coefficient $[\mathbf{x}^\gamma]G_n^\alpha(\mathbf{x}, t+k)$ counts the number of quadruples (π, A, c_1, c_2), where π, A, c_1, c_2 are as in the definition of $\mathcal{T}_\gamma^\alpha(t)$ except that c_2 might actually use only a subset of the colors $[k+t]$. Note however that all the colors in $[k]$ will necessarily be used by c_2, and that we can partition the quadruples according to the subset of colors used by c_2. This gives

$$[\mathbf{x}^\gamma]G_n^\alpha(\mathbf{x}, t+k) = \sum_{r \geq 0} \binom{t}{r} \#\mathcal{T}_\gamma^\alpha(r).$$

Now extracting the coefficient of $\mathbf{x}^\gamma$ in the right-hand side of (2.6.8) gives the same result. This completes the proof. □

In order to obtain an explicit expression for the series $G_n^\alpha(\mathbf{x}, t)$ it remains to enumerate the sets $\mathcal{T}_\gamma^\alpha(r)$ which is done below.

Proposition 2.6.9. *Let r be a nonnegative integer, let α be a composition of size m and length k, and let γ be a partition of size $n \geq m$ and length ℓ. Then the set $\mathcal{T}_\gamma^\alpha(r)$ specified by Definition 2.6.5 has cardinality*

$$\#\mathcal{T}_\gamma^\alpha(r) = \frac{n(n-\ell)!(n-k-r)!}{(n-k-\ell-r+1)!} \binom{n+k-1}{n-m-r}, \tag{2.6.10}$$

if $n-k-\ell-r+1 \geq 0$, and 0 otherwise.

The rest of this section is devoted to the proof of Proposition (2.6.9). In order to count the quadruples (π, A, c_1, c_2) satisfying Definition 2.6.5, we shall start by choosing π, c_1, c_2 before choosing the tuple A. For compositions $\gamma = (\gamma_1, \ldots, \gamma_\ell)$, $\delta = (\delta_1, \ldots, \delta_{\ell'})$ of n we denote by $\mathcal{B}_{\gamma,\delta}$ the set of triples (π, c_1, c_2), where π is a permutation of $[n]$, c_1 is a cycle coloring of π such that γ_i elements are colored i for all $i \in [\ell]$, and c_2 is a cycle coloring of the permutation $\pi \circ (1, 2, \ldots, n)$ such that δ_i elements are colored i for all $i \in [\ell']$. The number of such sets is exactly $c_{\gamma,\delta}$ from Definition 2.1.3. By Equation (2.2.27) [46] we have that

$$c_{\gamma,\delta} = \frac{n(n-\ell)!(n-\ell')!}{(n-\ell-\ell'+1)!}, \tag{2.6.11}$$

if $n - \ell - \ell' + 1 \geq 0$, and 0 otherwise.

Again, the striking features of the counting formula (2.6.11) is that it depends on the compositions γ, δ only through their lengths ℓ, ℓ'. This "symmetry" will prove particularly handy for enumerating $\mathcal{T}_\gamma^\alpha(r)$. Let r, α, γ be as in Proposition 2.6.9, and let $\delta = (\delta_1, \ldots, \delta_{k+r})$ be a composition of n of length $k + r$. We denote by $\mathcal{T}_{\gamma,\delta}^\alpha$ the set of quadruples (π, A, c_1, c_2) in $\mathcal{T}_\gamma^\alpha(r)$ such that the cycle coloring c_2 has δ_i elements colored i for all i in $[k + r]$ (equivalently, $(\pi, c_1, c_2) \in \mathcal{B}_{\gamma,\delta}$). We also denote $d_\delta^\alpha := \prod_{i=1}^k \binom{\delta_i}{\alpha_i}$. It is easily seen that for any triple $(\pi, c_1, c_2) \in \mathcal{B}_{\gamma,\delta}$, the number d_δ^α counts the tuples $A \in \mathcal{A}_n^\alpha$ such that $(\pi, A, c_1, c_2) \in \mathcal{T}_{\gamma,\delta}^\alpha$. Therefore,

$$\#\mathcal{T}_\gamma^\alpha(r) = \sum_{\delta \models n,\ \ell(\delta)=k+r} \#\mathcal{T}_{\gamma,\delta}^\alpha = \sum_{\delta \models n,\ \ell(\delta)=k+r} d_\delta^\alpha \, c_{\gamma,\delta},$$

where the sum is over all the compositions of n of length $k + r$. Using (2.6.11) then gives

$$\#\mathcal{T}_\gamma^\alpha(r) = \frac{n(n-\ell)!(n-k-r)!}{(n-k-\ell-r+1)!} \sum_{\delta \models n,\ \ell(\delta)=k+r} d_\delta^\alpha$$

if $n - k - \ell - r + 1 \geq 0$, and 0 otherwise. In order to complete the proof of Proposition 2.6.9, it only remains to prove the following lemma.

Lemma 2.6.12. *If α has size m and length k, then*

$$\sum_{\delta \models n,\ \ell(\delta)=k+r} d_\delta^\alpha = \binom{n+k-1}{n-m-r}.$$

Proof. We give a bijective proof illustrated in Figure 2-16. One can represent a composition $\delta = (\delta_1, \ldots, \delta_{k+r})$ as a sequence of rows of boxes (the ith row has δ_i boxes). With this representation, $d_\delta^\alpha := \prod_{i=1}^k \binom{\delta_i}{\alpha_i}$ is the number of ways of choosing α_i boxes in the ith row of δ for $i = 1, \ldots, k$. Hence $\sum_{\delta \models n,\ \ell(\delta)=k+r} d_\delta^\alpha$ counts α-**marked compositions** of size n and length $k + r$, that is, sequences of $k + r$ non-empty rows of boxes with some marked boxes in the first k rows, with a total of n boxes, and α_i marks in the ith row for $i = 1, \ldots, k$; see Figure 2-16. Now α-marked compositions of size n and length $k + r$ are clearly in bijection (by adding a marked box to each of the rows $1, \ldots, k$, and marking the last box of each of the rows $k + 1, \ldots, k + r$) with α'-marked compositions of size $n + k$ and length $k + r$ *such that the last box of each row is marked*, where $\alpha' = (\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_k + 1, 1, 1, \ldots, 1)$ is a composition of length $k + r$. Lastly, these objects are clearly in bijection (by concatenating all the rows) with sequences of $n+k$ boxes with $m+k+r$ marks, one of which is on the last box. There are $\binom{n+k-1}{n-m-r}$ such sequences, which concludes the proof of Lemma 2.6.12 and Proposition 2.6.9. $\square$

Figure 2-16: A $(2,1,2)$-marked composition of size $n = 12$ and length 5 and its bijective transformation into a sequence $n+k = 15$ boxes with $m+k+r = 5+3+2 = 10$ marks, one of which is on the last box.

2.6.3 Results on separation probabilities

In this section, we exploit Propositions 2.6.7 and 2.6.9 in order to derive our main results. All the results in this section will be consequences of the following theorem.

Theorem 2.6.13. *For any composition α of $m \leq n$ of length k, the generating function $G_n^\alpha(\boldsymbol{x}, t+k)$ in the variables t and $\boldsymbol{x} = \{x_1, x_2, \ldots\}$ has the following explicit expression in the bases $m_\lambda(\boldsymbol{x})$ and $\binom{t}{r}$:*

$$G_n^\alpha(\boldsymbol{x}, t+k) = \sum_{r=0}^{n-m} \binom{t}{r} \binom{n+k-1}{n-m-r} \sum_{\lambda \vdash n,\ \ell(\lambda) \leq n-k-r+1} \frac{n(n-\ell(\lambda))!(n-k-r)!}{(n-k-r-\ell(\lambda)+1)!} m_\lambda(\boldsymbol{x}). \tag{2.6.14}$$

Moreover, for any partition λ of n, one has $\#\mathcal{S}_\lambda^\alpha = [p_\lambda(\boldsymbol{x})] G_n^\alpha(\boldsymbol{x}, 1)$ and $\sigma_\lambda^\alpha = \dfrac{\#\mathcal{S}_\lambda^\alpha}{\binom{n}{\alpha_1, \alpha_2, \ldots, \alpha_k, n-m} \#\mathcal{C}_\lambda}$.

Theorem 2.6.13 is the direct consequence of Propositions 2.6.7 and 2.6.9. One of the striking features of (2.6.14) is that the expression of $G_n^\alpha(\mathbf{x}, t+k)$ depends on α only through its size and length. This "symmetry property" then obviously also holds for $\#\mathcal{S}_\lambda^\alpha = [p_\lambda(\mathbf{x})] G_n^\alpha(\mathbf{x}, 1)$, and translates into the formula (2.6.1) for separation probabilities as stated below.

Corollary 2.6.15. *Let λ be a partition of n, and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ be compositions of the same size m and length k. Then,*

$$\#\mathcal{S}_\lambda^\alpha = \#\mathcal{S}_\lambda^\beta, \tag{2.6.16}$$

or equivalently, in terms of separation probabilities, $\dfrac{\sigma_\lambda^\alpha}{\prod_{i=1}^k \alpha_i!} = \dfrac{\sigma_\lambda^\beta}{\prod_{i=1}^k \beta_i!}$.

We now derive explicit formulas for the separation probabilities for the product of a uniformly random permutation π, with particular constraints on its cycle type, with a uniformly random n-cycle. We focus on two constraints: the case where π is required to have p cycles, and the case where π is a fixed-point-free involution (for n even).

Case when π has exactly p cycles

Let $\mathcal{C}(n,p)$ denote the set of permutations of $[n]$ having p cycles. Recall that the numbers $c(n,p) = \#\mathcal{C}(n,p) = [x^p]x(x+1)(x+2)\cdots(x+n-1)$ are called the **signless Stirling numbers of the first kind**. We denote by $\sigma^\alpha(n,p)$ the probability that the product of a uniformly random permutation in $\mathcal{C}(n,p)$ with a uniformly random n-cycle is A-separated for a given set A in $\mathcal{A}_n^\alpha$. By a reasoning similar to the one used in the proof of (2.6.4), one gets

$$\sigma^\alpha(n,p) = \frac{1}{\binom{n}{\alpha_1,\alpha_2,\ldots,\alpha_k,n-m} c(n,p)} \sum_{\lambda\vdash n, \ell(\lambda)=p} \#\mathcal{S}_\lambda^\alpha. \tag{2.6.17}$$

We now compute the probabilities $\sigma^\alpha(n,p)$ explicitly.

Theorem 2.6.18. *Let α be a composition of m with k parts. Then,*

$$\sigma^\alpha(n,p) = \frac{(n-m)!\prod_{i=1}^k \alpha_i!}{c(n,p)} \sum_{r=0}^{n-m} \binom{1-k}{r}\binom{n+k-1}{n-m-r} \frac{c(n-k-r+1,p)}{(n-k-r+1)!}, \tag{2.6.19}$$

where $c(n,p)$ are signless Stirling numbers of the first kind.

For instance, Theorem 2.6.18 in the case $m = n$ gives the probability that the cycles of the product of a uniformly random permutation in $\mathcal{C}(n,p)$ with a uniformly random n-cycle refine a given set partition of $[n]$ having blocks of sizes $\alpha_1, \alpha_2, \ldots, \alpha_k$. This probability is found to be

$$\sigma^\alpha(n,p) = \frac{\prod_{i=1}^k \alpha_i!}{c(n,p)} \frac{c(n-k+1,p)}{(n-k+1)!}.$$

We now prove Theorem 2.6.18. Via (2.6.17), this amounts to enumerating $\mathcal{S}^\alpha(n,p) := \bigcup_{\lambda\vdash n,\ell(\lambda)=p} \mathcal{S}_\lambda^\alpha$, and using Theorem 2.6.13 one gets

$$\begin{aligned}
\#\mathcal{S}^\alpha(n,p) &= \sum_{\lambda\vdash n,\ell(\lambda)=p} [p_\lambda(\mathbf{x})]\, G_n^\alpha(\mathbf{x},1) \\
&= \sum_{r=0}^{n-m} \binom{1-k}{r}\binom{n+k-1}{n-m-r} \sum_{\ell=1}^{n-k-r+1} \frac{n(n-\ell)!(n-k-r)!}{(n-k-r-\ell+1)!} A(n,p,\ell),
\end{aligned} \tag{2.6.20}$$

where $A(n,p,\ell) := \sum_{\mu\vdash n,\ \ell(\mu)=p} [p_\mu(\mathbf{x})] \sum_{\lambda\vdash n,\ \ell(\lambda)=\ell} m_\lambda(\mathbf{x})$. The next lemma gives a formula for $A(n,p,\ell)$.

Lemma 2.6.21. *For any positive integers $p, \ell \leq n$*

$$\sum_{\mu\vdash n,\ \ell(\mu)=p} [p_\mu(\boldsymbol{x})] \sum_{\lambda\vdash n,\ \ell(\lambda)=\ell} m_\lambda(\boldsymbol{x}) = \binom{n-1}{\ell-1}\frac{(-1)^{\ell-p}c(\ell,p)}{\ell!}, \tag{2.6.22}$$

where $c(a,b)$ are the signless Stirling numbers of the first kind.

Proof. For this proof we use the principal specialization of symmetric functions, that is, their evaluation at $\mathbf{x} = 1^a := \{1,1,\ldots,1,0,0\ldots\}$ (a ones). Since $p_\gamma(1^a) = a^{\ell(\gamma)}$ for any positive integer a, one gets

$$\sum_{\lambda\vdash n,\ \ell(\lambda)=\ell} m_\lambda(1^a) = \sum_{p=1}^{n} a^p \sum_{\mu\vdash n,\ \ell(\mu)=p} [p_\mu(\mathbf{x})] \sum_{\lambda\vdash n,\ \ell(\lambda)=\ell} m_\lambda(\mathbf{x}).$$

The right-hand side of the previous equation is a polynomial in a, and by extracting the coefficient of a^p one gets

$$\sum_{\mu\vdash n,\ \ell(\mu)=p} [p_\mu(\mathbf{x})] \sum_{\lambda\vdash n,\ \ell(\lambda)=\ell} m_\lambda(\mathbf{x}) = [a^p] \sum_{\lambda\vdash n,\ \ell(\lambda)=\ell} m_\lambda(1^a).$$

Now, for any partition λ, $m_\lambda(1^a)$ counts the a-tuples of nonnegative integers such that the positive ones are the same as the parts of λ (in some order). Hence $\sum_{\lambda\vdash n,\ \ell(\lambda)=\ell} m_\lambda(1^a)$ counts the a-tuples of nonnegative integers with ℓ positive ones summing to n. This gives,

$$\sum_{\lambda\vdash n,\ \ell(\lambda)=\ell} m_\lambda(1^a) = \binom{n-1}{\ell-1}\binom{a}{\ell}.$$

Extracting the coefficient of a^p gives (2.6.22) since $[a^p]\dbinom{a}{\ell} = \dfrac{(-1)^{\ell-p}\,c(\ell,p)}{\ell!}$. □

Using Lemma 2.6.21 in (2.6.20) gives

$$\#\mathcal{S}^\alpha(n,p) = n! \sum_{r\geq 0}^{n-m} \binom{1-k}{r}\binom{n+k-1}{n-m-r} \sum_{\ell=1}^{n-k-r+1} \binom{n-k-r}{\ell-1} \frac{(-1)^{\ell-p}c(\ell,p)}{\ell!}, \tag{2.6.23}$$

which we simplify using the following lemma.

Lemma 2.6.24. *For any nonnegative integer a,* $\displaystyle\sum_{q=0}^{a} \binom{a}{q} \frac{(-1)^{q+1-p}\,c(q+1,p)}{(q+1)!} = \frac{c(a+1,p)}{(a+1)!}$.

Proof. The left-hand side equals $[x^p]\sum_{q=0}^{a}\binom{a}{q}\binom{x}{q+1}$. Using the Chu-Vandermonde identity this equals $[x^p]\binom{x+a}{a+1}$ which is precisely the right-hand side. □

Using Lemma 2.6.24 in (2.6.23) gives

$$\#\mathcal{S}^\alpha(n,p) = n! \sum_{r=0}^{n-m} \binom{1-k}{r}\binom{n+k-1}{n-m-r} \frac{c(n-k-r+1,p)}{(n-k-r+1)!}, \tag{2.6.25}$$

which is equivalent to (2.6.19) via (2.6.4). This completes the proof of Theorem 2.6.18. □

In the case $p = 1$, the expression (2.6.19) for the probability $\sigma^\alpha(1) = \sigma^\alpha_{(n)}$ can be written as a sum of $m-k$ terms instead. We state this below.

Corollary 2.6.26. *Let α be a composition of m with k parts. Then the separation probabilities $\sigma^\alpha_{(n)}$ (separation for the product of two uniformly random n-cycles) are*

$$\sigma^\alpha_{(n)} = \frac{(n-m)!\prod_{i=1}^k \alpha_i!}{(n+k)(n-1)!}\left(\frac{(-1)^{n-m}\binom{n-1}{k-2}}{\binom{n+m}{m-k}} + \sum_{r=0}^{m-k}\frac{(-1)^r\binom{m-k}{r}\binom{n+r+1}{m}}{\binom{n+k+r}{r}}\right).$$

The equation in Corollary 2.6.26, already stated in the introduction, is particularly simple when $m-k$ is small. For $\alpha = 1^k$ (i.e. $m = k$) one gets the result stated at the beginning of this paper:

$$\sigma^{1^k}_{(n)} = \begin{cases} \frac{1}{k!} & \text{if } n-k \text{ odd,} \\ \frac{1}{k!} + \frac{2}{(k-2)!(n-k+1)(n+k)} & \text{if } n-k \text{ even.} \end{cases} \tag{2.6.27}$$

In order to prove Corollary 2.6.26 we start with the expression obtained by setting $p = 1$ in (2.6.19):

$$\begin{aligned} \sigma^\alpha_{(n)} &= \frac{(n-m)!\prod_{i=1}^k \alpha_i!}{(n-1)!}\sum_{r=0}^{n-m}\binom{1-k}{r}\frac{1}{n-k-r+1}\binom{n+k-1}{n-m-r} \\ &= \frac{(n-m)!\prod_{i=1}^k \alpha_i!}{(n-1)!}[x^{n-m}](1+x)^{1-k}\sum_{r=0}^{n+k-1}\frac{x^r}{r+m-k+1}\binom{n+k-1}{r} \end{aligned} \tag{2.6.28}$$

We now use the following polynomial identity.

Lemma 2.6.29. *For nonnegative integers a, b, one has the following identity between polynomials in x:*

$$\sum_{i=0}^a \frac{x^i}{i+b+1}\binom{a}{i} = \frac{1}{(a+1)}\left(\frac{1}{\binom{a+b+1}{b}(-x)^{b+1}} - \sum_{i=0}^b \frac{\binom{b}{i}(x+1)^{a+i+1}}{\binom{a+i+1}{i}(-x)^{i+1}}\right). \tag{2.6.30}$$

Proof. It is easy to see that the left-hand side of (2.6.30) is equal to $\frac{1}{x^{b+1}}\int_0^x (1+t)^a t^b dt$. Now this integral can be computed via integration by parts. By a simple induction on b, this gives the right-hand side of (2.6.30). □

Now using (2.6.30) in (2.6.28), with $a = n+k-1$ and $b = m-k$, gives

$$\begin{aligned}\sigma^{\alpha}_{(n)} &= \frac{(n-m)!\prod_{i=1}^{k}\alpha_i!}{(n+k)(n-1)!}[x^{n-m}]\left(\frac{(1+x)^{1-k}}{\binom{n+m}{m-k}(-x)^{m-k+1}} - \sum_{r=0}^{m-k}\frac{\binom{m-k}{r}(1+x)^{n+r+1}}{\binom{n+k+r}{r}(-x)^{r+1}}\right)\\ &= \frac{(n-m)!\prod_{i=1}^{k}\alpha_i!}{(n+k)(n-1)!}\left(\frac{(-1)^{n-m}\binom{n-1}{k-2}}{\binom{n+m}{m-k}} + \sum_{r=0}^{m-k}\frac{(-1)^r\binom{m-k}{r}\binom{n+r+1}{m}}{\binom{n+k+r}{r}}\right).\end{aligned}$$

This completes the proof of Corollary 2.6.26. □

Case when π is a fixed-point-free involution

Given a composition α of $m \leq 2N$ with k parts, we define

$$H^{\alpha}_{N}(t) := \sum_{(\pi,A)\in\mathcal{S}^{\alpha}_{2N}} t^{\mathrm{excess}(\pi,A)},$$

where excess(π, A) is the number of cycles of the product $\pi\circ(1,2,\ldots,2N)$ containing none of the elements of A and where π is a fixed-point-free involution of $[2N]$. Note that $H^{\alpha}_{N}(t) = [p_{2^N}(\mathbf{x})]\, G^{\alpha}_{2N}(\mathbf{x},t)$. We now give an explicit expression for this series.

Theorem 2.6.31. *For any composition α of $m \leq 2N$ of length k, the generating series $H^{\alpha}_{N}(t+k)$ is given by*

$$H^{\alpha}_{N}(t+k) = N\sum_{r=0}^{\min(2N-m,N-k+1)}\binom{t}{r}\binom{2N+k-1}{2N-m-r}2^{k+r-N}\frac{(2N-k-r)!}{(N-k-r+1)!}. \tag{2.6.32}$$

Consequently the separation probabilities for the product of a fixed-point-free involution with a $2N$-cycle are given by

$$\sigma^{\alpha}_{2^N} = \frac{\prod_{i=1}^{k}\alpha_i!}{(2N-1)!(2N-1)!!}\sum_{r=0}^{\min(2N-m,N-k+1)}\binom{1-k}{r}\binom{2N+k-1}{2N-m-r}2^{k+r-N-1}\frac{(2N-k-r)!}{(N-k-r+1)!}. \tag{2.6.33}$$

Remark 2.6.34. *It is posible to show* (2.6.32) *directly following the idea of the proof of Theorem* (2.6.13)*. Since there is an analogue of* (2.6.8)*, namely*

$$H^{\alpha}_{2N}(t+k) = \sum_{r=0}^{2N-m}\binom{t}{r}\mathcal{U}^{\alpha}(r),$$

where $\mathcal{U}^{\alpha}(r)$ is the set of triples (π, A, c_2) where π is a fixed-point free involution of $[2N]$, A is in $\mathcal{A}^{\alpha}_{n}$ and c_2 is a a cycle coloring of the product $\pi\circ(1,2,\ldots,2N)$ in $[k+r]$ such that every color in $[k+r]$ is used and for all i in $[k]$ the elements in the subset A_i are colored i.

Then in an analogue to Proposition 2.6.9, we find that

$$\#\mathcal{U}^\alpha(r) = \frac{N(2N-k-r)!}{(N-k-r+1)!}\binom{2N+k-1}{2N-m-r},$$

where we use Lemma 2.6.12 and the following analogue of (2.6.11)*: given a composition $\delta = (\delta_1, \dots, \delta_\ell)$ of $2N$, let $\mathcal{C}_\delta$ be the set of triples (π, c_2), where π is a fixed-point free involution of $[2N]$ and c_2 is a cycle coloring of the permutation $\pi \circ (1, 2, \dots, 2N)$ such that δ_i elements are colored i for all $i \in [\ell]$ then*

$$\#\mathcal{C}_\delta = \frac{N(2N-\ell)!}{(N-\ell+1)!}2^{\ell-N}.$$

■

Remark 2.6.35. *One can interpret* (2.6.32) *in the case $m = k = 0$ (no marked edges) as follows:*

$$\sum_{M \in \mathcal{M}_N} t^{\#vertices} = H_N^{\emptyset}(t) = N\sum_{r=1}^{N+1}\binom{t}{r}2^{r-N}\frac{(2N-r)!}{(N-r+1)!}\binom{2N-1}{2N-r}. \tag{2.6.36}$$

This equation is exactly the celebrated Harer-Zagier formula [28]. ■

The rest of this section is devoted to the proof of Theorem 2.6.31. Since $H_N^\alpha(t) = [p_{2^N}(\mathbf{x})]\, G_{2N}^\alpha(\mathbf{x}, t)$, Theorem 2.6.13 gives

$$H_N^\alpha(t+k) = \tag{2.6.37}$$
$$\sum_{r=0}^{2N-m}\binom{t}{r}\binom{2N+k-1}{2N-m-r}\sum_{s=0}^{N-k-r+1}\frac{2N(N-s)!(2N-k-r)!}{(N-k-r-s+1)!}[p_{2^N}(\mathbf{x})]\sum_{\lambda\vdash 2N,\ \ell(\lambda)=N+s} m_\lambda(\mathbf{x}).$$

We then use the following result.

Lemma 2.6.38. *For any nonnegative integer $s \le N$,*

$$[p_{2^N}(\boldsymbol{x})]\sum_{\lambda\vdash 2N,\ \ell(\lambda)=N+s} m_\lambda(\boldsymbol{x}) \;=\; \frac{(-1)^s}{2^s s!(N-s)!}.$$

Proof. For partitions λ, μ of n, we denote $S_{\lambda,\mu} = [p_\lambda(\mathbf{x})]m_\mu(\mathbf{x})$ and $R_{\lambda,\mu} = [m_\lambda(\mathbf{x})]p_\mu(\mathbf{x})$. The matrices $S = (S_{\lambda,\mu})_{\lambda,\mu\vdash n}$ and $R = (R_{\lambda,\mu})_{\lambda,\mu\vdash n}$ are the transition matrices between the bases $\{p_\lambda\}_{\lambda,\vdash n}$ and $\{m_\lambda\}_{\lambda\vdash n}$ of symmetric functions of degree n, hence $S = R^{-1}$. Moreover the matrix R is easily seen to be lower triangular in the **dominance order** of partitions, that is, $R_{\lambda,\mu} = 0$ unless $\lambda_1 + \lambda_2 + \cdots + \lambda_i \le \mu_1 + \mu_2 + \cdots + \mu_i$ for all $i \ge 1$ ([57, Prop. 7.5.3]). Thus the matrix $S = R^{-1}$ is also lower triangular in the dominance order. Since the only partition of $2N$ of length $N + s$ that is not larger than the partition 2^N in the dominance order is $1^{2s}2^{N-s}$, one gets

$$[p_{2^N}(\mathbf{x})] \sum_{\lambda \vdash 2N,\ \ell(\lambda)=N+s} m_\lambda(\mathbf{x}) = [p_{2^N}(\mathbf{x})]\, m_{1^{2s}2^{N-s}}(\mathbf{x}). \tag{2.6.39}$$

To compute this coefficient we use the standard scalar product $\langle \cdot, \cdot \rangle$ on symmetric functions (see e.g. [57, Sec. 7]) defined by $\langle p_\lambda, p_\mu \rangle = z_\lambda$ if $\lambda = \mu$ and 0 otherwise, where $z_\lambda = \prod_i i^{n_i(\lambda)} n_i(\lambda)!$. From this definition one immediately gets

$$[p_{2^N}]\, m_{1^{2s}2^{N-s}} = \frac{1}{z_{2^N}} \langle p_{2^N}, m_{1^{2s}2^{N-s}} \rangle = \frac{1}{N!2^N} \langle p_{2^N}, m_{1^{2s}2^{N-s}} \rangle. \tag{2.6.40}$$

Let $\{h_\lambda\}$ denote the basis of the complete symmetric functions. It is well known that $\langle h_\lambda, m_\mu \rangle = 1$ if $\lambda = \mu$ and 0 otherwise, therefore $\langle p_{2^N}, m_{1^{2s}2^{N-s}} \rangle = [h_{1^{2s}2^{N-s}}] p_{2^N}$. Lastly, since $p_{2^N} = (p_2)^N$ and $p_2 = 2h_2 - h_1^2$ one gets

$$\langle p_{2^N}, m_{1^{2s}2^{N-s}} \rangle = [h_{1^{2s}2^{N-s}}] p_{2^N} = [h_1^{2s} h_2^{N-s}]\, (2h_2 - h_1^2)^N = 2^{N-s} (-1)^s \binom{N}{s}. \tag{2.6.41}$$

Putting together (2.6.39), (2.6.40) and (2.6.41) completes the proof. □

By Lemma 2.6.38, Equation (2.6.37) becomes

$$\begin{aligned} H_N^\alpha(t+k) &= \sum_{r=0}^{2N-m} \binom{t}{r} \binom{2N+k-1}{2N-m-r} \sum_{s=0}^{N-k-r+1} \frac{2N(N-s)!(2N-k-r)!}{(N-k-r-s+1)!} \frac{(-1)^s}{2^s s!(N-s)!} \\ &= 2N \sum_{r=0}^{2N-m} \binom{t}{r} \binom{2N+k-1}{2N-m-r} \frac{(2N-k-r)!}{(N-k-r+1)!} \sum_{s=0}^{N-k-r+1} \binom{N-k-r+1}{s} \frac{(-1)^s}{2^s} \\ &= 2N \sum_{r=0}^{\min(2N-m, N-k+1)} \binom{t}{r} \binom{2N+k-1}{2N-m-r} \frac{(2N-k-r)!}{(N-k-r+1)!} \frac{1}{2^{N-k-r+1}}, \end{aligned}$$

where the last equality uses the binomial theorem. This completes the proof of Equation (2.6.32). Equation (2.6.33) then immediately follows from the case $t = 1-k$ of (2.6.32) via (2.6.4). This completes the proof of Theorem 2.6.31. □

Remark 2.6.42. *In [7, Sec. 4] we also obtain a relation between the separation probabilities σ_λ^α and $\sigma_{\lambda'}^\alpha$, when the partition λ' is obtained from λ by adding some parts of size 1.* ■

Chapter 3

Flow polytopes and the Kostant partition function

This chapter is based on [44], joint work with K. Mészáros.

3.1 Introduction

In this chapter we use combinatorial techniques to establish the relationship between volumes of flow polytopes associated to signed graphs and the Kostant partition function. Our techniques yield a systematic method for computing volumes of flow polytopes associated to signed graphs. We study special families of polytopes in detail, such as the Chan-Robbins-Yuen polytope [15] and certain type C_{n+1} and D_{n+1} analogues of it. We also give several intriguing conjectures for their volume.

Our results on flow polytopes associated to signed graphs and the Kostant partition function specialize to the results of Baldoni and Vergne, in which they established the connection between type A_n flow polytopes and the Kostant partition function [4, 2]. Baldoni and Vergne use residue techniques, while in their unpublished work Postnikov and Stanley took a combinatorial approach [53, 58]. In our study of type A_n as well as type C_{n+1} and D_{n+1} flow polytopes we establish the above mentioned connections by entirely combinatorial methods.

Traditionally, flow polytopes are associated to loopless (and signless) graphs in the following way. Let G be a graph on the vertex set $[n+1]$, and let $\mathrm{in}(e)$ denote the smallest (initial) vertex of edge e and $\mathrm{fin}(e)$ the biggest (final) vertex of edge e. Think of fluid flowing on the edges of G from the smaller to the bigger vertices, so that the total fluid volume entering vertex 1 is one and leaving vertex $n+1$ is one, and there is conservation of fluid at the intermediate vertices. Formally, a **flow** f of size one on G is a function $f : E \to \mathbb{R}_{\geq 0}$ from the edge set E of G to the set of nonnegative real numbers such that

$$1 = \sum_{e \in E(G), \mathrm{in}(e)=1} f(e) = \sum_{e \in E(G), \mathrm{fin}(e)=n+1} f(e),$$

and for $2 \leq i \leq n$

$$\sum_{e\in E(G),\mathrm{fin}(e)=i} f(e) = \sum_{e\in E(G),\mathrm{in}(e)=i} f(e).$$

The **flow polytope** $\mathcal{F}_G$ associated to the graph G is the set of all flows $f : E \to \mathbb{R}_{\geq 0}$ of size one. A fascinating example is the flow polytope $\mathcal{F}_{K_{n+1}}$ of the complete graph K_{n+1}, which is also called the Chan-Robbins-Yuen polytope $CRYA_n$ [15] (Chan, Robbins and Yuen defined it in terms of matrices), and has kept the combinatorial community in its magic grip since its volume is equal to $\prod_{k=0}^{n-2} Cat(k)$, where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number. This was proved analytically by Zeilberger [64], yet, there is no combinatorial proof for this volume formula.

In their unpublished work [53, 58] Postnikov and Stanley discovered the following remarkable connection between the volume of the flow polytope and the Kostant partition function k_G:

Theorem 3.6.2 ([53, 58])**.** *Given a loopless (signless) connected graph G on the vertex set $[n+1]$, let $d_i = indeg_G(i) - 1$, for $i \in \{2, \ldots, n\}$. Then, the normalized volume* $\mathrm{vol}(\mathcal{F}_G)$ *of the flow polytope $\mathcal{F}_G$ associated to the graph G is*

$$\mathrm{vol}(\mathcal{F}_G) = k_G(0, d_2, \ldots, d_n, -\sum_{i=2}^{n} d_i). \tag{3.1.1}$$

The notation $indeg_G(i)$ stands for the indegree of vertex i in the graph G and k_G denotes the Kostant partition function associated to graph G.

In light of Theorem 3.6.2, Zeilberger's result about the volume of the Chan-Robbins-Yuen polytope $CRYA_n$ can be stated as:

$$k_{K_{n-1}}\left(1, 2, \ldots, n-2, -\binom{n-1}{2}\right) = \prod_{k=1}^{n-2} Cat(k). \tag{3.1.2}$$

Recall that the **Kostant partition function** k_G evaluated at the vector $\mathbf{a} \in \mathbb{Z}^{n+1}$ is defined as

$$k_G(\mathbf{a}) = \#\{(b_k)_{k\in[N]} \mid \sum_{k\in[N]} b_k \mathbf{v}_k = \mathbf{a} \text{ and } b_k \in \mathbb{Z}_{\geq 0}\}, \tag{3.1.3}$$

where $\{\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}\}$ is the multiset of vectors corresponding to the multiset of edges of G under the correspondence which associates an edge (i, j), $i < j$, of G with a positive type A_n root $e_i - e_j$, where e_i is the ith standard basis vector in $\mathbb{R}^{n+1}$.

In other words, $k_G(\mathbf{a})$ is the number of ways to write the vector $\mathbf{a}$ as a $\mathbb{N}$-linear combination of the positive type A_n roots (with possible multiplicities) corresponding to the edges of G, without regard to order. Note that for $k_G(\mathbf{a})$ to be nonzero, the partial sums of the coordinates of $\mathbf{a}$ have to satisfy $a_1 + \ldots + a_i \geq 0$, $i \in [n]$, and $a_1 + \ldots + a_{n+1} = 0$. Also, k_G has the following formal generating series:

$$\sum_{\mathbf{a}\in\mathbb{Z}^{n+1}} k_G(\mathbf{a}) x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} = \prod_{(i,j)\in E(G)} (1 - x_i x_j^{-1})^{-1}. \tag{3.1.4}$$

While endowed with combinatorial meaning, Kostant partition functions were introduced in and are a vital part of representation theory. For instance for classical Lie algebras, weight multiplicities and tensor product multiplicities (Littlewood-Richardson coefficients) can be expressed in terms of the Kostant partition function (see [29, 16] and Steinberg's formula in [31, Sec. 24.4]). Kostant partition functions also come up in toric geometry and approximation theory. A salient feature of $k_G(\mathbf{a})$ is that it is a *piecewise quasipolynomial function* in $\mathbf{a}$ if G is fixed [17, 62].

We generalize Theorem 3.6.2 to establish the connection between flow polytopes associated to loopless *signed* graphs and a **dynamic** Kostant partition function $k_G^{\mathrm{dyn}}(\mathbf{a})$ with the following formal generating series:

$$\sum_{\mathbf{a}\in\mathbb{Z}^{n+1}} k_G^{\mathrm{dyn}}(\mathbf{a})x_1^{a_1}\cdots x_{n+1}^{a_{n+1}} = \prod_{(i,j,-)\in E(G)}(1-x_ix_j^{-1})^{-1}\prod_{(i,j,+)\in E(G)}(1-x_i-x_j)^{-1}, \quad (3.1.5)$$

where G is a signed graph. By a signed graph we mean a graph where each edge has a positive or a negative sign associated to it. A signless graph can be thought of as a signed graph where all edges have a *negative* sign associated to them. The definition of a flow polytope associated to a signed graph generalizes the case of flow polytopes associated to signless graphs and can be found in Section 3.2.

We develop a systematic method for calculating volumes of flow polytopes of signed graphs. There are several ways to state and specialize our results; we highlight the next theorem as perhaps the most appealing special case.

Theorem 3.6.16. *Given a loopless connected signed graph G on the vertex set $[n+1]$, let $d_i = indeg_G(i) - 1$ for $i \in \{2,\ldots,n\}$, where $indeg_G(i)$ is the indegree of vertex i (the number of edges $(\cdot, i, -)$). The normalized volume $\mathrm{vol}(\mathcal{F}_G)$ of the flow polytope $\mathcal{F}_G$ associated to graph G is*

$$\mathrm{vol}(\mathcal{F}_G) = K_G^{dyn}(0, d_2, \ldots, d_n, d_{n+1}),$$

where K_G^{dyn} has the generating series given in Equation (3.1.5).

Inspired by the intriguing $CRYA_n$ polytope, we introduce its type C_{n+1} and D_{n+1} analogues, $CRYC_{n+1}$ and $CRYD_{n+1}$, prove that their number of vertices are 3^n and $3^n - 2^n$, respectively and we conjecture the following.

Conjecture 3.1.6. *The normalized volumes of the type C and type D analogues $CRYC_{n+1}$ and $CRYD_{n+1}$ of the Chan-Robbins-Yuen polytope $CRYA_n$ are*

$$\mathrm{vol}(CRYC_{n+1}) = 2^{(n-1)^2+n}\prod_{k=0}^{n-1} Cat(k),$$

$$\mathrm{vol}(CRYD_{n+1}) = 2^{(n-1)^2}\prod_{k=0}^{n-1} Cat(k),$$

where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.

Outline: In the first part of this chapter we introduce flow polytopes associated to signed graphs and characterize their vertices. In Section 3.2 the necessary background on signed graphs, Kostant partition functions and flows is given. We also define flow polytopes associated to signed graphs and remark that their Ehrhart functions can be expressed in terms of Kostant partition functions. In Section 3.3 we give a characterization of the vertices of flow polytopes associated to signed graphs, and prove that the vertices of a special family of flow polytopes associated to signed graphs are integral, noting that in general this is not the case. As an application of the results from this section we find nice formulas for the number of vertices of the type C and D generalizations of the Chan-Robbins-Yuen polytope.

The second part of the chapter is about subdivisions of flow polytopes. In Section 3.4 we show that certain operations on graphs, called reduction rules, are a way of encoding subdivisions of flow polytopes. Using the reduction rules, in Section 3.5 we state and prove the Subdivision Lemma, which is a key ingredient of our subsequent explorations. The Subdivision Lemma gives a hands on way of subdividing, and eventually triangulating, flow polytopes.

The last part of the chapter is about using the subdivision of flow polytopes to compute their volumes. In Section 3.6 we use the Subdivision Lemma to prove Theorems 3.6.2 and 3.6.16: namely that the volume of a flow polytope is equal to a value of the dynamic Kostant partition function. To do the above, we introduce the dynamic Kostant partition function in this section. The dynamic Kostant partition function specializes to the Kostant partition function in the case of signless graphs and has a nice and simple generating function, just like the Kostant partition function. We apply the above results in Section 3.7 to the study of volumes of the Chan-Robbins-Yuen polytope and its various generalizations. We conclude our chapter with several intriguing conjectures on the volumes of the type C and D generalizations of the Chan-Robbins-Yuen polytope. .

Supplementary code for calculating the volume of flow polytopes and for evaluating the (dynamic) Kostant partition function is available at the site:

`http://sites.google.com/site/flowpolytopes/`

3.2 Signed graphs, Kostant partition functions, and flows

In this section we define the concepts of graphs, Kostant partition functions and flows, all in the **signed** universe. One can think of these as the generalization of these concepts' signless counterparts from the type A_n (signless) root system to other types, such as C_{n+1} and D_{n+1}. We also define general flow polytopes, which are a main objects of this chapter. We conclude the section by a simple expression for the Ehrhart function of flow polytopes.

Throughout this section, the graphs G on the vertex set $[n+1]$ that we consider are signed, that is there is a sign $\epsilon \in \{+, -\}$ assigned to each of its edges. We allow loops and multiple edges. The sign of a loop is always $+$, and a loop at vertex i is denoted

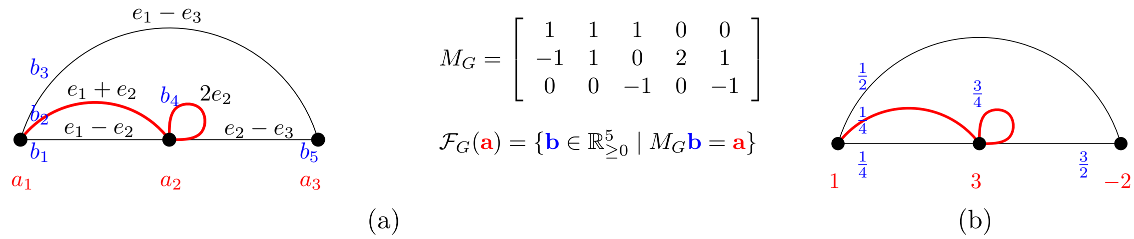


Figure 3-1: (a) A signed graph G on three vertices and the positive roots associated with each of the five edges. The columns of the matrix M_G correspond to these roots. The flow polytope $\mathcal{F}_G(\mathbf{a})$ consists of the flows $\mathbf{b} \in \mathbb{R}^4_{\geq 0}$ such that $M_G\mathbf{b} = \mathbf{a}$ where $\mathbf{a}$ is the netflow vector. The Kostant partition function $k_G(\mathbf{a})$ counts the lattice points of $\mathcal{F}_G(\mathbf{a})$, the number of ways of obtaining $\mathbf{a}$ as a $\mathbb{N}$-integer combination of the roots associated to G.
(b) A nonnegative flow on G with netflow vector $\mathbf{a} = (1, 3, -2)$. The flows on the edges are in blue.

by $(i, i, +)$. Denote by $(i, j, -)$ and $(i, j, +)$, $i < j$, a negative and a positive edge between vertices i and j, respectively. A positive edge, that is, an edge labeled by $+$, is **positively incident**, or, **incident with a positive sign**, to both of its endpoints. A negative edge is positively incident to its smaller vertex and **negatively incident** to its greater endpoint. See Figure 3-1(b) for an example of the incidences. Denote by m_{ij}^{ϵ} the multiplicity of edge (i, j, ϵ) in G, $i \leq j$, $\epsilon \in \{+, -\}$. To each edge (i, j, ϵ), $i \leq j$, of G, associate the positive type C_{n+1} root $\mathrm{v}(i, j, \epsilon)$, where $\mathrm{v}(i, j, -) = e_i - e_j$ and $\mathrm{v}(i, j, +) = e_i + e_j$. Let $\{\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}\}$ be the multiset of vectors corresponding to the multiset of edges of G (i.e., $\mathbf{v}_k = v(e_k)$). Note that $N = \sum_{1 \leq i \leq j \leq n+1} (m_{ij}^{-} + m_{ij}^{+})$.

The **Kostant partition function** k_G evaluated at the vector $\mathbf{a} \in \mathbb{Z}^{n+1}$ is defined as

$$k_G(\mathbf{a}) = \#\{(b_k)_{k \in [N]} \mid \sum_{k \in [N]} b_k \mathbf{v}_k = \mathbf{a} \text{ and } b_k \in \mathbb{Z}_{\geq 0}\}.$$

That is, $k_G(\mathbf{a})$ is the number of ways to write the vector $\mathbf{a}$ as an $\mathbb{N}$-linear combination of the positive type C_{n+1} roots corresponding to the edges of G, without regard to order.

Example 3.2.1. *For the signed graph G in Figure 3-1(a), $k_G(1, 3, -2) = 3$, since $(1, 3, -2) = (e1 - e_3) + (2e_2) + (e_2 - e_3) = (e_1 + e_2) + 2(e_2 - e_3) = (e_1 - e_2) + (2e_2) + 2(e_2 - e_3)$.*

Just like in the type A_n case, we would like to think of the vector $(b_i)_{i \in [N]}$ as a **flow**. For this we here give a precise definition of flows in the type C_{n+1} case, of which type A_n is of course a special case.

Let G be a signed graph on the vertex set $[n+1]$. Let $\{\{e_1, \ldots, e_N\}\}$ be the multiset of edges of G, and $X_G := \{\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}\}$ the multiset of positive type C_{n+1} roots corresponding to the multiset of edges of G. Also, let M_G be the $(n+1) \times N$ matrix whose columns are the vectors in X_G. Fix an integer vector $\mathbf{a} = (a_1, \ldots, a_n, a_{n+1}) \in \mathbb{Z}^{n+1}$.

An **a-flow** $\mathbf{f}_G$ on G is a vector $\mathbf{f}_G = (b_k)_{k\in[N]}$, $b_k \in \mathbb{R}_{\geq 0}$ such that $M_G\mathbf{f}_G = \mathbf{a}$. That is, for all $1 \leq i \leq n+1$, we have

$$\sum_{e\in E(G),\mathrm{inc}(e,v)=-} b(e) + a_v = \sum_{e\in E(G),\mathrm{inc}(e,v)=+} b(e) + \sum_{e=(v,v,+)} b(e), \tag{3.2.2}$$

where $b(e_k) = b_k$, $\mathrm{inc}(e,v) = -$ if $e = (g,v,-)$, $g < v$, and $\mathrm{inc}(e,v) = +$ if $e = (g,v,+)$, $g < v$, or $e = (v,j,\epsilon)$, $v < j$, and $\epsilon \in \{+,-\}$. When $\mathrm{inc}(e,v) = -$ (respectively, $\mathrm{inc}(e,v) = +$) we say that edge e is incident to v with a negative (respectively, positive) sign.

Example 3.2.3. *Figure 3-1(b) shows a signed graph G with three vertices with flow assigned to each edge. The netflow is* $\mathbf{a} = (1, 3, -2)$

Call $b(e)$ the **flow** assigned to edge e of G. If the edge e is negative, one can think of $b(e)$ units of fluid flowing on e from its smaller to its bigger vertex. If the edge e is positive, then one can think of $b(e)$ units of fluid flowing away both from e's smaller and bigger vertex to "infinity". Edge e is then a "leak" taking away $2b(e)$ units of fluid.

From the above explanation it is clear that if we are given an **a**-flow $\mathbf{f}_G$ such that $\sum_{e=(i,j,+),i\leq j} b(e) = y$, then $\mathbf{a} = (a_1, \ldots, a_n, 2y - \sum_{i=1}^n a_i)$.

An **integer a-flow** $\mathbf{f}_G$ on G is an **a**-flow $\mathbf{f}_G = (b_i)_{i\in[N]}$, with $b_i \in \mathbb{Z}_{\geq 0}$. It is a matter of checking the definitions to see that for a signed graph G on the vertex set $[n+1]$ and vector $\mathbf{a} = (a_1, \ldots, a_n, 2y - \sum_{i=1}^n a_i) \in \mathbb{Z}^{n+1}$, the number of integer **a**-flows on G are given by the Kostant partition function, as stated in the next lemma.

Lemma 3.2.4. *Given a signed graph G on the vertex set $[n+1]$ and a vector $\mathbf{a} = (a_1, \ldots, a_n, 2y - \sum_{i=1}^n a_i) \in \mathbb{Z}^{n+1}$, the integer* **a***-flows are in bijection with ways of writing* **a** *as a nonnegative linear combination of the roots associated to the edges of G. Thus $\#\{$integer $\mathbf{a}$-flows$\} = k_G(\mathbf{a})$.*

Define the **flow polytope** $\mathcal{F}_G(\mathbf{a})$ associated to a signed graph G on the vertex set $[n+1]$ and the integer vector $\mathbf{a} = (a_1, \ldots, a_{n+1})$ as the set of all **a**-flows $\mathbf{f}_G$ on G i.e., $\mathcal{F}_G = \{\mathbf{f}_G \in \mathbb{R}^N_{\geq 0} \mid M_G\mathbf{f}_G = \mathbf{a}\}$. The flow polytope $\mathcal{F}_G(\mathbf{a})$ then naturally lives in $\mathbb{R}^N$, where N is the number of edges of G.

Let X_G be the multiset of N vectors corresponding to the edges of G and assume they span an r-dimensional space. Also, let $C(X_G)$ be the *cone* generated by the vectors in X_G. A vector **a** is in the interior of $C(X_G)$ if and only if **a** can be expressed as $\mathbf{a} = \sum_{i=1}^N b_i\mathbf{v}_i$ where $b_i > 0$ for all i. We have the following proposition about the dimension of $\mathcal{F}_G(\mathbf{a})$.

Proposition 3.2.5. *The flow polytope $\mathcal{F}_G(\mathbf{a})$ is empty if* **a** *is not in the cone $C(X_G)$ and if* **a** *is in the interior of $C(X_G)$ then $\dim(\mathcal{F}_G(\mathbf{a})) = N - r$. This is also the dimension of the kernel of M_G.*

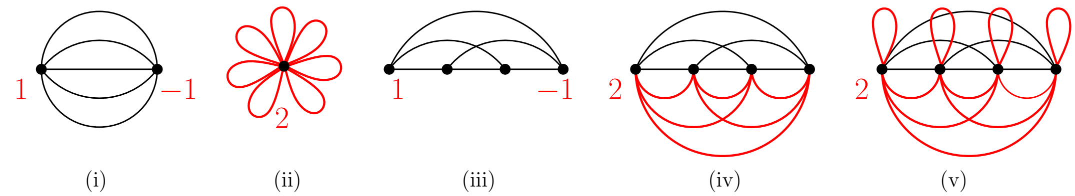


Figure 3-2: Graphs and netflow whose flow polytopes are: (i), (ii) simplices and (iii),(iv),(v) instances of $CRYA_n$, $CRYD_n$ and $CRYC_n$.

Remark 3.2.6. *For a signed connected graph G with vertex set $[n+1]$ and N edges, if $\mathbf{a}$ is in the interior of $C(X_G)$, then $\dim(\mathcal{F}_G(\mathbf{a})) = \#E(G) - \#V(G) + 1 = N - n$ if G only has negative edges (since X_G spans the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 0$) and $\dim(\mathcal{F}_G(\mathbf{a})) = \#E(G) - \#V(G) = N - n - 1$ otherwise.* ■

Next, we give the main examples of the flow polytopes we study (see Figure 3-2):

Examples 3.2.7.

(i) *Let G be the graph with vertices $\{1,2\}$ and edges $(1,2,-)$ with multiplicity m_{12}; and let $\mathbf{a} = (1,-1)$. Then $\mathcal{F}_G(1,-1)$ is a $(m_{12}-1)$-dimensional simplex.*

(ii) *Let G be the signed graph with one vertex $\{1\}$ and loops $(1,1,+)$ with multiplicity m_{11}; and let $\mathbf{a} = 2$. Then $\mathcal{F}_G(2)$ is a $(m_{11}-1)$-dimensional simplex.*

(iii) *Let $G = K_{n+1}$ be the complete graph with $n+1$ vertices (all edges $(i,j,-)$ $1 \leq i < j \leq n+1$) and $\mathbf{a} = e_1 - e_{n+1}$. Then $\mathcal{F}_{K_{n+1}}(e_1 - e_{n+1})$ is the type A_n Chan-Robbins-Yuen polytope or $CRYA_n$ [14, 15]. Such polytope is a face of the Birkhoff polytope of all $n \times n$ doubly stochastic matrices. It has dimension $\binom{n}{2}$, 2^{n-1} vertices, and Zeilberger [64] showed that its normalized volume is $\mathrm{vol}(CRYA_n) = \prod_{k=0}^{n-2} Cat(k)$ where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.*

(iv) *Let $G = K_n^D$ be the complete* signed *graph with n vertices (all edges $(i,j,\pm)$ $1 \leq i < j \leq n$) and $\mathbf{a} = 2e_1$. Then $CRYD_n = \mathcal{F}_{K_n^D}(2e_1)$ is a type D_n analogue of $CRYA_n$. We show it is integral (see Theorem 3.3.11) with dimension $n(n-2)$ and $3^{n-1} - 2^{n-1}$ vertices (see Proposition 3.3.14). We conjecture (see Conjecture 3.1.6) that its normalized volume is $2^{(n-2)^2} \cdot \mathrm{vol}(CRYA_n)$.*

(v) *Let $G = K_n^C$ be the complete* signed *graph with n vertices and with loops $(i,i,+)$ corresponding to the type C positive roots $2e_i$. Then $CRYC_n = \mathcal{F}_{K_n^B}(2e_1)$ is a type C_n analogue of $CRYA_n$. We show it is integral (see Theorem 3.3.11) with dimension $n(n-2)$ and 3^{n-1} vertices (see Proposition 3.3.15). We conjecture (see Conjecture 3.1.6) that its normalized volume is $2^{n-1} \cdot \mathrm{vol}(CRYD_n)$.*

There is a vast amount of research pertaining to flow polytopes associated to graphs with only negative edges and no loops and the special $\mathbf{a}$-vector $(1,0,\ldots,0,-1)$ [4, 1, 18]. Note that $(1,0,\ldots,0,-1)$ is the highest root in the type A_n root system, and the edges of graphs with only negative signs correspond to type A_n positive roots. In this light, it is natural to consider this generalization to signed graphs and special vectors $\mathbf{a}$, such as the highest roots of other root systems.

3.2.1 The Ehrhart function of the flow polytope $\mathcal{F}_G(\mathbf{a})$

It this subsection we explain how to write down the Ehrhart function of $\mathcal{F}_G(\mathbf{a})$ in terms of Kostant partition functions. This turns out to be an easy task.

Recall that given a polytope $\mathcal{P} \subset \mathbb{R}^N$, the t **dilate** of $\mathcal{P}$ is

$$t\mathcal{P} = \{(tx_1, \ldots, tx_N) \mid (x_1, \ldots, x_N) \in \mathcal{P}\}.$$

The number of lattice points of $t\mathcal{P}$ where t is a nonnegative integer and $\mathcal{P}$ is a convex polytope is given by the *Ehrhart function* $L_{\mathcal{P}}(t)$. If $\mathcal{P}$ has (rational) integral vertices then $L_{\mathcal{P}}(t)$ is a (quasi) polynomial (for background on the theory of Ehrhart polynomials see [5]).

Baldoni and Vergne [4] showed that $L_{\mathcal{F}_G(\mathbf{a})}(t)$ is given by the Kostant partition function.

Lemma 3.2.8. *([4]) For a signed graph G with no loops on the vertex set $[n+1]$ and a vector $\mathbf{a} \in \mathbb{Z}^{n+1}$, the Ehrhart function $L_{\mathcal{F}_G(\mathbf{a})}(t)$ of $\mathcal{F}_G(\mathbf{a})$ is*

$$L_{\mathcal{F}_G(\mathbf{a})}(t) = k_G(t\mathbf{a}). \tag{3.2.9}$$

, where k_G is the Kostant partition function associated to the signed graph G.

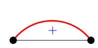
Proof. For any $t \in \mathbb{Z}_{\geq 0}$ the number of integer points of $t\mathcal{F}_G(\mathbf{a})$ is the number of integer $t\mathbf{a}$-flows on G. Thus, there are $k_G(t\mathbf{a})$ of them. Equation (3.2.9) follows. □

3.3 The vertices of the flow polytope $\mathcal{F}_G(\mathbf{a})$

3.3.1 Vertices of $\mathcal{F}_G(\mathbf{a})$

In this section we characterize the vertices of the flow polytope $\mathcal{F}_G(\mathbf{a})$. Remarkably, if G is a graph with only negative edges, then for any integer vector $\mathbf{a}$ the vertices of $\mathcal{F}_G(\mathbf{a})$ are integer. Such a statement is not true for signed graphs G in general. However, we show, using our characterization of the vertices of $\mathcal{F}_G(\mathbf{a})$ that for special integer vectors $\mathbf{a}$ the vertices of $\mathcal{F}_G(\mathbf{a})$ are integer. As an application of our vertex characterization, we show that the number of vertices of the type C_{n+1} and type D_{n+1} analogue of the Chan-Robbins-Yuen polytope from Examples 3.2.7 (iv),(v) are 3^n and $3^n - 2^n$, respectively.

That the vertices of $\mathcal{F}_G(\mathbf{a})$ are integer for any signless graph G and any integer vector $\mathbf{a}$ follows from the fact that the matrix M_G whose columns are the positive type A roots associated to the edges of G is totally unimodular. However, as mentioned above, for signed graphs G the polytope $\mathcal{F}_G(\mathbf{a})$ does not always have integer vertices as the following simple example shows.

Example 3.3.1. *Let G be the graph* 

the flow polytope $\mathcal{F}_G(1, 0)$ is a zero dimensional polytope with a vertex $(1/2, 1/2)$.

In the rest of the section G denotes a signed graph. Recall that we defined $\mathbf{a}$-flows to be nonnegative. In this section we use the term **nonzero signed** 0**-flow** to refer to a flow where we allow flows to be negative or positive or zero (as signified by signed), which is not zero everywhere (signified by nonzero) and where the net flow is 0.

Lemma 3.3.2. *An* $\mathbf{a}$*-flow* $\mathbf{f}_G$ *on* G *is a vertex of* $\mathcal{F}_G(\mathbf{a})$ *if and only if there is no nonzero signed* 0*-flow* $\mathbf{f}_G^\epsilon$ *such that* $\mathbf{f}_G - \mathbf{f}_G^\epsilon$ *and* $\mathbf{f}_G + \mathbf{f}_G^\epsilon$ *are flows on* G*.*

Lemma 3.3.2 follows from definitions, but since it is the starting point of the characterization of the vertices of $\mathcal{F}_G(\mathbf{a})$, we include a proof for clarity.

Proof of Lemma 3.3.2. If there is a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ such that $\mathbf{f}_G - \mathbf{f}_G^\epsilon$ and $\mathbf{f}_G + \mathbf{f}_G^\epsilon$ are flows on G (and thus $\mathbf{a}$-flows on G), then

$$\mathbf{f}_G = ((\mathbf{f}_G - \mathbf{f}_G^\epsilon) + (\mathbf{f}_G + \mathbf{f}_G^\epsilon))/2,$$

so $\mathbf{f}_G$ is not a vertex of $\mathcal{F}_G(\mathbf{a})$.

If $\mathbf{f}_G$ is not a vertex of $\mathcal{F}_G(\mathbf{a})$, then $\mathbf{f}_G$ can be written as

$$\mathbf{f}_G = (\mathbf{f}_G^1 + \mathbf{f}_G^2)/2,$$

for some $\mathbf{a}$-flows $\mathbf{f}_G^1$ and $\mathbf{f}_G^2$ on G. Thus,

$$\mathbf{f}_G^1 = \mathbf{f}_G - \mathbf{f}_G^\epsilon$$

and

$$\mathbf{f}_G^2 = \mathbf{f}_G + \mathbf{f}_G^\epsilon,$$

for some nonzero signed 0-flow $\mathbf{f}_G^\epsilon$. □

Lemma 3.3.3. *There is a nonzero signed* 0*-flow* $\mathbf{f}_G^\epsilon$ *such that* $\mathbf{f}_G - \mathbf{f}_G^\epsilon$ *and* $\mathbf{f}_G + \mathbf{f}_G^\epsilon$ *are flows on* G *if and only if there is a nonzero signed* 0*-flow* $\mathbf{f}_G^\epsilon$ *on* G *whose support is contained in the support of* $\mathbf{f}_G$*.*

Proof. One implication is trivial, and the other one follows by observing that given a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ on G whose support is contained in the support of $\mathbf{f}_G$, we can obtain another nonzero signed 0-flow $\mathbf{f}_G^{\epsilon'}$ on G whose support is contained in the support of $\mathbf{f}_G$ such that the absolute value of the values $\mathbf{f}_G^{\epsilon'}(e)$, for edges $e \in G$, is arbitrarily small, by simply letting $\mathbf{f}_G^{\epsilon'} = \mathbf{f}_G^\epsilon/M$, for some large value of M. Thus if there is a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ on G whose support is contained in the support of $\mathbf{f}_G$, then we can construct a nonzero signed 0-flow $\mathbf{f}_G^{\epsilon'}$ such that $\mathbf{f}_G - \mathbf{f}_G^{\epsilon'}$ and $\mathbf{f}_G + \mathbf{f}_G^{\epsilon'}$ are flows on G. □

Corollary 3.3.4. *An* $\mathbf{a}$*-flow* $\mathbf{f}_G$ *on* G *is a vertex of* $\mathcal{F}_G(\mathbf{a})$ *if and only if there is no nonzero signed* 0*-flow* $\mathbf{f}_G^\epsilon$ *on* G *whose support is contained in the support of* $\mathbf{f}_G$*.*

Proof. Corollary 3.3.4 follows from Lemmas 3.3.2 and 3.3.3. □

Lemma 3.3.5. *If* $H \subset G$ *is the support of a nonzero signed* 0*-flow* $\mathbf{f}_G^\epsilon$*, then* H *contains no vertices of degree 1.*

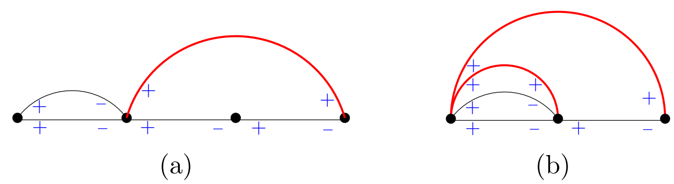


Figure 3-3: Regardless of how we order the edges above to form a cycle, the number of turns in the cycle will be 1 in (a) and even in (b). Thus, the resulting cycle in (a) is odd and in (b) is even.

Proof. If H contained a degree 1 vertex, $\mathbf{f}_G^\epsilon$ with support H could not be a 0-flow. $\square$

A **cycle** C is a sequence of oriented edges $e_1, \ldots, e_k$ such that the second vertex of e_i is the first vertex of e_{i+1} for $i \in [k]$ and with $k+1$ identified with 1. The number of **turns** in C is the number of times two consecutive edges meet at a vertex of C such that the edges of C are incident with the same sign to that vertex (repetition of vertices allowed). A cycle C of the graph G is called **even** if it has an even number of turns and **odd** otherwise. See Figure 3-3.

Lemma 3.3.6. *Given a set of edges which can be ordered to yield a cycle C, the parity of the number of turns of C is the same as that of any other cycle that the edges can be ordered to give.*

We leave the proof of Lemma 3.3.6 as an exercise to the reader. For examples see Figures 3-3(a) and 3-3(b).

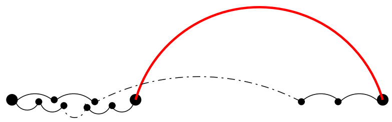
Lemma 3.3.7. *If $H \subset G$ is the support of a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$, then H contains an even cycle.*

Proof. Since by Lemma 3.3.5 H contains no vertices of degree 1, each edge of H is contained in at least one cycle. Let k be the number of linearly independent cycles in H. If $k = 1$ and the nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ has support H, then it follows by inspection that H is an even cycle. If $k > 1$ and the nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ has support H, let $P \subset H$ be a path such that $H - P$ contains $k-1$ linearly independent cycles and no vertices of degree 1. If P is contained in an even cycle in H, then we are done. If P is not contained in an even cycle of H, then there are two paths C_1 and C_2 in H such that $P + C_1$ and $P + C_2$ are cycles, but not even. Inspection shows that the cycle $C_1 + C_2$ is even. $\square$

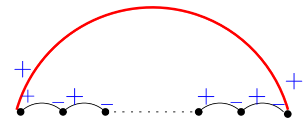
Lemma 3.3.8. *If $C \subset G$ is an even cycle, then there exists a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ with support C.*

Proof. Set $\mathbf{f}_G^\epsilon(e) = 0$ for $e \in G - C$ and $\mathbf{f}_G^\epsilon(e) \in \{+\epsilon, -\epsilon\}$ for $e \in C$. Note that since C is even there will be two such nonzero signed 0-flows $\mathbf{f}_G^\epsilon$. $\square$

Lemma 3.3.9. *There is a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ on G whose support is contained in the support of the $\mathbf{a}$-flow $\mathbf{f}_G$ if and only if the support of $\mathbf{f}_G$ contains an even cycle.*



(a)



(b)

Figure 3-4: Illustration of forms (i) and (ii) of Proposition 3.3.12

Proof. By Lemma 3.3.7 if there is a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ on G with support H, then H contains an even cycle. Thus, in particular, if there is a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ on G whose support is contained in the support of $\mathbf{f}_G$, then the support of $\mathbf{f}_G$ contains an even cycle. Conversely, by Lemma 3.3.8 if C is an even cycle contained in the support of $\mathbf{f}_G$, then there is a nonzero signed 0-flow $\mathbf{f}_G^\epsilon$ on G whose support is C, and thus contained in the support of $\mathbf{f}_G$. □

Theorem 3.3.10. *An* $\mathbf{a}$*-flow* $\mathbf{f}_G$ *on* G *is a vertex of* $\mathcal{F}_G(\mathbf{a})$ *if and only if the support of* $\mathbf{f}_G$ *contains no even cycle.*

Proof. Corollary 3.3.4 and Lemma 3.3.9 imply the statement of Theorem 3.3.10. □

Theorem 3.3.11. *If* $\mathbf{a} = (2, 0, \ldots, 0)$*, then the vertices of* $\mathcal{F}_G(\mathbf{a})$ *are integer. In particular, the set of vertices of* $\mathcal{F}_G(\mathbf{a})$ *is a subset of the set of integer* $\mathbf{a}$*-flows on* G*.*

By Theorem 3.3.10, in order to prove Theorem 3.3.11, it suffices to show that if the support of the $(2, 0, \ldots, 0)$-flow $\mathbf{f}_G$ contains no even cycle, then $\mathbf{f}_G$ is an integer flow. To achieve this, we characterize all possible odd cycles with no even subcycles in the support of a $(2, 0, \ldots, 0)$-flow $\mathbf{f}_G$. By a **subcycle** C' of a cycle C we mean a cycle C' whose edges are a subset of the edges of C.

Proposition 3.3.12. *A cycle* C *contained in the support of a* $(2, 0, \ldots, 0)$*-flow* $\mathbf{f}_G$ *contains no even subcycles if and only if its set of edges is of one of the three following forms:*

(i) $\{(v_1, v_2, -), \ldots, (v_{k-1}, v_k, -)\} \cup \{(w_1, w_2, -), \ldots, (w_{l-1}, w_l, -)\} \cup \{(w_l, v_k, +)\}$, *where* $v_1 = w_1$, $2 \le k, l$ *and* $v_1, \ldots, v_k, w_2, \ldots, w_l$ *are distinct. See Figure 3-4(a).*

(ii) $\{(v_1, v_2, -), \ldots, (v_{k-1}, v_k, -)\} \cup \{(v_1, v_k, +)\}$, *where* $v_1, \ldots, v_k$ *are distinct. See Figure 3-4(b).*

(iii) $\{(v_1, v_1, +)\}$

Proof. One direction is trivial.

To prove the other direction, let G' be the support of f_G. Observe that all vertices in G' must have a negative edge incident to them in order for the net flow to be 0 at all but the first vertex, unless G' is simply a loop at vertex 1. Note that a cycle with only negative edges is even. Note that a path of negative edges (which is not a cycle) can be contracted without affecting the parity of the number of turns of a cycle. The above observations together are sufficient to prove the non-trivial direction of the proposition. □

Proof of Theorem 3.3.11. Suppose that the $(2,0,\ldots,0)$-flow $\mathbf{f}_G$ is a vertex of $\mathcal{F}_G(2,0,\ldots,0)$. Let G' be the support of f_G. Theorem 3.3.10 and Proposition 3.3.12 imply that G' contains exactly one cycle C which contains no even subcycle and whose smallest vertex is v. If $v=1$ then $G'=C$ and if $v>1$ then G' is the union of C and a path $(1,z_1,-),(z_1,z_2,-),\ldots,(z_m,v,-)$. In both cases it is evident that the flow $\mathbf{f}_G$ has to be integer in order to be a $(2,0,\ldots,0)$-flow. □

Note that the proof of Theorem 3.3.11 characterizes all vertices of $\mathcal{F}_G(2,0,\ldots,0)$ very concretely. We summarize the results in Theorem 3.3.13.

Theorem 3.3.13. *A $(2,0,\ldots,0)$-flow $\mathbf{f}_G$ on G is a vertex of $\mathcal{F}_G((2,0,\ldots,0))$ if and only if it is the unique integer $(2,0,\ldots,0)$-flow on G with support of one of the following forms:*

(i) $\{(v_1,v_2,-),\ldots,(v_{k-1},v_k,-)\}\cup\{(w_1,w_2,-),\ldots,(w_{l-1},w_l,-)\}\cup\{(w_l,v_k,+)\}$, *where* $v_1=w_1=1$, $2\leq k,l$ *and* $v_1,\ldots,v_k,w_2,\ldots,w_l$ *are distinct.*

(ii) $\{(1,z_1,-),(z_1,z_2,-),\ldots,(z_m,v_1,-)\}\cup\{(v_1,v_2,-),\ldots,(v_{k-1},v_k,-)\}\cup$ $\cup\{(w_1,w_2,-),\ldots,(w_{l-1},w_l,-)\}\cup\{(w_l,v_k,+)\}$, *where* $v_1=w_1$, $2\leq k,l$ *and* $v_1,\ldots,v_k,w_2,\ldots,w_l$ *are distinct.*

(iii) $\{(v_1,v_2,-),\ldots,(v_{k-1},v_k,-)\}\cup\{(v_1,v_k,+)\}$, *where* $1=v_1,\ldots,v_k$ *are distinct.*

(iv) $\{(1,z_1,-),(z_1,z_2,-),\ldots,(z_m,v_1,-)\}\cup\{(v_1,v_2,-),\ldots,(v_{k-1},v_k,-)\}\cup\{(v_1,v_k,+)\}$, *where* $v_1,\ldots,v_k$ *are distinct.*

(v) $\{(v_1,v_1,+)\}$

(vi) $\{(1,z_1,-),(z_1,z_2,-),\ldots,(z_m,v_1,-)\}\cup\{(v_1,v_1,+)\}$

3.3.2 Vertices of the type C_{n+1} and type D_{n+1} Chan-Robbins-Yuen polytope

Theorem 3.3.13 gives a hands on characterization of the vertices of any type C_{n+1} and type D_{n+1} flow polytope. In this section we show how to use it to count the number of vertices of the type C_{n+1} and type D_{n+1} Chan-Robbins-Yuen polytopes $CRYD_{n+1}$ and $CRYC_{n+1}$.

Recall that the flow polytope $\mathcal{F}_{K_{n+1}}(1,0,\ldots,0,-1)$ of the complete graph K_{n+1} is the Chan-Robbins-Yuen polytope $CRYA_n$ [15]. One way to generalize $CRYA_n$ is to consider the complete signed graphs in type C_{n+1} and type D_{n+1}.

Let K^D_{n+1} be the complete signed graph on $n+1$ vertices of type D_{n+1} (all edges of the form $(i,j,\pm)$ for $1\leq i<j\leq n+1$ corresponding to all the positive roots in type D_{n+1}). Then the polytope $CRYD_{n+1}=\mathcal{F}_{K^D_{n+1}}(2,0,\ldots,0)$ is an analogue of the Chan-Robbins-Yuen polytope. The vector $(2,0,\ldots,0)$ is the highest root of type C_{n+1}, and we pick this vector as opposed to the highest root of type D_{n+1}, because we would like the vertices of $CRYD_{n+1}$ to be integral. If we were to study $\mathcal{F}_{K^D_{n+1}}(1,1,0,\ldots,0)$, where $(1,1,0,\ldots,0)$ is the highest root of type D_{n+1}, the vertices of this polytope would not be integral. Note that any signed graph on the vertex set $[n+1]$, including K^D_{n+1}, can be considered a type C_{n+1} graph, so that the choice of the highest root of C_{n+1} is not unnatural in any sense.

Let K^C_{n+1} be the complete signed graph together with loops $(i,i,+)$, $1\leq i\leq n+1$, corresponding to the type C_{n+1} positive roots $2e_i$ and let $CRYC_{n+1}=\mathcal{F}_{K^C_{n+1}}(2,0,\ldots,0)$.

Proposition 3.3.14. *The polytope $CRYD_{n+1}$ has $3^n - 2^n$ vertices.*

Proof. We prove the statement by induction. The base of induction is clear. Suppose that $CRYD_n$ has $3^{n-1} - 2^{n-1}$ vertices. Using Theorem 3.3.13 we see that the vertices of $CRYD_{n+1}$ have to be the unique integer $(2, 0, \ldots, 0)$-flows on G with support of the form:

- $\{(1, i, -)\} \cup S(i, n+1)$, where $2 \leq i \leq n$ and $S(i, n+1)$ is the support of a vertex of $CRYD_{n+2-i}$ where we consider the flow graph of $CRYD_{n+2-i}$ to be on the vertex set $\{i, i+1, \ldots, n+1\}$.
- $\{(v_1, v_2, -), \ldots, (v_{k-1}, v_k, -)\} \cup \{(w_1, w_2, -), \ldots, (w_{l-1}, w_l, -)\} \cup \{(w_l, v_k, +)\}$, where $v_1 = w_1 = 1$, $2 \leq k, l$ and $v_1, \ldots, v_k, w_2, \ldots, w_l$ are distinct.
- $\{(v_1, v_2, -), \ldots, (v_{k-1}, v_k, -)\} \cup \{(v_1, v_k, +)\}$, where $1 = v_1, \ldots, v_k$ are distinct.

Call the supports of the above forms of type I, II and II, respectively.
By induction, the number of vertices of $CRYD_{n+1}$ of type I is

$$\sum_{i=2}^{n}(3^{n+1-i} - 2^{n+1-i}).$$

By inspection, the number of vertices of $CRYD_{n+1}$ of type II is

$$\sum_{1<i<j\leq n+1}(3^{i-2}2^{j-i-1}).$$

Finally, the number of vertices of $CRYD_{n+1}$ of type III is

$$\sum_{i=2}^{n+1}2^{i-2}.$$

It is a matter of simple algebra to show that

$$\sum_{i=2}^{n}(3^{n+1-i} - 2^{n+1-i}) + \sum_{1<i<j\leq n+1}(3^{i-2}2^{j-i-1}) + \sum_{i=2}^{n+1}2^{i-2} = 3^n - 2^n.$$

□

Proposition 3.3.15. *The polytope $CRYC_{n+1}$ has 3^n vertices.*

Proof. Using Theorem 3.3.13 we see that the set of vertices of $CRYC_{n+1}$ is equal to the set of vertices of $CRYD_{n+1}$ together with the vertices which are the unique integer $(2, 0, \ldots, 0)$-flows on G with support of the form:

- $\{(v_1, v_1, +)\}$
- $\{(1, z_1, -), (z_1, z_2, -), \ldots, (z_m, v_1, -)\} \cup \{(v_1, v_1, +)\}$

By Proposition 3.3.14 the number of vertices of $CRYD_{n+1}$ is $3^n - 2^n$ and the number of vertices of the form described above is 2^n. Thus, Proposition 3.3.15 follows.

□

3.4 Reduction rules of the flow polytope $\mathcal{F}_G(\mathbf{a})$

In this section we propose an algorithmic way of triangulating the flow polytope $\mathcal{F}_G(\mathbf{a})$. This also yields a systematic way to calculate the volume of $\mathcal{F}_G(\mathbf{a})$ by summing the volumes of the simplices in the triangulation. The process of triangulation of $\mathcal{F}_G(\mathbf{a})$ is closely related to the triangulation of root polytopes by subdivision algebras, as studied by Mészáros in [42, 43].

Given a signed graph G_0 on the vertex set $[n+1]$, if we have two edges incident to vertex i with opposite signs, *e.g.* $(a, i, -), (i, b, +)$ with flows p and q, we will add a new edge not incident to i, *e.g.* $(a, b, +)$, and discard one or both of the original edges to obtain graphs G_1, G_2, and G_3 respectively. We then reassign flows to preserve the original netflow on the vertices. We look at all possible cases and obtain the reduction rules (R1)-(R6) in Figure 3-5.

3.4.1 Reduction rules for signed graphs

Given a graph G_0 on the vertex set $[n+1]$ and $(a, i, -), (i, b, -) \in E(G_0)$ for some $a < i < b$, let G_1, G_2, G_3 be graphs on the vertex set $[n+1]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0)\backslash\{(a,i,-)\} \cup \{(a,b,-)\}, \\ E(G_2) &= E(G_0)\backslash\{(i,b,-)\} \cup \{(a,b,-)\}, \\ E(G_3) &= E(G_0)\backslash\{(a,i,-)\}\backslash\{(i,b,-)\} \cup \{(a,b,-)\}. \end{aligned} \tag{R1}$$

Given a graph G_0 on the vertex set $[n+1]$ and $(a, i, -), (i, b, +) \in E(G_0)$ for some $a < i < b$, let G_1, G_2, G_3 be graphs on the vertex set $[n+1]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0)\backslash\{(a,i,+)\} \cup \{(a,b,+)\}, \\ E(G_2) &= E(G_0)\backslash\{(i,b,-)\} \cup \{(a,b,+)\}, \\ E(G_3) &= E(G_0)\backslash\{(a,i,-)\}\backslash\{(i,b,+)\} \cup \{(a,b,+)\}. \end{aligned} \tag{R2}$$

Given a graph G_0 on the vertex set $[n+1]$ and $(a, i, -), (b, i, +) \in E(G_0)$ for some $a < b < i$, let G_1, G_2, G_3 be graphs on the vertex set $[n+1]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0)\backslash\{(a,i,-)\} \cup \{(a,b,+)\}, \\ E(G_2) &= E(G_0)\backslash\{(b,i,+)\} \cup \{(a,b,+)\}, \\ E(G_3) &= E(G_0)\backslash\{(a,i,-)\}\backslash\{(b,i,+)\} \cup \{(a,b,+)\}. \end{aligned} \tag{R3}$$

Given a graph G_0 on the vertex set $[n+1]$ and $(a,i,+),(b,i,-) \in E(G_0)$ for some $a<b<i$, let G_1, G_2, G_3 be graphs on the vertex set $[n+1]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0)\backslash\{(a,i,+)\} \cup \{(a,b,+)\}, \\ E(G_2) &= E(G_0)\backslash\{(b,i,-)\} \cup \{(a,b,+)\}, \\ E(G_3) &= E(G_0)\backslash\{(a,i,+)\}\backslash\{(b,i,-)\} \cup \{(a,b,+)\}. \end{aligned} \tag{R4}$$

Given a graph G_0 on the vertex set $[n+1]$ and $(a,i,-),(a,i,+) \in E(G_0)$ for some $a<i$, let G_1, G_2, G_3 be graphs on the vertex set $[n+1]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0)\backslash\{(a,i,+)\} \cup \{(a,a,+)\}, \\ E(G_2) &= E(G_0)\backslash\{(a,i,-)\} \cup \{(a,a,+)\}, \\ E(G_3) &= E(G_0)\backslash\{(a,i,+)\}\backslash\{(a,i,+)\} \cup \{(a,a,+)\}. \end{aligned} \tag{R5}$$

Given a graph G_0 on the vertex set $[n+1]$ and $(a,i,-),(i,i,+) \in E(G_0)$ for some $a<i$, let G_1, G_2, G_3 be graphs on the vertex set $[n+1]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0)\backslash\{(a,i,-)\} \cup \{(a,i,+)\}, \\ E(G_2) &= E(G_0)\backslash\{(i,i,+)\} \cup \{(a,i,+)\}, \\ E(G_3) &= E(G_0)\backslash\{(a,i,-)\}\backslash\{(i,i,+)\} \cup \{(a,i,+)\}. \end{aligned} \tag{R6}$$

We say that G_0 **reduces** to G_1, G_2, G_3 under the reduction rules (R1)-(R6). Figure 3-5 shows these reduction rules graphically.

Proposition 3.4.1. *Given a signed graph G_0 on the vertex set $[n+1]$, a vector $\mathbf{a} \in \mathbb{Z}^{n+1}$, and two edges e_1 and e_2 of G_0 on which one of the reductions (R1)-(R6) can be performed yielding the graphs G_1, G_2, G_3, then*

$$\mathcal{F}_G(\mathbf{a}) = \mathcal{F}_{G_1}(\mathbf{a}) \bigcup \mathcal{F}_{G_2}(\mathbf{a}), \quad \mathcal{F}_{G_1}(\mathbf{a}) \bigcap \mathcal{F}_{G_2}(\mathbf{a}) = \mathcal{F}_{G_3}(\mathbf{a}), \quad \textit{and } \mathcal{F}_{G_1}(\mathbf{a})^\circ \bigcap \mathcal{F}_{G_2}(\mathbf{a})^\circ = \varnothing,$$

where $\mathcal{P}^\circ$ denotes the interior of $\mathcal{P}$.

The proof of Proposition 3.4.1 is left to the reader. Figure 3-5 and the definition of a flow polytope is all that is needed!

3.5 Subdivision of flow polytopes

In this section we use the reduction rules for signed graphs given in Section 3.4, following a specified order, to subdivide flow polytopes. The main result of this section is the Subdivision Lemma as stated below, and again in Lemma 3.5.7. While the notation of this lemma seems complicated at first, the subsections below contain all the definitions and explanations necessary to understand it. This lemma is key in all our pursuits: it lies at the heart of the relationship between flow polytopes and Kostant partition functions. It also is a tool for systematic subdivisions, and as such calculating volumes of particular flow polytopes.

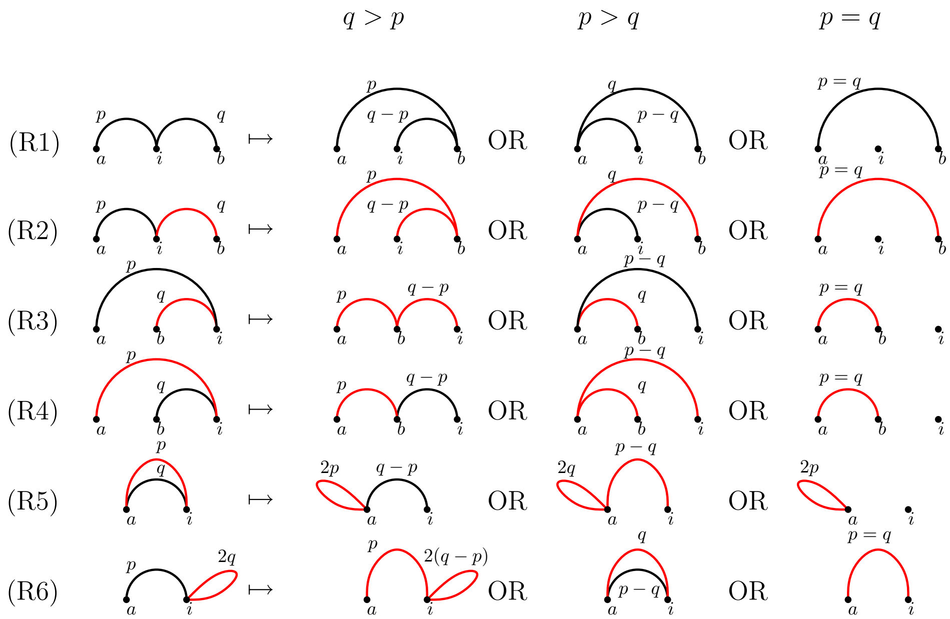


Figure 3-5: Reduction rules from Equations (R1)-(R6).

Subdivision Lemma. *Let G be a signed graph on the vertex set $[n+1]$ and $\mathcal{F}_G(\mathbf{a})$ be its flow polytope for $\mathbf{a} \in \mathbb{Z}^{n+1}$ with $a_i = 0$, for a fixed $i \in [n]$. Then the flow polytope subdivides as:*

$$\mathcal{F}_G(\mathbf{a}) = \bigcup_{T \in \mathcal{T}^{\pm}_{\mathcal{I}_i, \mathcal{O}_i}(\mathcal{O}_i^+)} \mathcal{F}_{G_T^{(i)}}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, 2y - \sum_{i=1}^{n} a_i), \tag{3.5.1}$$

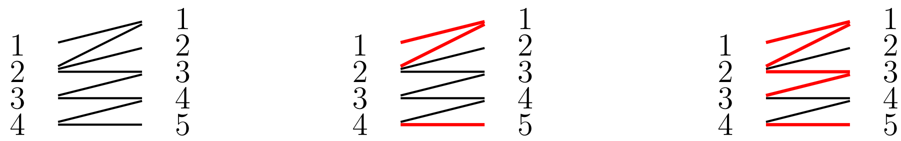
where $G_T^{(i)}$ are graphs on the vertex set $[n+1] \backslash \{i\}$ as defined in Section 3.5.2; and and $\mathcal{T}^{\pm}_{\mathcal{I}_i, \mathcal{O}_i}(\mathcal{O}_i^+)$ is the set of signed trees (signed trees are defined in Section 3.5.1.

First we define the trees or equivalently weak compositions that are important for the subdivision (Sections 3.5.1 and 3.5.2), we then define the order of application of reduction rules and restate and prove the Subdivision Lemma (Section 3.5.3). In the next section we use this lemma to compute volumes of flow polytopes for both signless graphs H and signed graphs G.

3.5.1 Noncrossing trees

The subdivisions mentioned above are encoded by bipartite trees with negative and positive edges that are noncrossing. We start by defining such trees.

A **negative bipartite noncrossing tree** T with left vertices $x_1, \ldots, x_\ell$ and right vertices $x_{\ell+1}, \ldots, x_{\ell+r}$ is a bipartite tree of negative edges that has no pair of edges $(x_p, x_{\ell+q}, -), (x_t, x_{\ell+u}, -)$ where $p < t$ and $q > u$. If L and R are the ordered sets $(x_1, \ldots, x_\ell)$ and $(x_{\ell+1}, \ldots, x_{\ell+r})$, let $\mathcal{T}^-_{L,R}$ be the set of such noncrossing bipartite trees. Note that $\#\mathcal{T}^-_{L,R} = \binom{\ell+r-2}{\ell-1}$, since they are in bijection with weak compositions of $\ell - 1$ into r parts. Namely, a tree T corresponds to the composition of indegrees



(a) $T_1 \in \mathcal{T}^-_{L,R}$ (b) $T_2 \in \mathcal{T}^\pm_{L,R}(\{1,5\})$ (c) $T_3 \in \mathcal{T}^\pm_{L,R}(\{1,3,5\})$

Figure 3-6: Examples of bipartite noncrossing trees that are: (a) negative (composition $(1,0,1,1,0)$), (b) signed with $R^+ = \{1,5\}$ (composition $(1^+,0^-,1^-,1^-,0^+)$), (c) signed with $R^+ = \{1,3,5\}$ (composition $(1^+,0^-,1^+,1^-,0^+)$).

of the right vertices: $(b_1,\ldots,b_r)$ where $b_i = \mathit{indeg}_T(x_{\ell+i}) - 1$. See Figure 3-6 (a) for an example of such a tree.

A **signed bipartite noncrossing tree** is a bipartite noncrossing tree T with negative $(\cdot,\cdot,-)$ and positive $(\cdot,\cdot,+)$ edges such that the right vertices are either incident to only negative edges or only positive edges. Let $\mathcal{T}^\pm_{L,R}(R^+)$ be the set of such trees with $\#L$ left vertices, the ordered set R^+ of right vertices incident to only positive edges, and $\#R - \#R^+$ right vertices incident to only negative edges. Note that for fixed R^+, $\#\mathcal{T}^\pm_{L,R}(R^+) = \#\mathcal{T}^-_{L,R}$, and we can encode such trees with a signed composition $(b_1^\pm, b_2^\pm, \ldots, b_r^\pm)$ indicating whether the incoming edges to each right vertex are all positive or all negative. See Figure 3-6 (b)-(c) for an example of such trees.

If both L and R are empty, the set $\mathcal{T}^\pm_{\varnothing,\varnothing}$ consists of one element: the empty tree.

3.5.2 Removing vertex i from a signed graph G

One of the points of the Subdivision Lemma is to start by a graph G on the vertex set $[n+1]$ and to subdivide the flow polytope of G into flow polytopes of graphs on a vertex set smaller than $[n+1]$. In this section we show the mechanics of this. We take a signed graph G and replace incoming and outgoing edges of a fixed vertex i by edges that avoid i and come from a noncrossing tree T. The outcome is a graph we denote by $G^{(i)}_T$ on the vertex set $[n+1]\backslash\{i\}$. To define this precisely we first introduce some notation:

Given a signed graph G and one of its vertices i, let $\mathcal{I}_i = \mathcal{I}_i(G)$ be the multiset of incoming edges to i (negative edges of the form $(\cdot,i,-)$). Let $\mathcal{O}_i = \mathcal{O}_i(G)$ be the multiset of outgoing edges from i (edges of the form $(\cdot,i,+)$ and $(i,\cdot,\pm)$). And let $\mathcal{O}^\pm_i$ be the signed refinement of $\mathcal{O}_i$. Note that $\mathit{indeg}_G(i) = \#\mathcal{I}_i(G)$.

Fix a tree $T \in \mathcal{T}^\pm_{\mathcal{I}_i,\mathcal{O}_i}(\mathcal{O}^+_i)$ with $\#\mathcal{I}_i$ left vertices, $\#\mathcal{O}_i$ right vertices and $\#\mathcal{O}^+_i$ right vertices incident only to positive edges. For each tree-edge (e_1,e_2) of T where $e_1 = (r,i,-) \in \mathcal{I}_i$ and $e_2 \in \mathcal{O}_i$ ($e_2 = (i,s,\pm)$ or $(t,i,\pm)$), let $\mathit{edge}(e_1,e_2)$ be the following signed edge:

$$\mathit{edge}(e_1,e_2) = \begin{cases} (r,s,\pm) & \text{if } e_2 = (i,s,\pm), \\ (r,t,+) & \text{if } e_2 = (t,i,+). \end{cases} \tag{3.5.2}$$

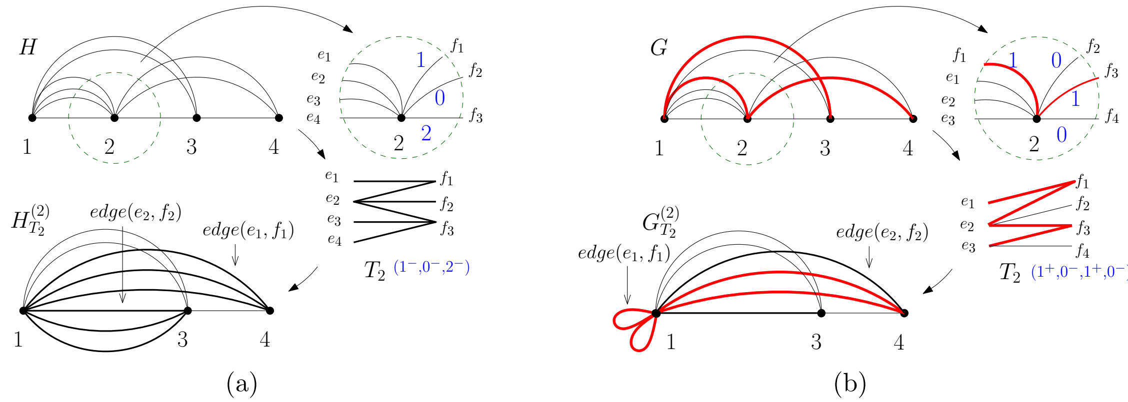


Figure 3-7: Replacing the incident edges of vertex 2 in (a) a graph H, of only negative edges, by a noncrossing tree T encoded by the composition $(1^-, 0^-, 2^-)$ of $3 = indeg_H(2) - 1$. (b) a signed graph G by a signed noncrossing tree T encoded by the composition $(1^+, 0^-, 1^+, 0^-)$ of $2 = indeg_G(2) - 1$.

Note that if $e_1 = (r, i, -)$ and $e_2 = (r, i, +)$ then we allow $e(e_1, e_2)$ to be the loop $(r, r, +)$.

Then $G^{(i)}_T$ is the graph obtained from $G \backslash i$ (G with vertex i removed) by adding to it the edges $\{edge(e_1, e_2) \mid (e_1, e_2) \in E(T)\}$. See Figure 3-7 for examples of $G^{(i)}_T$ for a graph of only negative edges and a signed graph.

Remark 3.5.3. *If T is given by a weak composition of $\#\mathcal{I}_i - 1$ into $\#\mathcal{O}_i$ parts, say $(b_e)_{e \in \mathcal{O}_i}$. Then:*

(i) we record this composition by labelling the edges e in $\mathcal{O}_i$ of G with the corresponding part b_e. We can view this labelling as assigning a flow $b(e) = b_e$ to edges e of $\mathcal{O}_i$ in G.

(ii) The $b_e + 1$ edges $(\cdot, e)$ in T will correspond to $b_e + 1$ edges $edge(\cdot, e)$ in $G^{(i)}_T$. We think of these $b_e + 1$ edges as one edge coming from the original edge e in G, and b_e new ones.

■

The following is an easy consequence of the construction of $G^{(i)}_T$.

Proposition 3.5.4. *Given a graph G on the vertex set $[n+1]$, the incoming and outgoing edges of vertex j of the graph $G^{(i)}_T$ on the vertex set $[n+1]\backslash\{i\}$ built above are:*

$$\mathcal{I}_j(G^{(i)}_T) = \begin{cases} \mathcal{I}_j(G) \cup \{\textit{new edges } (k, j, -) \mid k < i < j\} & \textit{if } j > i, \\ \mathcal{I}_j(G) & \textit{if } j < i; \end{cases} \tag{3.5.5}$$

$$\mathcal{O}_j(G^{(i)}_T) = \begin{cases} \mathcal{O}_j(G) \cup \{\textit{new edges } (k, j, +) \mid k < i < j\} & \textit{if } j > i, \\ \mathcal{O}_j(G) & \textit{if } j < i. \end{cases} \tag{3.5.6}$$

Next, we give a subdivision of the flow polytope $\mathcal{F}_G$ of a signed graph G in terms of flow polytopes $\mathcal{F}_{G^{(i)}_T}$ of graphs $G^{(i)}_T$.

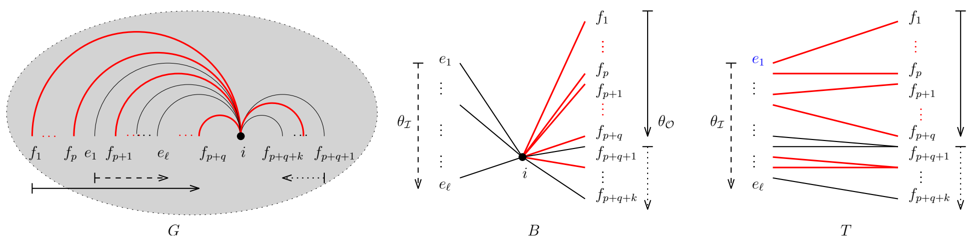


Figure 3-8: Setting of Lemma 3.5.7 for edges incident to vertex i. We fix total orders $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$ on $\mathcal{I}_i(G)$ and $\mathcal{O}_i(G)$ respectively. The resulting bipartite trees are in $\mathcal{T}^{\pm}(L,R)^{R"}$ where $L = \theta_{\mathcal{I}}(\mathcal{I})$, $R = \theta_{\mathcal{O}}(\mathcal{O})$ and $R^+ = \theta_{\mathcal{O}}(\mathcal{O}^+)$.

3.5.3 Subdivision Lemma

In this subsection we are ready to state again the Subdivision Lemma, now with all the terminology defined, and prove it. We want to subdivide the flow polytope of a graph G on the vertex set $[n+1]$. To do this we apply the reduction rules to incoming and outgoing edges of a vertex i in G with zero flow. Then by repeated application of reductions to this vertex, we can essentially delete this vertex from the resulting graphs, and as a result get to graphs on the vertex set $[n+1]\backslash\{i\}$. The Subdivision Lemma tells us exactly what these graphs, with a smaller vertex set, are.

We have to specify in which order we do the reduction at a given vertex i, since at any given stage there might be several choices of pairs of edges to reduce. First we fix a linear order $\theta_{\mathcal{I}}$ on the multiset $\mathcal{I}_i(G)$ of incoming edges to vertex i, and a linear order $\theta_{\mathcal{O}}$ on the multiset $\mathcal{O}_i(G)$ of outgoing edges from vertex i. Recall that $\mathcal{O}_i(G)$ also includes edges $(a,i,+)$ where $a<i$. We choose the pair of edges to reduce in the following way: we pick the first available edge from $\mathcal{I}_i(G)$ and from $\mathcal{O}_i(G)$ according to the orders $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$. At each step of the reduction, one outcome will have one fewer incoming edge and the other outcome will have one fewer outgoing edge. In each outcome, when we choose the next pair of edges to reduce we pick the next edge from $I_i(G)$ and from $\mathcal{O}_i(G)$ that is still available.

The Subdivision Lemma shows that when we follow this order to apply reductions to a vertex with zero flow; the full dimensional outcomes are encoded by signed bipartite noncrossing trees.

Lemma 3.5.7 (Subdivision Lemma)**.** *Let G be a signed graph on the vertex set $[n+1]$ and $\mathcal{F}_G(\mathbf{a})$ be its flow polytope for $\mathbf{a} \in \mathbb{Z}^{n+1}$ with $a_i = 0$. Fix linear orders $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$ on $\mathcal{I}_i(G)$ and $\mathcal{O}_i(G)$ respectively. If we apply the reduction rules to edges incident to vertex i following the linear orders, then the flow polytope subdivides as:*

$$\mathcal{F}_G(\mathbf{a}) = \bigcup_{T \in \mathcal{T}^{\pm}_{L,R}(R^+)} \mathcal{F}_{G^{(i)}_T}(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n, 2y - \sum_{i=1}^{n} a_i), \tag{3.5.8}$$

where $G^{(i)}_T$ is as defined in Section 3.5.2; and $\mathcal{T}^{\pm}_{L,R}(R^+)$ is the set of signed trees with $L = \theta_{\mathcal{I}}(\mathcal{I}_i), R = \theta_{\mathcal{O}}(\mathcal{O}_i)$ and $R^+ = \theta_{\mathcal{O}}(\mathcal{O}^+_i)$.

Proof. We apply the reduction rules (see Figure 3-5 for the rules) to pairs of edges incident to i following the orders $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$. Each step of the reduction takes a graph G and gives two graphs G' and G'' like G but where an edge incident to i has been replaced with a new positive or negative edge not incident to vertex i. This new edge will not take part of any other reduction on vertex i. We continue the reduction until we obtain graphs with only one edge incident to vertex i. The flow on that single edge is forced to be zero since $a_i = 0$ so we can "remove" the edge from the graph and as obtain a graph with no incident edges to vertex i. We call such graphs the **final outcomes** of the reduction.

We only deal with the edges incident to vertex i, so for clarity we carry out the reductions on a graph B representing these edges ordered by $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$; see Figure 3-8(b). The graph B has left vertices L, a middle vertex i, and right vertices R; and edges $\{(e_t, i, -) \mid e_t \in L\} \cup \{(i, f_t, \pm) \mid f_t \in R\}$ where $\pm$ depends on the sign of f_t. At the end of the reduction the lingering edges will form a noncrossing tree T and the corresponding full outcome will be $G_T^{(i)}$.

Let $\mathbf{a}' = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots) \in \mathbb{Z}^n$. Since the netflow on vertex i is zero, a partial outcome G' where $\#\mathcal{I}_i(G') + \#\mathcal{O}_i(G') > 1$ and either $\mathcal{I}_i(G')$ or $\mathcal{O}_i(G')$ is empty is a priori lower dimensional. This is because all the edges on the nonempty multiset of $\mathcal{I}_i(G')$ or $\mathcal{O}_i(G')$ are forced to have zero flow. We call such outcomes **bad**. We show by induction on $c_G := \#\mathcal{I}_i(G) + \#\mathcal{O}_i(G)$, that the good (non bad) full outcomes of the reduction are exactly the graphs $G_T^{(i)}$ for all noncrossing bipartite trees T in $\mathcal{T}^{\pm}_{L,R}(R^+)$ where $L = \theta_{\mathcal{I}}(\mathcal{I}_i), R = \theta_{\mathcal{O}}(\mathcal{O}_i)$ and $R^+ = \theta_{\mathcal{O}}(\mathcal{O}_i^+)$. Recall that such trees are in bijection with signed compositions $(b_e)_{e\in R}$ of $\#\mathcal{I}_i(G) - 1$ into $\#\mathcal{O}_i(G)$ parts.

The base case, when $c_G = 1$ consists of a single edge e incident to vertex i that is incoming $(a, i, -)$ or outgoing $(i, b, \pm)$. Since the netflow on vertex i is zero, then the flow on e is forced to be zero. We obtain $G\backslash\{i\}$ which we can identify with $G_T^{(i)}$ where $T = \emptyset$ is the empty tree.

For $c_G > 1$ if either $\mathcal{I}_i(G)$ is empty or $\mathcal{O}_i(G)$ is empty then G is already a bad outcome. If both $\mathcal{I}_i(G)$ and $\mathcal{O}_i(G)$ are nonempty then from the linear orders $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{O}}$ we pick the next available pair of edges to reduce. The pair will be an incoming negative edge $e_1 = (a, i, -)$ and an outgoing edge $f_1 = (i, b, \pm)$. We do the reduction (R1) or (R2) in Figure 3-5 and obtain graphs G' and G'' with a new edge $(a, b, \pm)$ and without $(i, b, \pm)$ or $(a, i, -)$ respectively (see Figure 3-9). For both G' and G'' we have $c_{G'} = c_{G''} = c_G - 1$ ($\mathcal{O}_i(G') = \mathcal{O}_i(G)\backslash(i, b, \pm)$ and $\mathcal{I}_i(G'') = \mathcal{I}_i(G)\backslash(a, i, -)$). By induction, the final outcomes of the reduction on G' are $G'^{(i)}_{T'}$ where T' are noncrossing bipartite trees in $\mathcal{T}^{\pm}_{L\backslash a, R\backslash b}$. But $T' \cup (a, b, \pm)$ is still a noncrossing bipartite tree (since we follow the orders $\theta_{\mathcal{I}_i(G)}$ and $\theta_{\mathcal{O}_i(G)}$), $G'^{(i)}_{T'} = G^{(i)}_{T'\cup(a,b,\pm)}$ and the set $\{T' \cup (a, b, \pm) \mid T' \in \mathcal{T}^{\pm}_{L\backslash a, R\backslash b}\}$ are exactly the trees in $\mathcal{T}^{\pm}_{L,R}(R^+)$ with $b_1 = indeg(f_1) - 1 = 0$ (see Figure 3-9). Let $\mathcal{T}^{(b_1=0)}$ be the set of these trees. Similarly, by induction, the final outcomes of the reduction on G'' are the graphs $G_T^{(i)}$ for all trees T in $\mathcal{T}^{\pm}_{L,R}(R^+)$ where $b_1 = indeg(f_1) - 1 > 0$. Let $\mathcal{T}^{(b_1>0)}$ be the set of these trees. Since $\mathcal{T}^{\pm}_{L,R}(R^+) = \mathcal{T}^{(b_1=0)} \cup \mathcal{T}^{(b_1>0)}$ where the union is disjoint, then from G we obtain the full outcomes

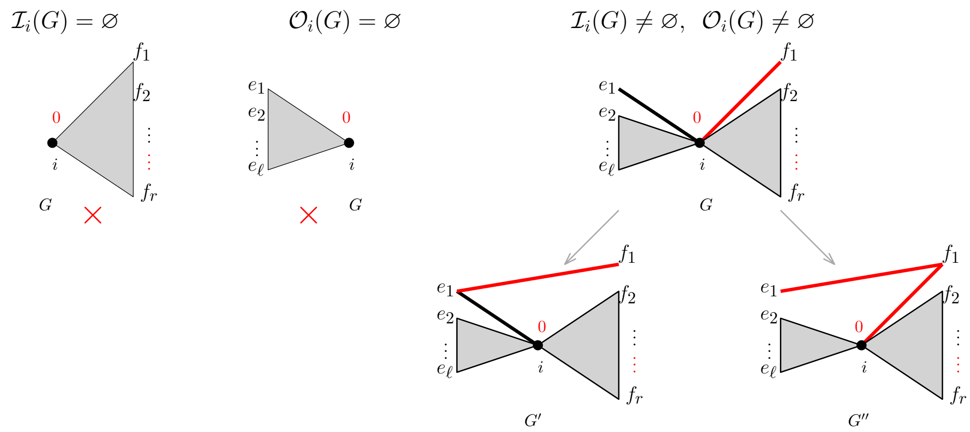


Figure 3-9: Inductive step in proof of the Subdivision Lemma.

$G_T^{(i)}$ where $T \in \mathcal{T}^{\pm}_{L,R}(R^+)$.

So from the reduction we obtain flow polytopes $\mathcal{F}_{G_T^{(i)}}(\mathbf{a})$ where T is in $\mathcal{T}^{\pm}_{L,R}(R^+)$. Thus, by repeated application of Proposition 3.4.1, it will follow that $\mathcal{F}_G(\mathbf{a})$ subdivides as a union of $\mathcal{F}_{G_T^{(i)}}(\mathbf{a})$ for all trees T in $\mathcal{T}^{\pm}_{L,R}(R^+)$ as desired. $\square$

See Figure 3-10 for an example of a subdivision into final outcomes that are indexed by noncrossing bipartite trees.

In the next section, we apply Lemma 3.5.7 to compute the volume of the flow polytope $\mathcal{F}_G(\mathbf{a})$ where G is a signed graph and $\mathbf{a} = (2, 0, \ldots, 0)$, the highest root of the root system C_{n+1}. As a motivation and to highlight differences, we first use a special case of the Subdivision Lemma, as done by Postnikov and Stanley [53, 58], to compute the volume of the polytope $\mathcal{F}_H(1, 0, \ldots, 0, -1)$ where H is a graph with only negative edges.

3.6 Volume of flow polytopes

In this section we use the Subdivision Lemma (Lemma 3.5.7) on flow polytopes $\mathcal{F}_H(1, 0, \ldots, 0, -1)$, where H is a graph consisting of only negative edges, and on $\mathcal{F}_G(2, 0, \ldots, 0)$, where G is a signed graph, to prove the formulae for their volume given in Theorem 3.6.2 ([53, 58]) and Theorem 3.6.16, respectively. To establish the connection between the volume of flow polytopes and Kostant partition functions for signed graphs, we introduce the notion of dynamic Kostant partition functions, which specializes to Kostant partition functions in the case graphs with only negative edges.

3.6.1 A correspondence between integer flows and simplices in a triangulation of $\mathcal{F}_H(1, 0, \ldots, 0, -1)$, where H only has negative edges

Let H be a graph on the vertex set $[n+1]$ and *only* negative edges, and $\mathcal{F}_H(1, 0, \ldots, 0, -1)$ be its flow polytope where $(1, 0, \ldots, 0, -1) \in \mathbb{Z}^{n+1}$. We apply Lemma 3.5.7 succes-

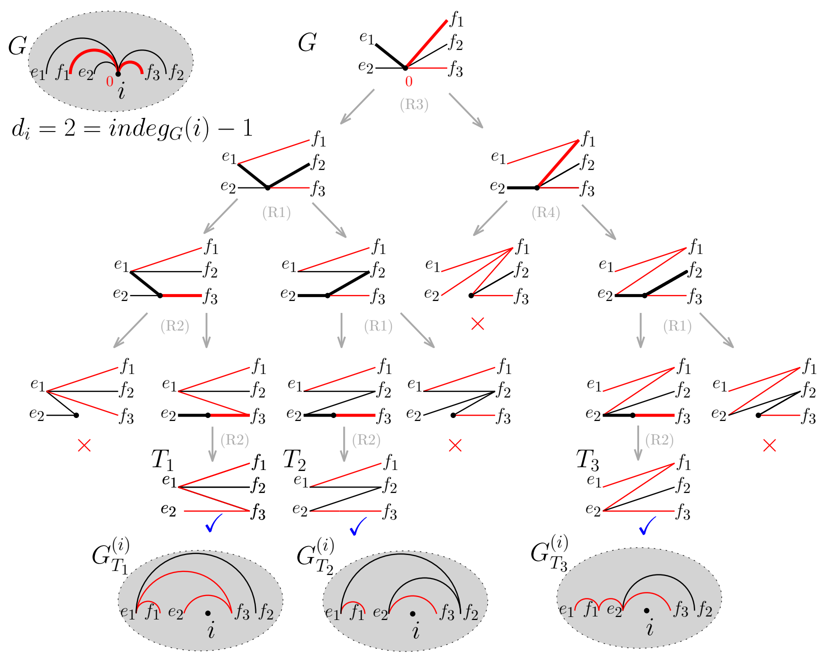


Figure 3-10: Example of a subdivision (the selected edges to reduce are **bold**). The outcomes indicated by × are bad outcomes since they are priori lower dimensional. The final outcomes indicated by ✓ are indexed by signed trees in $\mathcal{T}^{\pm}_{\{e_1,e_2\},\{f_1,f_2,f_3\}}(f_1)$ or equivalently the compositions $(0^-,0^-,1^+),(0^-,1^-,0^+)$, and $(1^-,0^-,0^+)$.

sively to vertices $2,3,\ldots,n$. At the end we obtain the subdivision:

$$\mathcal{F}_H(1,0,\ldots,0,-1) = \bigcup_{T_n^-}\cdots\bigcup_{T_3^-}\bigcup_{T_2^-}\mathcal{F}_{((\cdots(H^{(2)}_{T_2^-})^{(3)}_{T_3^-}\cdots)^{(n)}_{T_n^-}}(1,-1), \tag{3.6.1}$$

where T_i^- are noncrossing trees with only negative edges. See Figure 3-12 (a) for an example of a subdivision of an instance of $\mathcal{F}_H(1,0,\ldots,0,-1)$. Then $H_n := ((\cdots(H^{(2)}_{T_2^-})^{(3)}_{T_3^-}\cdots)^{(n)}_{T_n^-}$ is a graph consisting of two vertices, 1 and $n+1$ and $\#E(H)-n+1$ edges between them. Then $\mathcal{F}_{H_n}(1,-1)$ is an $(\#E(H)-n)$-dimensional simplex with normalized unit volume. Therefore, $\mathrm{vol}(\mathcal{F}_{H_n}(1,0,\ldots,0,-1))$ is the number of choices of bipartite noncrossing trees $T_2^-,\ldots,T_n^-$ where T_{i+1}^- encodes a composition of $\#\mathcal{I}_{i+1}(H_i)-1$ with $\#\mathcal{O}_{i+1}(H_i)$ parts. The next result by Postnikov and Stanley [53, 58] shows that this number of tuples of trees is also the number of certain integer flows on H. We reproduce their proof to motivate and highlight the differences with the case of signed graphs discussed in the next subsection.

Theorem 3.6.2 ([53, 58]). *Given a loopless connected (signless) graph H on the vertex set $[n+1]$, let $d_i = indeg_H(i)-1$ for $i\in\{2,\ldots,n\}$. Then, the normalized* *volume* $\mathrm{vol}(\mathcal{F}_H(1,0,\ldots,0,-1)$ *of the flow polytope associated to graph* H *is*

$$\mathrm{vol}(\mathcal{F}_H(1,0,\ldots,0,-1)) = k_H(0,d_2,\ldots,d_n,-\sum_{i=2}^{n} d_i),$$

where k_H *is the Kostant partition function of* H.

Proof. For this proof, let $H_i := (\cdots(H^{(2)}_{T_2^-})^{(3)}_{T_3^-}\cdots)^{(i)}_{T_i^-}$ for $i = 2,\ldots,n$. From Equation (3.6.1) and the discussion immediately after, we have that $\mathrm{vol}(\mathcal{F}_H(1,0,\ldots,0,-1))$ is the number of choices of noncrossing bipartite trees $(T_2^-,\ldots,T_n^-)$ where T_{i+1}^- encodes a weak composition of $\#\mathcal{I}_{i+1}(H_i)-1$ with $\#\mathcal{O}_{i+1}(H_i)$ parts. We give a correspondence between $\big(H;(T_2^-,\ldots,T_n^-)\big)$ and integer $\mathbf{a}$-flows on H where $\mathbf{a} = (0,d_2,\ldots,d_n,-\sum_{i=2}^n d_i)$. The proof is then complete since by Lemma 3.2.4 these integer flows are counted by $k_H(0,d_2,\ldots,d_n,-\sum_{i=2}^n d_i)$.

To give the correspondence between $\big(H;(T_2^-,\ldots,T_n^-)\big)$ and integer $\mathbf{a}$-flows on H where $\mathbf{a} = (0,d_2,\ldots,d_n,-\sum_{i=2}^n d_i)$, note that the tree T_{i+1}^- is given by a weak composition $(b_e^{(i+1)})_{e\in\mathcal{O}_{i+1}(H_i)}$ of $\#\mathcal{I}_{i+1}(H_i)-1$ into $\#\mathcal{O}_{i+1}(H_i)$ parts. By Remark 3.5.3 (i), we can encode this weak composition by assigning a flow $b(e) = b_e^{(i+1)}$ to edges e in $\mathcal{O}_{i+1}(H_i)$ to H_i. But since H and H_i consist only of negative edges, iterating Proposition 3.5.4 we see that

$$\mathcal{O}_{i+1}(H_i) = \mathcal{O}_{i+1}(H). \tag{3.6.3}$$

Therefore, we can also encode the weak compositions on the edges of H. So, for $i = 2,\ldots,n$ we record weak compositions $(b_e^{(i)})_{e\in\mathcal{O}_i(H)}$ (and thus the trees T_i^-) as flows $b(e) = b_e^{(i)}$ on $e \in \mathcal{O}_i(H)$ of H. For $i = 1$, we assign flows $b(e) = 0$ for $e \in \mathcal{O}_1(H)$. Next we calculate the netflow on vertex $i+1$ of H:

$$\sum_{e\in\mathcal{O}_{i+1}(H)} b(e) = \#I_{i+1}(H_i)-1, \tag{3.6.4}$$

$$\sum_{e\in\mathcal{I}_{i+1}(H)} b(e) = \#\{\text{new edges } (\cdot, i+1, -)\}, \tag{3.6.5}$$

Where Equation (3.6.4) follows since $(b_e^{(i)})_{e\in\mathcal{O}_{i+1}(H)}$ is a weak composition of $\#I_{i+1}(H_i)-1$. Equation (3.6.5) follows from Remark 3.5.3 (ii). Then by Proposition 3.5.4

$$\begin{aligned} a_{i+1} &= \sum_{e\in\mathcal{O}_{i+1}(H)} b(e) - \sum_{e\in\mathcal{I}_{i+1}(H)} b(e) \\ &= (\#\mathcal{I}_{i+1}(H_i)-1) - \#\{\text{new edges } (\cdot,i+1,-)\} \\ &= (\#\mathcal{I}_{i+1}(H) + \#\{\text{new edges } (\cdot,i+1,-)\}-1) - \#\{\text{new edges } (\cdot,i+1,-)\}. \end{aligned}$$

So $a_{i+1} = \#\mathcal{I}_{i+1}(H)-1 = indeg_H(i+1)-1 = d_{i+1}$. Thus we have a map from $\big(H;(T_2^-,\ldots,T_n^-)\big)$ to an integer $\mathbf{a}$-flow in H where $\mathbf{a} = (0,d_2,\ldots,d_n,-\sum_{i=2}^n d_i)$.

Next we show this map is bijective. Given such an integer flow on H, we read

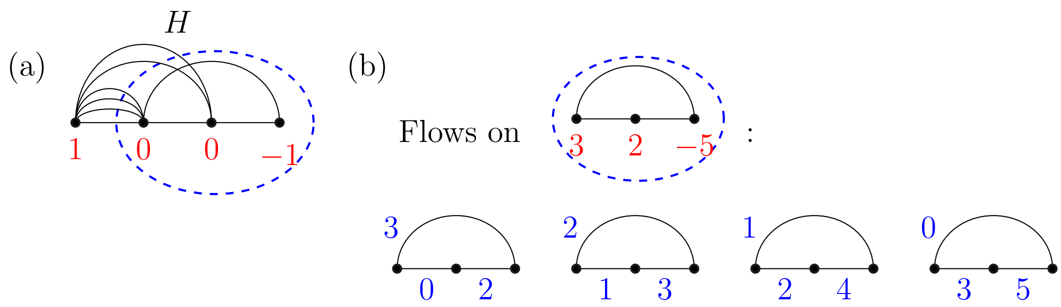


Figure 3-11: Example of Theorem 3.6.2 to find vol$\mathcal{F}_H(1,0,0,-1) = k_H(0,3,2,-5) =$ 4: (a) Graph H with negative edges, (b) the four flows on H with netflow $(0, d_2, d_3, d_4) = (0,3,2,-5)$ where $d_i = indeg_H(i) - 1$.

off the flows on the edges of $\mathcal{O}_i(H)$ for $i = 2, \ldots, n$ in clockwise order and obtain a weak composition of $\sum_{e \in \mathcal{O}_i(H)} b(e) =: N_i$ with $\#\mathcal{O}_i(H)$ parts. Next, we encode each of these compositions as noncrossing trees T_i^-. We know that $\mathcal{O}_{i+1}(H) = \mathcal{O}_{i+1}(H_i)$ and it is not hard to show by induction on i that $N_{i+1} = \#\mathcal{I}_{i+1}(H_i) - 1$ where $H_i = (\cdots (H^{(2)}_{T_2^-})^{(3)}_{T_3^-} \cdots)^{(i)}_{T_i^-}$.

This shows the map described above is the correspondence we desired. $\square$

Example 3.6.6 (Application of Theorem 3.6.2)**.** *The flow polytope $\mathcal{F}_H(1,0,0,-1)$ for the negative graph H in Figure 3-11 (a) has normalized volume 4. This is the number of flows on H with netflow $(0, d_2, d_3, d_4) = (0,3,2,-5)$ where $d_i = indeg_H(i) - 1$,* i.e. *$k_H(0,3,2,-5) = 4$. The four flows are in Figure 3-11 (b).*

We now look at computing the normalized volume of $\mathcal{F}_G(\mathbf{a})$ where G is a signed graph and $\mathbf{a} = (2,0,\ldots,0)$.

3.6.2 A correspondence between dynamic integer flows and simplices in a triangulation of $\mathcal{F}_G(2,0,\ldots,0)$, where G is a signed graph

Let G be a signed graph on the vertex set $[n+1]$ and $\mathbf{a} = (2,0,\ldots,0)$. In order to subdivide the polytope $\mathcal{F}_G(\mathbf{a})$, we follow the same first steps as in the previous case. Mainly:

We apply Lemma 3.5.7 successively to vertices $2, 3, \ldots, n+1$. At the end we obtain:

$$\mathcal{F}_G(2,0,\ldots,0) = \bigcup_{T_{n+1}} \cdots \bigcup_{T_3} \bigcup_{T_2} \mathcal{F}_{(\cdots(G^{(2)}_{T_2})^{(3)}_{T_3}\cdots)^{(n+1)}_{T_{n+1}}}(2). \tag{3.6.7}$$

See Figure 3-12 (b) for an example of a subdivision of an instance of $\mathcal{F}_G(2,0,\ldots,0)$. In this case, $G_{n+1} := (\cdots (G^{(2)}_{T_2})^{(3)}_{T_3} \cdots)^{(n+1)}_{T_{n+1}}$ is a graph consisting of one vertex with $\#E(G) - n$ positive loops. Thus, $\mathcal{F}_{G_{n+1}}(2)$ is an $(\#E(G) - n - 1)$-dimensional simplex with normalized unit volume. Therefore, vol$(\mathcal{F}_G(2,0,\ldots,0))$ is the number of choices of signed noncrossing bipartite trees $T_2, T_3, \ldots, T_{n+1}$ where T_{i+1} encodes a

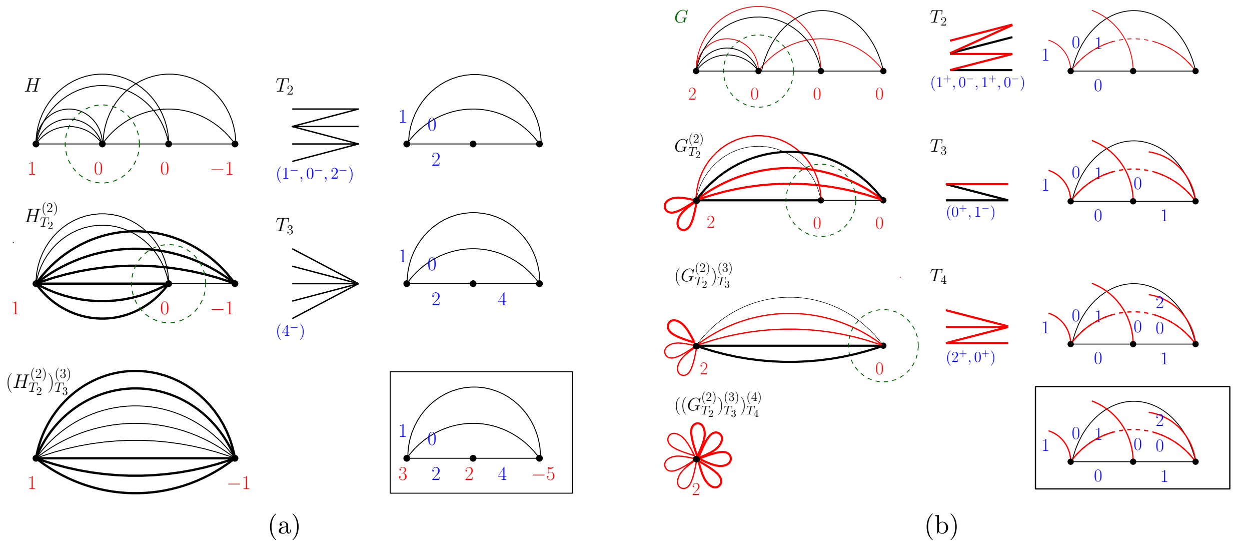


Figure 3-12: Example of the subdivision to find the volume of (a) $\mathcal{F}_H(1,0,0,-1)$ for H with only negative edges and of (b) $\mathcal{F}_G(2,0,0,0)$ for signed G. The subdivision is encoded by noncrossing trees T_{i+1} that are equivalent to compositions $(b_1,\ldots,b_r)$ of $\#\mathcal{I}_{i+1}(H_i)-1$ ($\#\mathcal{I}_{i+1}(G_i)-1$) with $\#\mathcal{O}_{i+1}(H_i)$ ($\#\mathcal{O}_{i+1}(G_i)$) parts. These trees or compositions are recorded by the integer (dynamic) flow on $H\backslash\{1\}$ ($G\backslash\{1\}$) in the box with netflow $(d_2,d_3,-d_2-d_3)=(3,2,-5)$ where $d_i=indeg_i(H)$ ($(d_2,d_3,d_4)=(2,1,1)$ where $d_i=indeg_i(G)$).

weak composition of $\#\mathcal{I}_{i+1}(G_i)-1$ with $\#\mathcal{O}_{i+1}(G_i)$ parts. However, instead of a correspondence between $(G;(T_2,T_3,\ldots,T_{n+1}))$ and the usual integer flows on G, there is a correspondence with a special kind of integer flow on G that we call **dynamic integer flow**.

Next, we motivate the need of these new integer flows. Let $G_i := (\cdots(G^{(2)}_{T_2})\cdots)^{(i)}_{T_i}$. The tree T_{i+1} is given by a signed weak composition $(b^{(i+1)}_e)_{e\in\mathcal{O}_{i+1}(G_i)}$ of $\#\mathcal{I}_{i+1}(G_i)-1$ into $\#\mathcal{O}_{i+1}(G_i)$ parts. And again, by Remark 3.5.3 (i), we can encode the composition by assigning a flow $b(e)=b^{(i+1)}_e$ for $e\in\mathcal{O}_{i+1}(G_i)$ to G_i. However, contrary to Equation (3.6.3), iterating Proposition 3.5.4 we get

$$\mathcal{O}_{i+1}(G_i)\supseteq\mathcal{O}_{i+1}(G), \tag{3.6.8}$$

and in general these multisets are not equal (*e.g.* in Figure 3-7 (b), $\mathcal{O}_4(G)=\varnothing$ but $\mathcal{O}_4(G^{(2)}_T)=\{(1,4,+),(1,4,+)\}$). Thus, we cannot encode the compositions as flows on a *fixed* graph G but rather on a graph G *and* additional positive edges added according to the flows assigned to previous positive edges. This is what we mean by dynamic flow. The next definition makes this precise.

Definition 3.6.9 (Dynamic integer flow)**.** *Given a signed graph G and an edge $e=(i,j,+)$ of G, we will regard $e=(i,j,+)$ as two positive* half-edges $(i,\varnothing,+)$ *and $(\varnothing,j,+)$ that still have "memory" of being together (see Figure 3-13 (a)). Thus, we assign* integer *flows $b_\ell(e)$ and $b_r(e)$ to the left and right halves of the positive edge,*

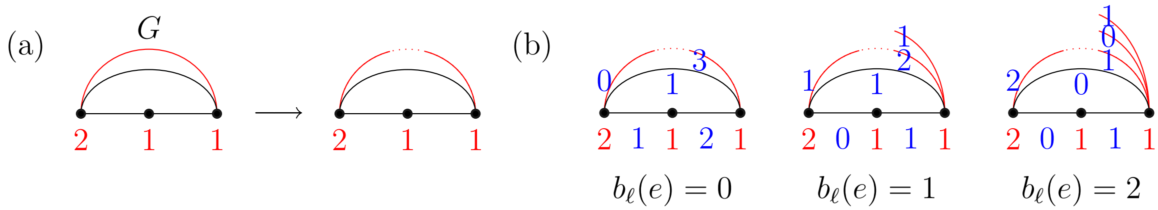


Figure 3-13: Example of dynamic flow: (a) signed graph G with positive edge e split into two half-edges, (b) three of the 17 dynamic integer flows where $b_\ell(e) = 0, 1$, and 2 so that zero, one and two right positive half-edges are added.

starting at the left half-edge. Once we assign $b_\ell(e)$ units of flow, we add $b_\ell(e)$ new right positive half-edges e' incident to j *that can also be assigned integer flows $b_r(e')$. When we assign an integer flow to a right positive half-edges no edges are added.*

An analogue of Equation (3.2.2) *still holds:*

$$\sum_{e\in E(G),inc(e,v)=-} b(e) + a_v = \sum_{e\in E(G),inc(e,v)=+} (b_\ell(e) + b_r(e)) + \sum_{e',new\ right\ (+)\ half\ edges} b_r(e'), \tag{3.6.10}$$

where a_v is the outflow at vertex v and $inc(e, v) = -$ if $e = (g, v, -)$, $g < v$, and $inc(e, v) = +$ if $e = (g, v, +)$, $g < v$, or $e = (v, j, \epsilon)$, $v < j$, and $\epsilon \in \{+, -\}$. We call these integer **a***-flows* **dynamic**.

Definition 3.6.11 (Dynamic Kostant partition function)**.** *Given a signed graph G on the vertex set $[n + 1]$ and* **a** *a vector in $\mathbb{Z}^{n+1}$, the* **dynamic Kostant partition function** $k_G^{dyn}(\mathbf{a})$ *is the number of integer dynamic* **a***-flows in G.*

Example 3.6.12. *For the signed graph G in Figure 3-13 (a) with only one positive edge $e = (1, 3, +)$, we give three of its 20 integer dynamic flows with netflow $(2, 1, 1)$ where we add $b_\ell(e) = 0, 1$ and 2 right half edges respectively.*

Next, we give the generating series of the dynamic Kostant partition function $k_G^{\mathrm{dyn}}(\mathbf{a})$.

Proposition 3.6.13. *The generating series of the dynamic Kostant partition function is*

$$\sum_{\mathbf{a}\in\mathbb{Z}^{n+1}} k_G^{dyn}(\mathbf{a})\mathbf{x}^{\mathbf{a}} = \prod_{(i,j,-)\in E(G)} (1 - x_i x_j^{-1})^{-1} \prod_{(i,j,+)\in E(G)} (1 - x_i - x_j)^{-1}, \tag{3.6.14}$$

where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_{n+1}^{a_{n+1}}$.

Proof. By Definition 3.6.9 of the integer dynamic flow, if the left half-edge of a positive edge $e = (i, j, +)$ has flow $k \in \mathbb{N}$ then we add k new right half-edges incident to j besides the existing half-edge. In this case the contribution to the generating series of the dynamic integer flows is $x_i^k (1 - x_j)^{-k-1}$. Thus the total contribution to the generating series from $e=(i,j,+)$ in G is

$$\sum_{k\geq 0} x_i^k(1-x_j)^{-k-1} = (1-x_j)^{-1}(1-x_i(1-x_j)^{-1})^{-1}$$
$$= (1-x_i-x_j)^{-1}.$$

In addition, the contributions of negative edges $e=(i,j,-)$ is $(1-x_ix_j^{-1})^{-1}$. Taking the product of these contributions for each of the edges of G gives the stated generating series $\sum_{\mathbf{a}\in\mathbb{Z}^{n+1}} k_G^{\mathrm{dyn}}(\mathbf{a})\mathbf{x}^{\mathbf{a}}$. □

Remark 3.6.15. *By assigning the possible integer flows to left half-edges and adding the appropriate number of right half-edges it is possible to write the dynamic Kostant partition function $k_G^{dyn}(\mathbf{a})$ as a finite sum of Kostant partition functions. For example for the graph G in Figure 3-13 $k_G^{dyn}(2,1,1) = k_{G_1}(2,1,1)+k_{G_2}(1,1,1)+k_{G_3}(0,1,1) = 6+8+6$ where for $i=1,2,3$, G_i is obtained from G by setting the flow on the left half-edge $(1,\varnothing,+)$ to be $i-1$ and adding $i-1$ right half edges $(\varnothing,4,+)$. This observation implies that $k_G^{dyn}(\mathbf{a})$ is a sum of piecewise quasipolynomial functions. It would be of interest to study the chamber structure of k_G^{dyn}.*

We are now ready to state and prove the main product of the broad technique we developed.

Theorem 3.6.16. *Given a loopless connected signed graph G on the vertex set $[n+1]$, let $d_i = indeg_G(i)$ for $i\in\{2,\ldots,n\}$. The normalized volume $\mathrm{vol}(\mathcal{F}_G)$ of the flow polytope associated to graph G is*

$$\mathrm{vol}(\mathcal{F}_G(2,0,\ldots,0)) = k_G^{dyn.}(0,d_2,\ldots,d_n,d_{n+1}).$$

Proof. Recall from the argument right before Definition 3.6.9, that $\mathrm{vol}(\mathcal{F}_G(2,0,\ldots,0))$ is the number of tuples $(T_2,\ldots,T_{n+1})$ of bipartite trees each tree T_{i+1} encoding a composition of $\#\mathcal{I}_{i+1}(G_i)-1$ with $\#\mathcal{O}_{i+1}(G_i)$ parts (where G_i is the graph $(\cdots(G_{T_2}^{(2)})_{T_3}^{(3)}\cdots)_{T_i}^{(i)}$). By construction, we encode each of these compositions using dynamic integer flows on G. The netflow on vertex $i+1$ will be:

$$\begin{aligned} a_{i+1} &= \#\mathcal{I}_{i+1}^{-}(G_i) - 1 - \#\{\text{new neg. edges incident to } i+1\} \\ &= \big(\#\mathcal{I}_{i+1}^{-}(G) + \#\{\text{new edges } (\cdot,i+1,-)\} - 1\big) - \#\{\text{new edges } (\cdot,i+1,-)\} \\ &= \#\mathcal{I}_{i+1}^{-}(G) - 1 = indeg_G^{-}(i+1). \end{aligned}$$

Thus we have a map from $(G;(T_2,\ldots,T_{n+1}))$ to an integer dynamic $\mathbf{a}$-flow in G where $\mathbf{a}=(0,d_2,d_3,\ldots,d_{n+1})$ where $d_i = indeg_G(i)$.

Next we show this map is bijective. Given such an integer dynamic flow in G, we read off the flows on the edges of $\mathcal{O}_i(G)$ and the new right positive half-edges e' incident to i for $i=2,\ldots,n+1$ in clockwise order. We obtain a weak composition of $\sum_{e\in\mathcal{O}_i(G)} b(e) + \sum_{\text{new right (+) half edges } e'} b_r(e') := N_i$ into $\#\mathcal{O}_i(G) + \#\{\text{new right (+) half edges}\}$ parts. Next, we encode these weak compositions as signed noncrossing trees. We know that $\#\mathcal{O}_i(G) + \#\{\text{new right (+) half edges}\} =$

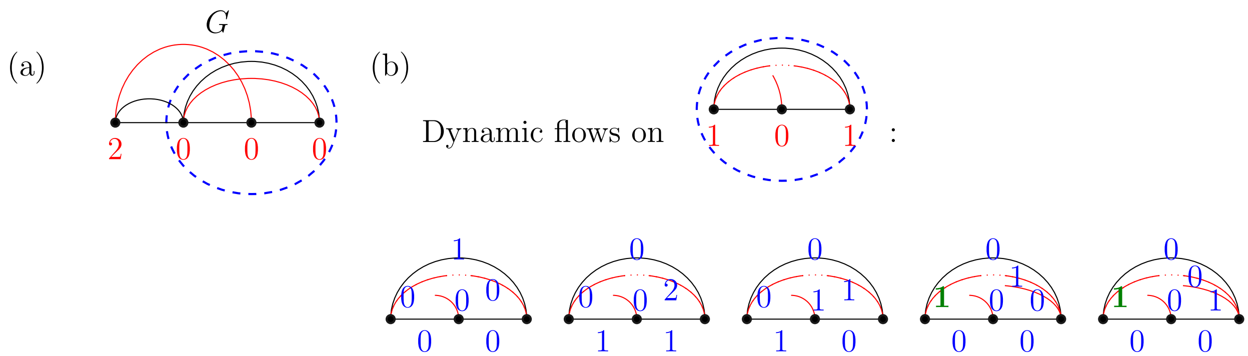


Figure 3-14: Example of Theorem 3.6.16 to find vol$\mathcal{F}_G(2,0,0,0) = k_G^{\mathrm{dyn}}(0,1,0,1) = 5$: (a) Signed graph G, (b) the five dynamic flows on G with netflow $(0, d_2, d_3, d_4) = (0,1,0,1)$ where $d_i = indeg_G(i) - 1$ (the last two flows have an additional right positive half-edge).

$\#\mathcal{O}_{i+1}(G_i)$ and it is not hard to show by induction that $N_{i+1} = \#\mathcal{I}_{i+1}(G_i) - 1$ where $G_i = (\cdots(G^{(2)}_{T_2})^{(3)}_{T_3}\cdots)^{(i)}_{T_i}$.

This shows the map is the correspondence we desired. $\square$

Example 3.6.17 (Application of Theorem 3.6.16)**.** *The flow polytope $\mathcal{F}_G(2,0,0,0)$ for the signed graph G in Figure 3-14 (a) has normalized volume* 5*. This is the number of dynamic integer flows on G with netflow $(0,d_2,d_3,d_4) = (0,1,0,1)$ where $d_i = indeg_G(i) - 1$. The five dynamic integer flows are in Figure 3-14 (b).*

3.7 The volumes of the (signed) Chan-Robbins-Yuen polytopes

When $H = K_{n+1}$, the complete graph on $n+1$ vertices, $\mathcal{F}_{K_{n+1}}(1,0,\ldots,0,-1)$ is also known as the Chan-Robbins-Yuen polytope $CRYA_n$ [14, 15]. This polytope is a face of the Birkhoff polytope of all $n \times n$ doubly stochastic matrices. Zeilberger computed in [64] the volume of this polytope using the *Morris identity* [49, Thm. 4.13]. This polytope has drawn much attention with its combinatorial volume $\prod_{i=1}^{n-2} Cat(i)$, and the lack of a combinatorial proof of this formula. In this section we study $CRYA_n$ and its type C_{n+1} and D_{n+1} generalizations.

3.7.1 Chan-Robbins-Yuen polytope of type A_n

We reproduce an equivalent proof of Zeilberger's result using Theorem 3.6.2. First we mention the version of the identity used in [64] and a special value of it which gives a product of consecutive Catalan numbers. Then we use Theorem 3.6.2 to show that the volume of the polytope reduces to this value of the identity.

Lemma 3.7.1 (Morris Identity [64][1]). *For a positive integers m, a, and b, and positive half integers c, let*

$$H(a,b,c;x_1,x_2,\dots,x_m) := \prod_{i=1}^{m} x_i^{-a}(1-x_i)^{-b} \prod_{1\le i<j\le m}(x_j-x_i)^{-2c},$$

and let $M_m(a,b,c) = CT_{x_m}\cdots CT_{x_1}H(a,b,c;x_1,x_2,\dots,x_m)$, where CT_{x_i} mean the constant term in the expansion of the variable x_i. Then

$$M_m(a,b,c) = \frac{1}{m!}\prod_{j=1}^{m-1}\frac{\Gamma(a+b+(m-1+j)c)\Gamma(c)}{\Gamma(a+jc+1)\Gamma(b+jc)\Gamma(c+jc)}, \tag{3.7.2}$$

where $\Gamma(\cdot)$ is a gamma function ($\Gamma(j)=(j-1)!$ when $j\in\mathbb{N}$).

Next we give a special value of this identity.

Corollary ([64]). *For the constant term $M_m(a,b,c)$ defined above, we have*

$$M_m(2,0,1/2) = M_m(1,1,1/2) = \prod_{k=1}^{m} Cat(k), \tag{3.7.3}$$

where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.

Proof. By Equation (3.7.2) if $c=1/2$ and either if $a=2, b=0$ or $a=b=1$ then

$$M_m(2,0,1/2) = M_m(1,1,1/2) = \frac{1}{m!}\prod_{j=1}^{m-1}\frac{\Gamma((m+3+j)/2)\Gamma(1/2)}{\Gamma(j/2+1)\Gamma(j/2+1/2)\Gamma(j/2+1/2)}.$$

Then M_m/M_{m-1} is

$$M_m/M_{m-1} = \frac{1}{m}\frac{\Gamma(m+1/2)\Gamma(m+1)}{\Gamma(m/2+1)}\frac{\Gamma(1/2)}{\Gamma(m/2+1/2)\Gamma(m/2)\Gamma(m/2+3/2)},$$

using the duplication formula of gamma functions, $\Gamma(z)\Gamma(z+1/2) = \Gamma(2z)\Gamma(1/2)/2^{2z-1}$, on the product $\Gamma(m/2)\Gamma(m/2+1/2)$ gives

$$M_m/M_{m-1} = \frac{1}{m}\frac{\Gamma(m+1/2)\Gamma(m+1)}{\Gamma(m/2+1)}\frac{2^{m-1}}{\Gamma(m)\Gamma(n/2+3/2)},$$

using the duplication formula on $\Gamma(m+1/2)\Gamma(m+1)$ and on $\Gamma(m/2+3/2)$ and $\Gamma(m/2+1)$ gives

$$M_m/M_{m-1} = \frac{1}{m}\frac{\Gamma(2m+1)}{\Gamma(m+2)\Gamma(m)} = \frac{1}{m+1}\binom{2m}{m} = Cat(m).$$

[1] In the original Morris identity the powers a,b,c are negative. In [64] Zeilberger *shadowed* the expression to extend it to the case when a,b,c are positive.

And so $M_m(1,1,1/2) = Cat(0)Cat(1)\cdots Cat(m)$. $\square$

Corollary ([64])**.** *For $n \geq 1$, let K_{n+1} be the complete graph on $n+1$ vertices. Then the volume of the flow polytope $\mathcal{F}_{K_{n+1}}(1,0,\ldots,0,-1)$ is*

$$\mathrm{vol}(\mathcal{F}_{K_{n+1}}(1,0,\ldots,0,-1)) = \prod_{k=0}^{n-2} Cat(k),$$

where $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.

Proof. If $H = K_{n+1}$, by Theorem 3.6.2 we have that

$$\begin{aligned}\mathrm{vol}(\mathcal{F}_{K_{n+1}}(1,0,\ldots,0,-1)) &= k_{K_{n+1}}(0,0,1,2,\ldots,n-2,-(n-2)(n-1)/2)\\ &= k_{K_{n-1}}(1,2,\ldots,n-2,-(n-2)(n-1)/2).\end{aligned}$$

Where we reduced from K_{n+1} to K_{n-1} since the outflow on the first two vertices of K_{n+1} is zero. Then from the generating series of the Kostant partition function (3.1.4):

$$k_{K_{n-1}}(1,2,\ldots,n-2,-(n-2)(n-1)/2) = [x_1^1 x_2^2 \cdots x_{n-2}^{n-2}] \prod_{1\leq i<j\leq n-1} (1 - x_i x_j^{-1})^{-1}\Bigg|_{x_{n-1}=1} \tag{3.7.4}$$

where we have set $x_{n-1} = 1$ since its power is determined by the power of the other variables. Since $1/(1 - x_i x_j^{-1}) = x_j/(x_j - x_i)$ then

$$\begin{aligned}\mathrm{vol}\mathcal{F}_{K_{n+1}}(1,0,\ldots,0,-1) &= [x_1^1 x_2^2 \cdots x_{n-2}^{n-2}] x_1^0 x_2^1 x_3^2 \cdots x_{n-2}^{n-3} \prod_{i=1}^{n-2} (1-x_i)^{-1} \prod_{1\leq i<j\leq n-2} (x_j - x_i)^{-1}\\ &= [x_1 x_2 \cdots x_{n-2}] \prod_{i=1}^{n-2} (1-x_i)^{-1} \prod_{1\leq i<j\leq n-2} (x_j - x_i)^{-1}.\end{aligned} \tag{3.7.5}$$

Since $[x]f(x) = CT_x \frac{1}{x} f(x)$ we get

$$\mathrm{vol}\mathcal{F}_{K_{n+1}}(1,0,\ldots,0,-1) = CT_{x_{n-2}} CT_{x_{n-3}} \cdots CT_{x_1} \prod_{i=1}^{n-2} x_i^{-1}(1-x_i)^{-1} \prod_{1\leq i<j\leq n-2} (x_j - x_i)^{-1}. \tag{3.7.6}$$

Note that the right-hand-side above is $M_{n-2}(1,1,1/2)$. Then by 3.7.3 the result follows. $\square$

Remark 3.7.7. *(i) Note that in this case of $H = K_{n+1}$, the multiset $\{\{\alpha_i\}\}$ of roots corresponding to the edges of K_{n+1} are all the positive type A_n roots, and the netflow vector $(1,0,\ldots,0,-1)$ is the highest root in type A_n. The volumes of*

$\mathcal{F}_{K_{n+1}}(\mathbf{a})$ for generic positive roots in A_n do not appear to have nice product formulas. (ii) There is no combinatorial proof for the formula of the normalized volume of $\mathcal{F}_{K_{n+1}}(1,0,\ldots,0,-1)$. Another proof of this formula using residues was given by Baldoni and Vergne [2, 3]. ■

3.7.2 Volumes of Chan-Robbins-Yuen polytopes of type C_n and type D_n.

Recall from Examples 3.2.7 (iv) that K_n^D is the complete signed graph on n vertices (all edges of the form $(i,j,\pm)$ for $1 \le i < j \le n$ corresponding to all the positive roots in type D_n), and $CRYD_n = \mathcal{F}_{K_n^D}(2,0,\ldots,0)$ is an analogue of the Chan-Robbins-Yuen polytope. Next, using Theorem 3.6.16 and Proposition 3.6.13 we express the volume of this polytope as the constant of a certain rational function. This is an analogue of (3.7.6).

Proposition 3.7.8. *Let $CRYD_n$ be the flow polytope $\mathcal{F}_{K_n^D}(2,0,\ldots,0)$ where K_n^D is the complete signed graph with n vertices (all edges of the form $(i,j,\pm)$, $1 \le i < j \le n$). Then*

$$\operatorname{vol}(CRYD_n) = CT_{x_{n-2}} \cdots CT_{x_1} \prod_{i=1}^{n-2} x_i^{-1}(1-x_i)^{-2} \prod_{1\le i<j\le n-2} (x_j - x_i)^{-1}(1-x_j-x_i)^{-1} \tag{3.7.9}$$

Proof. By Theorem 3.6.16 if $G = K_n^D$ we have that

$$\operatorname{vol}(\mathcal{F}_{K_n^D}(2,0,\ldots,0)) = k_{K_n^D}^{dyn.}(0,0,1,2,\ldots,n-2),$$

and by Proposition 3.6.13 in terms of the generating series of $K_{K_n^D}^{dyn.}$ this volume is given by

$$\operatorname{vol}(\mathcal{F}_{K_n^D}(2,0,\ldots,0)) = [x_3^1 x_4^2 \cdots x_n^{n-2}] \prod_{1\le i<j\le n} (1 - x_i x_j^{-1})^{-1}(1-x_i-x_j)^{-1}.$$

Since the outflow on the first two vertices is zero, we can plug in $x_1 = x_2 = 0$ above. Then relabellings the variables $x_m \mapsto x_{m-2}$ for clarity gives:

$$\operatorname{vol}(\mathcal{F}_{K_n^D}(2,0,\ldots,0)) = [x_1^1 x_2^2 \cdots x_{n-2}^{n-2}] \prod_{1\le i<j\le n-2} (1-x_i x_j^{-1})^{-1}(1-x_i-x_j)^{-1} \prod_{1\le i\le n-2} (1-x_i)^{-2}.$$

In addition, just as we did with $CRYA_{n+1}$ in (3.7.4)-(3.7.6) the above equation is equivalent to the desired expression:

$$\operatorname{vol}(\mathcal{F}_{K_n^D}(2,0,\ldots,0)) = CT_{x_{n-2}} CT_{x_{n-3}} \cdots CT_{x_1} \prod_{i=1}^{n-2} x_i^{-1}(1-x_i)^{-2} \prod_{1\le i<j\le n-2} (x_j-x_i)^{-1}(1-x_j-x_i)^{-1}.$$

□

We get the following for $v_n = \mathrm{vol}(CRYD_n)$ either through dynamic flows, or using (3.7.9), or direct volume computation (using the `Maple` package `convex` [20]):

n	2	3	4	5	6	7
v_n	1	2	32	5120	9175040	197300060160
$\frac{v_n}{v_{n-1}}$		2	$2^3 \cdot \mathbf{2}$	$2^5 \cdot \mathbf{10}$	$2^7 \cdot \mathbf{14}$	$2^9 \cdot \mathbf{42}$

which suggests the following conjecture:

Conjecture 3.7.10. *Let $CRYD_n$ be the flow polytope $\mathcal{F}_{K_n^D}(2,0,\ldots,0)$ where K_n^D is the complete signed graph with n vertices (all edges of the form $(i,j,\pm)$, $1 \le i < j \le n$). Then the normalized volume of $CRYD_n$ is*

$$\mathrm{vol}(CRYD_n) = 2^{(n-2)^2} \prod_{k=0}^{n-2} Cat(k).$$

Remark 3.7.11. *The right-hand-side of* (3.7.9) *looks like the right-hand-side of the following generalization of the Morris identity (Lemma 3.7.1):*

$$CT_{x_m} \cdots CT_{x_1} \prod_{i=1}^{m} x_i^{-a}(1-x_i)^{-b} \prod_{1 \le i < j \le m} (x_j - x_i)^{-2c}(1 - x_i - x_j)^{-2d}.$$

for positive integers $m, a,$ and b and positive half integers c and d. We were unable to find a formula in terms of m, a, b, c for such a generalization. ■

Finally, we briefly consider the flow polytopes: (i) $\mathcal{F}_{K_n^C}(2,0,\ldots,0)$ where K_n^C is the complete signed graph with loops $(i,i,+)$ corresponding to the type C positive roots $2e_i$, (ii) $\mathcal{F}_{K_n^B}(2,0,\ldots,0)$ where K_n^B is the complete signed graph with loops $(i,i,+)$ corresponding to the type B positive root e_i, (iii) $\mathcal{F}_{K_n^C}(1,1,0,\ldots,0)$, and (iv) $\mathcal{F}_{K_n^B}(1,1,0,\ldots,0)$. These polytopes also appear to have interesting volumes:

Conjecture 3.7.12. *Let K_n^D, K_n^B, K_n^C be the signed complete graphs whose edges correspond to the positive roots in type D_n, B_n and C_n as defined above then*

$$\mathrm{vol}\mathcal{F}_{K_n^C}(2,0,\ldots,0) = 2^{n-2} \cdot \mathrm{vol}(CRYD_n) \tag{3.7.13}$$

and except for $n=2$ (where $\mathrm{vol}\mathcal{F}_{K_n^D}(2,0) = \mathrm{vol}\mathcal{F}_{K_n^D}(1,1)$),

$$\mathrm{vol}\mathcal{F}_{K_n^{\{B,C,D\}}}(2,0,\ldots,0) = 2 \cdot \mathrm{vol}\mathcal{F}_{K_n^{\{B,C,D\}}}(1,1,0,\ldots,0). \tag{3.7.14}$$

Chapter 4

Counting matrices over finite fields with restricted support

This chapter is based on [33], joint work with A.J. Klein, J.B. Lewis and [39].

4.1 Introduction

We study certain q-analogues of permutations with restricted positions, or equivalently of placements of non-attacking rooks. The q-analogue of permutations we work with is invertible $n \times n$ matrices over the finite field $\mathbf{F}_q$ with q elements, as in [60, Ch. 1]. Then the analogue of permutations with restricted positions is invertible matrices over $\mathbf{F}_q$ with some entries required to be zero.

Specifically, given a subset S of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, let $\mathrm{mat}_q(n, S, r)$ be the number of $n \times n$ matrices over $\mathbf{F}_q$ with rank r, none of whose nonzero entries lie in S. This is clearly an analogue (in the plain English meaning) of the problem of counting permutations whose permutation matrix has no 1 in the position of any entry of S, but actually much more can be said. In [39, Prop. 5.1] it was shown that $\mathrm{mat}_q(n, S, r)/(q-1)^r$ is in fact an *enumerative* q-analogue of permutations with restricted positions; that is, its value, modulo $(q-1)$, counts the placements of r non-attacking rooks on the complement of S.

The function $\mathrm{mat}_q(n, S, r)$ can exhibit a variety of different behaviors, as seen in the following three examples.

Examples 4.1.1. *1. When $S = \varnothing$, $\mathrm{mat}_q(n, \varnothing, n)$ is the number of $n \times n$ invertible matrices over $\mathbf{F}_q$, which is*

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) = q^{\binom{n}{2}}(q-1)^n \prod_{i=1}^{n} (1 + q + \cdots + q^{i-1}).$$

The last term in the product is a polynomial with positive coefficients, and in fact is the generating series for permutations in $\mathfrak{S}_n$ by number of inversions.

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & a_{17} \\ a_{21} & 0 & a_{23} & 0 & 0 & a_{26} & 0 \\ a_{31} & 0 & 0 & a_{34} & a_{35} & 0 & 0 \\ 0 & a_{42} & a_{43} & 0 & a_{45} & 0 & 0 \\ 0 & a_{52} & 0 & a_{54} & 0 & a_{56} & 0 \\ 0 & 0 & a_{63} & a_{64} & 0 & 0 & a_{67} \\ 0 & 0 & 0 & 0 & a_{75} & a_{76} & a_{77} \end{bmatrix}$$

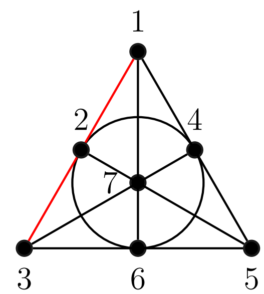


Figure 4-1: A representative matrix counted in $\mathrm{mat}_q(7, F, 7)$ where F is the complement of the point-line incidence matrix of the Fano plane, shown at right. Stembridge [61] showed this to be the smallest example of the form $\mathrm{mat}_q(n, S, n)$ that is not a polynomial in q.

2. *When $n = 3$ and S is the diagonal $\{(1,1),(2,2),(3,3)\}$ we have*

$$\mathrm{mat}_q(3, \{(1,1),(2,2),(3,3)\}, 3) = (q-1)^3(q^3 + 2q^2 - q).$$

   The number of invertible $n \times n$ matrices for general n over $\mathbf{F}_q$ with zero diagonal was computed in [39, Prop. 2.2]; as in this example, it is of the form $(q-1)^n f(q)$ for a polynomial f with both positive and negative coefficients.

3. *When $n = 7$, Stembridge [61] found a set F with 28 elements (shown in Figure 4-1) such that $\mathrm{mat}_q(7, F, 7)$ is given by a quasi-polynomial in q, that is, by two distinct polynomials depending on whether q is even or odd. The set F is the complement of the incidence matrix of the* Fano plane.

From the examples above we see that $\mathrm{mat}_q(n, S, r)$ is not necessarily a polynomial in q, and if it is a polynomial in q it might or might not be of the form $(q-1)^r f(q)$ where $f(q)$ is a polynomial with nonnegative integer coefficients. Then a natural question to ask is for which families of sets S is $\mathrm{mat}_q(n, S, r)/(q-1)^r$ not a polynomial in q, or a polynomial in q, or a polynomial in q with nonnegative integer coefficients.

Question 4.1.2. *What families of sets S are there such that $\mathrm{mat}_q(n, S, r)/(q-1)^r$ is (i) not a polynomial in q, (ii) a polynomial in q, or (iii) a polynomial in q with nonnegative integer coefficients?*

In the remainder of this introduction, we give a summary of our progress towards answering this question.

Outline and summary of results

In Section 4.2, we give the definitions and notation that will be used throughout the chapter including the definition and some properties of q-rook numbers.

In Section 4.3, we look at question of general conditions on r and S under which the function $\mathrm{mat}_q(n, S, r)$ is always a polynomial in q. We show that if $r = 1$ then

$\mathrm{mat}_q(n, S, 1)$ is a polynomial in q for any set S, though not necessarily with nonnegative coefficients. (It is an open question whether there is a set S such that $\mathrm{mat}_q(n, S, 2)$ is non-polynomial in q.)

In the rest of the chapter, we discuss special families of sets S such that $\mathrm{mat}_q(n, S, r)/(q-1)^r$ is a polynomial in q with nonnegative integer coefficients. Haglund [27] showed that if the set S is a *s*traight shape then $\mathrm{mat}_q(n, \overline{S}, r)/(q-1)^r$ is a polynomial with nonnegative integer coefficients.

Our first main result, proved in Section 4.4, is to extend this to complements of *s*kew shapes.

Corollary 4.4.6. *For any skew shape $S_{\lambda/\mu}$,*

$$\mathrm{mat}_q(n, \overline{S_{\lambda/\mu}}, r) = (q-1)^r f(q),$$

where $f(q)$ is a polynomial with nonnegative integer coefficients.

In fact, we show that this is true for an even larger class of shapes than skew shapes, namely those that have what we call the *North-East Property.* Also, because $\mathrm{mat}_q(n, S, r)$ is invariant under permuting rows and columns we have that $\mathrm{mat}_q(n, \overline{S}, r)/(q-1)^r$ is a polynomial with nonnegative integer coefficients for any set S that is a straight or skew shape after permuting rows and columns.

In Sections 4.5 and 4.6 we study another natural family of diagrams: the collection of **Rothe diagrams** of permutations, which appear in the study of Schubert calculus. The Rothe diagram R_w of a permutation w is a subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ whose cardinality is equal to the number of inversions of w; it is given by

$$R_w = \{(i, j) \mid 1 \le i, j \le n,\ w(i) > j,\ w^{-1}(j) > i\}.$$

See Figure 4-5 for some examples of Rothe diagrams. Lascoux and Schützenberger showed in [37] that the Rothe diagram R_w of a permutation w is a straight shape up to permutation of rows and columns if and only if w, written as a word $w_1 w_2 \cdots w_n$, avoids the permutation pattern 2143 (i.e., there is no sequence $i < j < k < l$ such that $w_j < w_i < w_l < w_k$). Our second main result is to give an analogous criterion for the case of complements of skew shapes.

Theorem 4.5.4. *The Rothe diagram R_w of a permutation w is, up to permuting its rows and columns, the complement of a skew shape if and only if w can be decomposed as $w = a_1 a_2 \ldots a_k b_1 b_2 \ldots b_{n-k}$ where $a_i < b_j$ for all i and j, and both $a_1 a_2 \ldots a_k$ and $b_1 b_2 \ldots b_{n-k}$ are* 2143*-avoiding.*

We also show that this condition is equivalent to the statement that w avoids the nine patterns 24153, 25143, 31524, 31542, 32514, 32541, 42153, 52143, and 214365, and we express the generating series for these permutations in terms of the generating series for vexillary permutations.

By Corollary 4.4.6, if w satisfies the condition above then $\mathrm{mat}_q(n, R_w, r)/(q-1)^r$ is a polynomial with nonnegative integer coefficients. Surprisingly, computer calculations for $n \le 7$ [34] suggest that in the top rank case $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is a polynomial with nonnegative integer coefficients for *all* permutations w (see Conjecture 4.5.1). Moreover, computer calculations also suggest that when w avoids the permutation patterns $1324, 24153, 31524$, and 426153 we have that $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is (up to a power of q) the *Poincaré polynomial* $P_w(q) = \sum_{u \geq w} q^{\ell(u)}$, where the sum is over all permutations u of n above w in the *strong Bruhat order* (see Conjecture 4.6.6). Interestingly, these four patterns have appeared in related contexts [22, 52, 56, 30].

Supplementary code for calculating $\mathrm{mat}_q(n, S, r)$ and other related objects and data generated by this code to test the conjectures in Section 4.6 is available at the following website:

`http://sites.google.com/site/matrixfinitefields/`

4.2 Definitions

We denote $[n] = \{1, 2, \ldots, n\}$. The **support** of a matrix A is the set of indices (i, j) of the nonzero entries $A_{i,j} \neq 0$. Fix integers n and r such that $n \geq 1$ and $n \geq r \geq 0$, and let S be a subset of $[n] \times [n]$. We define $\mathrm{mat}_q(n, S, r)$ to be the number of $n \times n$ matrices over $\mathbf{F}_q$ with rank r and support contained in $\overline{S}$, the complement of S. That is, $\mathrm{mat}_q(n, S, r)$ counts matrices A of rank r such that if $(i, j) \in S$ then $A_{ij} = 0$. We consider the problem of computing $\mathrm{mat}_q(n, S, r)$.

We now define several special types of diagrams that will be important to us in what follows. Examples of these diagrams are given in Figure 4-2. We say that $S \subseteq [n] \times [n]$ is a **straight shape** if its elements form a Young diagram of a partition. (Throughout this chapter we use *English notation* and matrix coordinates for partitions.) Thus, to every integer partition λ with at most n parts and with largest part at most n (i.e., to each sequence of integers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$) there is an associated set $S = S_\lambda = \{(i, j) \mid 1 \leq j \leq \lambda_i\}$. We denote by $|\lambda|$ the **size** $\lambda_1 + \lambda_2 + \cdots$ of the shape λ. This is also the number of entries in S_λ. Similarly, if λ and μ are partitions such that $S_\mu \subseteq S_\lambda$ then we say that the set $S_\lambda \setminus S_\mu$ is a **skew shape** and we denote it by $S_{\lambda/\mu}$. Lastly, we say that a set $S \subseteq [n] \times [n]$ has the **North-East (NE) Property** if for all $i, i', j, j' \in [n]$ such that $i' < i$ and $j < j'$ we have that if $(i, j), (i', j)$, and (i, j') are in S, then so is (i', j'). Note that for any partitions λ and μ, $S_\lambda, \overline{S_\lambda}$, and $S_{\lambda/\mu}$ have the NE Property. But $\overline{S_{\lambda/\mu}}$ in general does not have this property.

We denote by $\mathfrak{S}_n$ the group of permutations on $[n]$. We write permutations as words $w = w_1 w_2 \cdots w_n$ where w_i is the image of w at i. Let $\mathrm{inv}(w)$ denote the number of **inversions** $\#\{(i, j) \mid i < j, w_i > w_j\}$ of w. We also identify each permutation w with its **permutation matrix**, the $n \times n$ 0-1 matrix with ones in positions (i, w_i).

We think of the 1s in a permutation matrix as n non-attacking rooks on $[n] \times [n]$. In this case, the number of inversions of the permutation is exactly the number of elements in $[n] \times [n]$ that are not directly below/south (in the same column) or to the right/east (in the same row) of any placed rook. We generalize this as follows. Given a subset B of $[n] \times [n]$ (sometimes called a *b*oard) and a rook placement C of r non-attacking rooks on B, the **SE-inversion number** $\mathrm{inv}_{\mathsf{SE}}(C, B)$ is the number of elements in B *not* directly south (in the same column) or to the east (in the same row) of any placed rook. Then the rth (SE) q**-rook number** of Garsia and Remmel [21] is

$$R_r^{(\mathsf{SE})}(B,q) = \sum_C q^{\mathrm{inv}_{\mathsf{SE}}(C,B)},$$

where the sum is over all rook placements C of r non-attacking rooks on B. We define the north east inversion number $\mathrm{inv}_{\mathsf{NE}}(C,B)$ and rook polynomial $R_r^{(\mathsf{NE})}(B,q)$ analogously.

Proposition 4.2.1 ([21])**.** *Given an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, set $S_\lambda = \{(i,j) \mid 1 \leq i \leq n, 1 \leq i \leq \lambda_j\}$. The Garsia-Remmel q-rook number $R_n^{(\mathsf{SE})}(S_\lambda, q)$ is*

$$R_n^{(\mathsf{SE})}(S_\lambda, q) = \prod_{i=1}^{n} [\lambda_i - i + 1]_q, \tag{4.2.2}$$

where $[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$.

Remark 4.2.3. *We will see as a corollary of Theorems 4.4.1 and 4.4.2 that for a straight shape λ, $R_r^{(\mathsf{SE})}(S_\lambda, q) = R_r^{(\mathsf{NE})}(S_\lambda, q)$. However, this is not true for all shapes. For example, if $\lambda/\mu = 4432/31$, we have $R_3^{(\mathsf{SE})}(S_{4432/31}, q) = 1{+}6q^2{+}5q^3{+}3q^4{+}2q^5{+}q^6$ and $R_3^{(\mathsf{NE})}(S_{4432/31}, q) = 2q + 8q^2 + 7q^3 + q^4$. But for skew shapes in the case of n rooks we have do have an analogous relation.* ■

Proposition 4.2.4. *For a skew shape $S_{\lambda/\mu}$ we have*

$$R_n^{(\mathsf{SE})}(S_{\lambda/\mu}, q) = q^{\binom{n}{2} - |\mu|} \cdot R_n^{(\mathsf{NE})}(S_{\lambda/\mu}, q^{-1}).$$

Proof. For each rook placement of n rooks on $S_{\lambda/\mu}$, the number of SE-inversions is equal to the number of inversions of the associated permutation w minus the size of μ. On the other hand, the number of NE-inversions of this rook placement on $S_{\lambda/\mu}$ is $\binom{n}{2}$ minus the number of inversions of w. The result follows. □

4.3 Polynomial formula for the rank-one case $\mathrm{mat}_q(n, S, 1)$

In Figure 4-1 we showed an example by Stembridge [61] of a set $S \subseteq [7] \times [7]$ such that $\mathrm{mat}_q(7, S, 7)$ is not a polynomial in q. In this chapter, we mainly focus on studying certain families of sets S where $\mathrm{mat}_q(n, S, r)$ is a polynomial in q. But before looking at particular sets S, we show that when $r = 1$, for any set $S \subseteq [m] \times [n]$, the function $\mathrm{mat}_q(m \times n, S, 1)$ is always a polynomial in q.

Proposition 4.3.1. *For any m and n and any set $S \subseteq [m] \times [n]$, $\mathrm{mat}_q(m \times n, S, 1)$ is a polynomial in q.*

$$
\begin{array}{ccc}
\text{(i)}\quad S_{(4,3,2)} & \text{(ii)}\quad S_{(5,5,4,3,1)/(2,2,1)} & \text{(iii)}\quad S \\
\begin{bmatrix} 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}, &
\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}, &
\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 & a_{45} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} \end{bmatrix} \\
\text{(iv)}\quad \overline{S_{(4,3,2)}} & \text{(v)}\quad \overline{S_{(5,5,4,3,1)/(2,2,1)}} & \text{(vi)}\quad \overline{S} \\
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, &
\begin{bmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & a_{25} \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 0 \\ a_{51} & 0 & 0 & 0 & 0 \end{bmatrix}, &
\begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & a_{43} & a_{44} & 0 \\ 0 & 0 & a_{53} & 0 & 0 \end{bmatrix}
\end{array}
$$

Figure 4-2: Representative matrices from $\mathrm{mat}_q(5, S, r)$ when S is (i) a straight shape, (ii) a skew shape, (iii) a set with the NE Property; and their respective complements (iv),(v),(vi).

Proof. Fix m, n and S. We count matrices with a given collection of nonzero rows. Given a nonempty subset $T \subseteq [m]$ of rows, let $a_S(T)$ be the number of columns with no entries which are both in one of the rows of T and in S. Then there are exactly $(q^{a_S(T)} - 1)(q-1)^{\#T-1}$ matrices of rank 1 over $\mathbf{F}_q$ whose support avoids S and whose nonzero rows are exactly those in T. It follows immediately that

$$\mathrm{mat}_q(m \times n, S, 1) = \sum_{\substack{T \subseteq [m] \\ \text{nonempty}}} (q^{a_S(T)} - 1)(q-1)^{\#T-1}$$

is a polynomial in q. $\square$

Example 4.3.2. *Take the* 4×4 *shape* $S = \{(i,i) \mid 1 \le i \le 4\}$. *Then*

$$\begin{aligned} \mathrm{mat}_q(4 \times 4, S, 1) &= \sum_{k=1}^{4} \binom{4}{k} (q^{4-k} - 1)(q-1)^{k-1} \\ &= (q-1) \cdot 2(7q^2 - 2q + 1). \end{aligned}$$

(In fact one can show that if S *is the diagonal* $\{(i,i) \mid 1 \le i \le n\}$ *then* $\mathrm{mat}_q(n \times n, S, 1) = \frac{1}{q-1}((2q-1)^n - 2q^n + 1)$*.)*

Remark 4.3.3. *In later sections of this chapter, we show that for certain diagrams* S *(straight shapes, skew shapes, and conjecturally Rothe diagrams of permutations), the function* $\mathrm{mat}_q(n, S, r)/(q-1)^r$ *is not only a polynomial in* q *but also has nonnegative coefficients. However, this is not the case for matrices of rank* 1*: although each summand is a power of* $q-1$ *times a polynomial with positive coefficients, the powers of* $q-1$ *differ. So, as in Example 4.3.2, negative coefficients can turn up for certain*

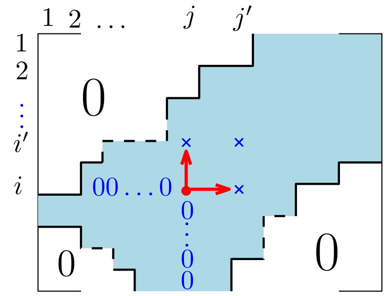


Figure 4-3: NE elimination on a representative matrix counted in $\mathrm{mat}_q(n, \overline{B}, r)$ with a pivot on (i, j) where B has the NE Property.

choices of S. It is a potentially interesting question to classify shapes S such that $\mathrm{mat}_q(m \times n, S, 1)/(q-1)$ *has positive coefficients.* ■

4.4 Formula for $\mathrm{mat}_q(n, \overline{B}, r)$ when B has NE Property

In [27], Haglund proved the following result.

Theorem 4.4.1 ([27, Thm. 1])**.** *For every straight shape S_λ we have*

$$\mathrm{mat}_q(n, \overline{S_\lambda}, r) = (q-1)^r q^{|\lambda|-r} R_r^{(\mathsf{SE})}(S_\lambda, q^{-1}).$$

We now extend this result (using the same proof technique) to all shapes with the NE Property, that is, with the property that for any $i' < i$, $j < j'$, if (i, j), (i', j) and (i, j') belong to B, then (i', j') does as well.

Theorem 4.4.2. *Fix any n and r and any set $S \subseteq [n] \times [n]$ with the NE Property. The number of $n \times n$ matrices over $\mathbf{F}_q$ of rank r whose support is contained in B is*

$$\mathrm{mat}_q(n, \overline{B}, r) = (q-1)^r q^{\#S-r} R_r^{(\mathsf{NE})}(B, q^{-1}). \tag{4.4.3}$$

Proof. Choose a matrix A counted in $\mathrm{mat}_q(n, \overline{B}, r)$, that is, whose support is in B, and perform Gaussian elimination in the following (north-east) order: traverse each column from bottom to top, starting with the leftmost (i.e., first) column. When you come to a nonzero entry (i.e., a pivot), use it to eliminate the entries to its north in the same column and to its east in the same row. See Figure 4-3 for an example of this stage of the elimination process. Then move on to the next column and repeat until there is at most one nonzero entry in every row and column.

By the NE Property, at each stage of the elimination process just described we obtain another matrix counted in $\mathrm{mat}_q(n, \overline{B}, r)$. After elimination, the positions of the pivots are a placement of r non-attacking rooks on B.

Given a fixed placement of r non-attacking rooks on B, let a be the number of cells in B that are directly north or directly east of a rook. There are $(q-1)^r q^a$ matrices of

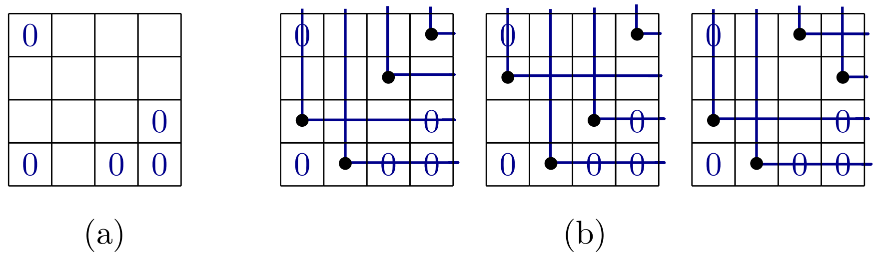


Figure 4-4: (a) Set B with the NE Property. (b) Example of computing $\mathrm{mat}_q(n, \overline{B}, r)$ when B has the NE Property. There are three placements of four rooks in B with 0, 1 and 1 NE-inversions respectively. By Theorem 4.4.2, $\mathrm{mat}_q(4, \overline{B}, 4) = (q-1)^4 q^{11-4}(1+2q^{-1})$.

rank r whose support is in B that give this placement after the elimination procedure described above. It is not hard to see that $a = \#B - r - \mathrm{inv}_{\mathsf{NE}}(C, B)$. Thus, summing over all placements or r non-attacking rooks, we obtain

$$\mathrm{mat}_q(n, \overline{B}, r) = (q-1)^r q^{\#B-r} \sum_C (q^{-1})^{\mathrm{inv}_{\mathsf{NE}}(C,B)} = (q-1)^r q^{\#B-r} R_r^{(\mathsf{NE})}(B, q^{-1}),$$

as desired. $\square$

Note that *a priori* it is not clear that the expression on the right-hand side of Equation (4.4.3) is a polynomial. However, this expression is a polynomial for the following reason: for any rook placement, there cannot be any more inversions than there are empty cells without rooks in them. There are $\#B$ cells unoccupied by zeros, and, of these, r have rooks in them. So the maximum value of $\mathrm{inv}_{\mathsf{NE}}(C, B)$ is $\#B - r$. Since this is the power of q at the beginning of the formula, there will not be any q^{-1} terms, and $\mathrm{mat}_q(n, \overline{B}, r)$ is a polynomial.

Example 4.4.4. *For $n = 4$ and $r = 4$, the set $B = ([4]\times[4])\setminus\{(1,1),(3,4),(4,1),(4,3),(4,4)\}$ has the NE Property (as in Figure 4-4(a)) and there are three placements of four rooks on B (as in Figure 4-4(b)). The number of NE-inversions of these placements are 0, 1 and 1 respectively. Thus*

$$\begin{aligned}\mathrm{mat}_q(4, \overline{B}, 4) &= (q-1)^4 q^{11-4}(1+2q^{-1}) \\ &= (q-1)^4(q^7 + 2q^6).\end{aligned}$$

We give two corollaries of Theorem 4.4.2. First, since a straight shape S_λ has the NE Property, by comparing Haglund's result and Theorem 4.4.2 we see that the (NE) and (SE) q-rook numbers of S_λ agree.

Corollary 4.4.5. *For any straight shape S_λ we have*

$$R_r^{(\mathsf{NE})}(S_\lambda, q) = R_r^{(\mathsf{SE})}(S_\lambda, q).$$

(Recall that in general the (NE) and (SE) q-rook numbers of a general board do not agree; see for example Remark 4.2.3.)

Second, since any skew shape $S_{\lambda/\mu}$ has the NE Property, we have the following corollary:

Corollary 4.4.6. *For any skew shape $S_{\lambda/\mu}$,*

$$\mathrm{mat}_q(n, \overline{S_{\lambda/\mu}}, r) = (q-1)^r f(q),$$

where $f(q)$ is a polynomial with nonnegative integer coefficients.

Example 4.4.7. *For $\lambda/\mu = 4432/31$, we have*

$$\begin{aligned}\mathrm{mat}_q(4\times 4, \overline{S_{4432/31}}, 3) &= (q-1)^3 q^{9-3}(2q^{-1} + 8q^{-1} + 7q^{-3} + q^{-4})\\ &= (q-1)^3 q^2 (q+1)(2q^2+6q+1).\end{aligned}$$

In general, for skew shapes $S_{\lambda/\mu}$ there is no product formula for $\mathrm{mat}_q(n, \overline{S_{\lambda/\mu}}, r)$ analogous to (4.2.2), even when $r = n$.

4.5 Studying $\mathrm{mat}_q(n, S, r)$ when S is a Rothe diagram

Given a permutation $w \in \mathfrak{S}_n$ written as a word $w = w_1 w_2 \cdots w_n$ where w_i is the image of w at i, the **Rothe diagram** R_w is the set

$$R_w = \{(i,j) \mid 1 \leq i, j \leq n,\ w(i) > j,\ w^{-1}(j) > i\}.$$

Equivalently R_w is the set of elements in $[n] \times [n]$ that do not lie directly south or directly east of entries (i, w_i) of the permutation matrix of w. See Figure 4-5 for some examples of Rothe diagrams. Note that $\#R_w$ is the number of inversions of w, that is, the number of pairs (i,j) such that $i < j$ but $w_i > w_j$. Also, R_w has the following property: if (i,j) and (k,ℓ) are in R_w and $i > k$, $j < \ell$ then the entry (i,ℓ) is also in R_w. We call this the **Le property** of Rothe diagrams [52, Sec. 6].

The main conjecture for Rothe diagrams, which has been verified for $n \leq 6$ [34], is the following:

Conjecture 4.5.1. *If R_w is the Rothe diagram of a permutation w in $\mathfrak{S}_n$ then $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is a polynomial in q with nonnegative integer coefficients.*

In Subsection 4.5.1 we give properties of Rothe diagrams that help in calculating $\mathrm{mat}_q(n, R_w, r)$. In Subsection 4.5.2 we study Conjecture 4.5.1 for two families of permutations. The first family is the set of permutations w such that R_w is a straight shape or the complement of a skew shape (after permuting rows and columns). The conjecture holds for such permutations by Theorem 4.4.1 and Corollary 4.4.6. In Theorem 4.5.4, we characterize these permutations. The second family is the set of

$$w = 41523 \qquad\qquad w = 21534 \qquad\qquad w = 31524$$

$$\begin{bmatrix} 0 & 0 & 0 & \underline{a_{14}} & a_{15} \\ \underline{a_{21}} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & 0 & 0 & a_{34} & \underline{a_{35}} \\ a_{41} & \underline{a_{42}} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & \underline{a_{53}} & a_{54} & a_{55} \end{bmatrix} \quad \begin{bmatrix} 0 & \underline{a_{12}} & a_{13} & a_{14} & a_{15} \\ \underline{a_{21}} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & \underline{a_{35}} \\ a_{41} & a_{42} & \underline{a_{43}} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & \underline{a_{54}} & a_{55} \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \underline{a_{13}} & a_{14} & a_{15} \\ \underline{a_{21}} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & 0 & a_{33} & 0 & \underline{a_{35}} \\ a_{41} & \underline{a_{42}} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & \underline{a_{54}} & a_{55} \end{bmatrix}$$

Figure 4-5: Representative matrices counted by $\mathrm{mat}_q(5, R_w, r)$ where R_w is a Rothe diagram and w is (i) 41523 (vexillary), (ii) 21534 (skew-vexillary), (iii) $w = 31524$ (not skew-vexillary). The entries $a_{i\,w_i}$ are in red.

permutations $w \in \mathfrak{S}_n$ such that there is some $u \in \mathfrak{S}_n$ and some positive integer s such that

$$\mathrm{mat}_q(n, R_w, r)/(q-1)^r = q^s \sum_{u \preceq v} q^{-\mathrm{inv}(v)},$$

where the (partial) order $\preceq$ is the *strong Bruhat order* (see e.g. [41, Sec. 2.1.2]) on $\mathfrak{S}_n$ and $\mathrm{inv}(v)$ is the number of (SE) inversions of v as defined in Section 4.2.

4.5.1 Properties of $\mathrm{mat}_q(n, S, r)$ when S is a Rothe diagram

In this section we give some simple properties of $\mathrm{mat}_q(n, S, r)$ when $S = R_w$ is the Rothe diagram of a permutation w. These properties are useful to simplify the size of computations involved in empirically confirming conjectures about $\mathrm{mat}_q(n, R_w, r)$ like Conjectures 4.5.1 and 4.6.6.

If the permutation w is the word $w_1 w_2 \cdots w_n$, the **reverse** of w is the permutation $re(w) = w_n w_{n-1} \cdots w_1$. The **complement** of w is the permutation $c(w) = u_1 u_2 \cdots u_n$ where $u_i = n + 1 - i - w_i$. In addition, the **reverse complement** of w is the permutation $rc(w) = v_1 v_2 \cdots v_n$ where $v_i = n + 1 - w_{n+1-i}$. Lastly, the **left-to-right maxima** of w are the values w_i such that $w_i > w_j$ for all j such that $1 \leq j < i$.

Proposition 4.5.2. *Given a permutation w in $\mathfrak{S}_n$ and its Rothe diagram R_w, we have*

(i) $\mathrm{mat}_q(n, R_w, r) = \mathrm{mat}_q(n, R_{w^{-1}}, r)$ *and*
(ii) $\mathrm{mat}_q(n, R_w, r) = \mathrm{mat}_q(n, R_{rc(w)}, r)$.

Proof. It is easy to see that for any permutation w, the diagram $R_{w^{-1}}$ is the transpose of R_w, and the first statement follows immediately. We now consider the second statement.

Fix a permutation w with Rothe diagram R_w. Each element (i, j) of R_w corresponds to the inversion of w formed by the entries with matrix coordinates (i, w_i) and (w_j^{-1}, j). In $rc(w)$, these elements of w are transformed to $(n+1-i, n+1-w_i)$ and $(n+1-w_j^{-1}, n+1-j)$ and still form an inversion; in $R_{rc(w)}$, this inversion corresponds to the element with coordinates $(n+1-w_j^{-1}, n+1-w_i)$. It follows immediately that the diagram $R_{rc(w)}$ is the result of taking the transpose of R_w, rearranging rows and columns by multiplying on both sides by the permutation matrix of w, and rotating the result by 180°. $\square$

Next we show that the indices of the columns (rows) of $[n] \times [n]$ contained in $\overline{R_w}$ correspond to the left-to-right maxima of w (of w^{-1}). This is useful for computation because it is easy to express $\mathrm{mat}_q(n, S, r)$ in terms of values of mat_q for sets obtained by removing rows or columns that contain no elements of S.

Proposition 4.5.3. *The kth column (row) of $[n] \times [n]$ is contained in $\overline{R_w}$ if and only if k is a left-to-right maximum of w (of w^{-1}).*

Proof. This follows from the definitions of R_w and of the left-to-right maxima. $\square$

4.5.2 Skew-vexillary permutations

A permutation w is **vexillary** if its Rothe diagram, up to a permutation of its rows and columns, is the diagram of a partition. Call this partition $\lambda(w)$. Then by Haglund's Theorem 4.4.1, for vexillary permutations w we have that

$$\mathrm{mat}_q(n, R_w, r) = \mathrm{mat}_q(n, S_{\lambda(w)}, r) = (q-1)^r q^{n^2 - \mathrm{inv}(w) - r} R_r^{(\mathsf{NE})}(\overline{S_{\lambda(w)}}, q^{-1}).$$

It is well-known that w is vexillary if and only if w avoids 2143 [37], i.e., there is no sequence $1 \leq i < j < k < l \leq n$ in w such that $w_j < w_i < w_l < w_k$.

Next we give a characterization of permutations whose Rothe diagram, up to a permutation of rows and columns, is the complement of a skew shape. For such a permutation w, we have by Corollary 4.4.6 that $\mathrm{mat}_q(n, R_w, r)/(q-1)^r$ is a polynomial with nonnegative integer coefficients. So Conjecture 4.5.1 holds for these permutations.

For the proof we need the following definition: we say that a skew shape $S_{\lambda/\mu}$ in $[n] \times [n]$ is **non-overlapping** if there is no row nor column that contains entries from both S_μ and $\overline{S_\lambda}$.

Theorem 4.5.4. *The Rothe diagram of $w = w_1 w_2 \cdots w_n$ is, up to permuting its rows and columns, the complement of a skew shape if and only if w can be decomposed as $a_1 a_2 \cdots a_k b_1 b_2 \cdots b_{n-k}$ where $a_i < b_j$ and each of $a_1 a_2 \cdots a_k$ and $b_1 b_2 \cdots b_{n-k}$ is 2143-avoiding.*

Proof. First we prove the "if" direction. This argument is illustrated in Figure 4-6. Suppose that w can be decomposed into $a = a_1 a_2 \cdots a_k$ and $b = b_1 b_2 \cdots b_{n-k}$ as in the theorem statement. Then the Rothe diagram R_w is block-diagonal, i.e., it consists of some entries in the upper-left $k \times k$ block and some in the lower-right $(n-k) \times (n-k)$ block, with no entries in the upper-right $k \times (n-k)$ block or lower-left $(n-k) \times k$ block. Furthermore, note that the upper-left and lower-right subdiagrams are identical to the Rothe diagrams of the permutations order-isomorphic to $a_1 a_2 \cdots a_k$ and $b_1 b_2 \cdots b_{n-k}$, respectively.

Since both of these permutations are 2143-avoiding, and their Rothe diagrams in the upper-left and lower-right corners do not share any rows or columns in common,

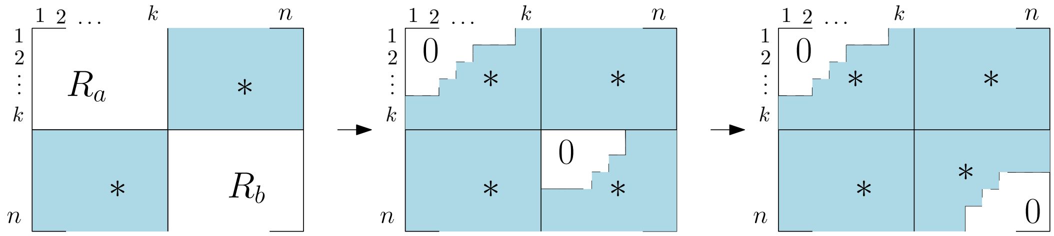


Figure 4-6: If w can be decomposed as $a_1a_2\cdots a_kb_1b_2\cdots b_{n-k}$ where $a_i < b_j$ and both $a = a_1a_2\cdots a_k$ and $b = b_1b_2\cdots b_{n-k}$ are 2143 avoiding then $\overline{R_w}$ can be rearranged into a skew shape.

they can be rearranged independently to form two separate straight shapes. We may then rotate the straight shape corresponding to R_b by 180° via permuting rows and columns (without changing the rearranged upper-left corner) to get a straight shape in the upper-left corner and an upside-down straight shape in the lower-right. This is the outside of a skew shape, as desired.

Second, we prove the "only if" direction of the theorem. Suppose that the diagram R_w, when rearranged, forms the complement of a skew shape $S_{\lambda/\mu}$. This skew shape contains the column that was previously (i.e., before rearrangement) given by $\{(j, w_1) \mid j \geq 1\}$. Likewise, it contains the row that was previously given by $\{(w_1^{-1}, j) \mid j \geq 1\}$. It follows that the skew shape $S_{\lambda/\mu}$ is non-overlapping. After rearrangement, every entry of R_w either belongs to S_μ or $\overline{S_\lambda}$. We use this partition of the elements of R_w to identify the appropriate decomposition of w.

We color an entry of R_w blue if it belongs to S_μ after rearrangement, otherwise we color it red. We show the following claim: for every entry w_i of w, the elements of R_w in the same row or column as (i, w_i) are either all blue or all red.

Since $S_{\lambda/\mu}$ is non-overlapping, the entries of R_w in each row have the same color, and likewise for columns. If there is an entry (i, w_i) with elements (i, j) and (k, w_i) of R_w then by the Le property of Rothe diagrams (k, j) is also in R_w. Therefore all three entries have the same color, and the claim follows.

By the argument of the preceding paragraph, we may color the elements of w as follows: if (i, w_i) is in the same row or column as a red entry of R_w then we color w_i red, whereas if (i, w_i) is in the same row or column as a blue entry of R_w then we color w_i blue, and otherwise we leave w_i uncolored. We observe a few properties of the colored and uncolored elements of the permutation. First, inversions of w can only happen between elements of the same color. Second, w_i is uncolored if and only if w_i is not involved in any inversions. And third, the subword of the blue (respectively, red) elements of w is 2143-avoiding. This is because by definition, the entries of R_w in the same row or column as (i, w_i) for blue (respectively, red) w_i are exactly the entries in S_μ (respectively, $\overline{S_\lambda}$) after rearrangement. This is equivalent to saying that the subword of the blue (respectively, red) elements of w is vexillary and thus 2143-avoiding.

From the three observations above it follows that the permutation w decomposes as $u_1c_1u_2c_2u_3$ where (i) the u_i are (possibly empty) blocks of uncolored elements, c_1

is the block of elements of one color of w, and c_2 is the block of elements of the other color of w; (ii) the entries of each block are smaller than the entries of the following blocks, and (iii) the blocks c_1 and c_2 are 2143-avoiding. Finally, if we set $a_1a_2\cdots a_k = u_1c_1$ and $b_1b_2\cdots b_{n-k} = u_2c_2u_3$ we get a desired decomposition of w where $a_i < b_j$ and $a_1a_2\cdots a_k$ and $b_1b_2\cdots b_{n-k}$ are 2143-avoiding. $\square$

We call the above permutations **skew-vexillary**[1] and we denote by $\lambda/\mu(w)$ the skew shape whose complement is the rearrangement of R_w.[2]

Corollary 4.5.5. *By Theorem 4.4.2, if w is skew-vexillary then* $\mathrm{mat}_q(n, R_w, r)/(q-1)^r$ *is equal to* $q^{n^2-\mathrm{inv}(w)-r}R_r^{(\mathsf{NE})}(S_{\lambda/\mu(w)}, q^{-1})$*, a polynomial with nonnegative integer coefficients. In particular, Conjecture 4.5.1 holds for skew-vexillary permutations.*

If w is a skew-vexillary permutation then every subpermutation of w is, as well. This implies that skew-vexillarity may be rephrased as a pattern-avoidance condition. We do this now.

Proposition 4.5.6. *The permutation $w \in \mathfrak{S}_n$ can be decomposed as $w = a_1\cdots a_kb_1\cdots b_{n-k}$ such that $a_i < b_j$ for all i and j and the permutations $a_1\cdots a_k$ and $b_1\cdots b_{n-k}$ avoid* 2143 *if and only if w avoids the nine patterns* 24153*,* 25143*,* 31524*,* 31542*,* 32514*,* 32541*,* 42153*,* 52143 *and* 214365*.*

Proof. Call the decomposition in question an "SV-decomposition" (for **S**kew-**V**exillary). First, we show that if w contains any of the nine patterns listed in the statement of the theorem, it does not have an SV-decomposition.

Let p be any of the eight patterns of length 5; it's easy to check that p is indecomposable, i.e., we cannot write $p = uv$ with u, v nonempty and $u_i < v_j$ for all i, j. Thus, if we write $w = ab$ with $a_i < b_j$ we must have either p contained in a or p contained in b. Since p contains 2143, it follows that either a or b contains 2143, so this decomposition is not SV, as desired.

Now consider the case of the pattern 214365. Any decomposition of w decomposes 214365, and it's easy to see that in any of the four decompositions of 214365, one piece or the other contains a copy of 2143. This completes the proof that any permutation containing the given patterns has no SV-decomposition.

Now consider the converse. Suppose that w is not SV-decomposable. There are two cases.

If w is indecomposable and contains 2143, then w contains a minimal indecomposable permutation that contains 2143. The minimal 2143-containing indecomposable permutations are precisely the eight permutations of length 5 that we consider.

Finally, we show by induction that every decomposable but not SV-decomposable permutation contains one of the nine patterns. Choose a such w, and write $w = ab$ with $a_i < b_j$. Without loss of generality, in this decomposition we have that a contains

[1] Note that in the literature [11, Prop. 2.3] there is another meaning of the term "skew-vexillary permutation" which does not seem to be related to our definition.

[2] The "function" $\lambda/\mu(w)$ is not actually well-defined most of the time since you can switch the upper-left and lower-right corners by permuting rows and columns. Luckily nothing we use it for depends on this choice.

2143. If b has a descent, it follows immediately that w contains 214365. Otherwise, $w = a_1 \cdots a_k(k+1)(k+2)\cdots n$. Observe that a permutation of this form has an SV-decomposition if and only if the shorter permutation $a = a_1 \cdots a_k$ has an SV-decomposition; thus, a has no SV-decomposition. If a is indecomposable, we have by the preceding paragraph that a contains one of the nine patterns; if a is decomposable, we have the same result by induction.

Putting the two cases together, every permutation that is not SV-decomposable contains at least one of the nine patterns, as desired. $\square$

Remark 4.5.7. *Vexillary permutations have many more interesting properties than just* 2143*-avoiding permutations (see for example [41, Sections 2.6.5 and 2.8.1]). Do any of these properties carry over to skew-vexillary permutations?* $\blacksquare$

If w is skew-vexillary then $\mathrm{mat}_q(n, R_w, r)/(q-1)^r$ is of the form

$$q^{n^2-\mathrm{inv}(w)-r} \sum_{\text{some } u\in\mathfrak{S}_n} q^{-\mathrm{inv}(u)}.$$

Another polynomial with this form is the *Poincaré polynomial* of the strong Bruhat order in $\mathfrak{S}_n$. In the next subsection we study the connections between these and $\mathrm{mat}_q(n, R_w, n)$.

4.6 Poincaré polynomials, $\mathrm{mat}_q(n, R_w, n)$ and q-rook numbers

A natural question when faced with a family of polynomials with positive integer coefficients is whether they count some nice combinatorial object. In this section, we investigate connections between our polynomials $\mathrm{mat}_q(n, R_w, n)$ (note in particular that we focus on the case of full rank) and certain well-known polynomials we define now.

As before, let $\mathrm{inv}(w)$ denote the number of inversions $\#\{(i,j) \mid i<j, w_i > w_j\}$ of w. Recall the notion of the **strong Bruhat order** $\prec$ on the symmetric group [12, Ch. 2]: if t_{ij} is the transposition that switches i and j, we have as our basic relations that $u \prec u\cdot t_{ij}$ in the strong Bruhat order when $\mathrm{inv}(u)+1 = \mathrm{inv}(u\cdot t_{ij})$, and we extend by transitivity. Let $P_w(q) = \sum_{u\succeq w} q^{\mathrm{inv}(u)}$ be the (upper) **Poincaré polynomial** of w, where we sum over all permutations u that succeed w in the strong Bruhat order. Equivalently, $P_w(q)$ is the rank generating function of the interval $[w, w_0]$ in the strong Bruhat order where w_0 is the largest element $n\, n-1\, \ldots\, 21$ of this order.

Example 4.6.1. *If* $w = 3412$*, then the permutations in* $\mathfrak{S}_4$ *that succeed* w *in the Bruhat order are* $\{3412, 3421, 4312, 4321\}$*. The generating polynomial for this set by number of inversions is* $P_{2143}(q) = q^6 + 2q^5 + q^4$*.*

In [56], Sjöstrand gave necessary and sufficient conditions for $P_w(q)$ to be equal to a q-rook number of a skew shape associated to w. Namely, the **left hull** $H_L(w)$ of w is the smallest skew shape that covers w. Equivalently, $H_L(w)$ is the union over

$$\begin{array}{cc} R_{35142} & H_L(35142) \\ \begin{bmatrix} 0 & 0 & \underline{a_{13}} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & 0 & \underline{a_{25}} \\ \underline{a_{31}} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & 0 & a_{43} & \underline{a_{44}} & a_{45} \\ a_{51} & \underline{a_{52}} & a_{53} & a_{54} & a_{55} \end{bmatrix} & \begin{bmatrix} 0 & 0 & \underline{a_{13}} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & \underline{a_{25}} \\ \underline{a_{31}} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & \underline{a_{44}} & 0 \\ a_{51} & \underline{a_{52}} & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 4-7: Matrices indicating the (i) Rothe diagram and (ii) left hull of $w = 35142$. The matrix entries $a_{i\,w_i}$ are in red.

non-inversions (i, j) of w of the rectangles $\{(k, \ell) \mid w_i \leq k \leq w_j, i \leq \ell \leq j\}$. See Figure 4-7 for an example of the left hull of a permutation.

The following special case of a result by Sjöstrand characterizes when $P_w(q)$ is equal to the rook polynomial of the left hull of the permutation w.

Theorem 4.6.2 ([56, Cor. 3.3]). *The Bruhat interval $[w, w_0]$ in $\mathfrak{S}_n$ equals the set of rook placements in the left hull $H_L(w)$ of w (and in particular $R_n^{(\mathsf{SE})}(H_L(w), q) = q^{|\mu|}P_w(q)$ where μ is the shape such that $H_L(w) = S_{\lambda/\mu}$ for some λ) if and only if w avoids the patterns* 1324, 24153, 31524, *and* 426153.

If w is a skew-vexillary then by Corollary 4.5.5 we know that $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is (up to a power of q) a q-rook number. Next we show that this q-rook number is essentially a q-rook number of the left hull of a permutation v that avoids the four patterns above. Therefore by Theorem 4.6.2 $\mathrm{mat}(n, R_w, n)/(q-1)^n$ is (up to a power of q) a Poincaré polynomial $P_v(q)$.

4.6.1 $\mathrm{mat}_q(n, R_w, n)$ for skew-vexillary permutations is a Poincaré polynomial

In this section we use Sjöstrand's result (Theorem 4.6.2) to show that for skew-vexillary permutations w, the function $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is not only a polynomial with nonnegative coefficients but, up to a power of q, is a Poincaré polynomial.

Proposition 4.6.3. *If w is skew-vexillary then*

$$\mathrm{mat}_q(n, R_w, n) = q^{\binom{n}{2} - \mathrm{inv}(w)}(q-1)^n \cdot P_v(q)$$

for some permutation $v \in \mathfrak{S}_n$.

Proof. If w is skew-vexillary, then by Corollary 4.5.5 we know that

$$\mathrm{mat}_q(n, R_w, n)/(q-1)^n = q^{n^2 - \mathrm{inv}(w) - n} R_n^{(\mathsf{NE})}(S_{\lambda/\mu(w)}, q^{-1}),$$

where $R^{(\mathsf{NE})}(S_{\lambda/\mu(w)}, q)$ is the rook polynomial of $S_{\lambda/\mu(w)}$, the non-overlapping skew shape whose complement is the rearrangement of R_w. We will show that this polynomial is the Poincaré polynomial $P_v(q)$ of a permutation v.

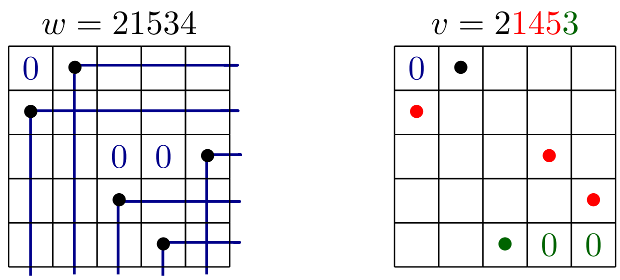


Figure 4-8: Example of Proposition 4.6.3. For the permutations w and v shown we have that $\mathrm{mat}_q(5, R_w, 5)/(q-1)^5 = q^{\binom{5}{2}-\mathrm{inv}(w)} \cdot P_v(q)$.

Define the permutation matrix of v as follows: let $w = a_1a_2\cdots a_k b_1 b_2 \cdots b_{n-k}$ be the decomposition promised by Theorem 4.5.4. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots)$. For $i = 1, \ldots, k$, let $v_i = \min(([n] \setminus [\mu_i]) \setminus \{v_1, \ldots, v_{i-1}\})$ and for $j = 1, \ldots, n-k$ let $v_{n+1-j} = \max([\lambda_{n-j}] \setminus \{v_{n-j+1}, \ldots, v_n\})$. This defines a 0-1 matrix with exactly one 1 in every row; it follows from the proof of Theorem 4.5.4 that this matrix is in fact a permutation matrix (with $\{v_1, \ldots, v_k\} = [k]$ and $\{v_{k+1}, \ldots, v_n\} = \{n-k+1, \ldots, n\}$). See Figure 4-8 for an example of this construction. It is clear that $S_{\lambda/\mu(w)} = H_L(v)$. Also by Proposition 4.2.4 we have that $q^{\binom{n}{2}-|\mu|} R_n^{(\mathsf{NE})}(S_{\lambda/\mu}(w), q^{-1}) = R_n^{(\mathsf{SE})}(H_L(v), q)$.

By construction the prefix $v_1 \cdots v_k$ avoids 132 and the suffix $v_{k+1} \cdots v_n$ avoids 213 and $v_j > v_i$ for $j \geq k+1$ and $i \leq k$. It is easy to see that the set of permutations with such a decomposition is closed under containment of patterns, and does not contain any of the permutations 1324, 24153, 31524, and 426153. Therefore, every permutation in this set, and in particular v, avoids these four patterns. By Theorem 4.6.2 it follows that $q^{|\mu|} R_n^{(\mathsf{SE})}(H_L(v), q) = P_v(q)$. Thus

$$\begin{aligned}\mathrm{mat}_q(n, R_w, n)/(q-1)^n &= q^{n^2-\mathrm{inv}(w)-n} R_n^{(\mathsf{NE})}(S_{\lambda/\mu(w)}, q^{-1}) \\ &= q^{\binom{n}{2}+|\mu|-\mathrm{inv}(w)} R_n^{(\mathsf{SE})}(H_L(v), q) \\ &= q^{\binom{n}{2}-\mathrm{inv}(w)}(q-1)^n \cdot P_v(q),\end{aligned}$$

as desired. □

Example 4.6.4. *By Theorem 4.5.4, the permutation $w = 21534$ is skew-vexillary. After rearranging rows and columns (see Figure 4-8), the skew shape $S_{\lambda/\mu(w)}$ is $S_{55553/1}$. The associated v is 21453 and we have*

$$\begin{aligned}\mathrm{mat}_q(5, R_{21534}, 5) &= q^{10-3}(q-1)^5 P_{21453}(q) \\ &= q^7(q-1)^5(q^{10}+4q^9+9q^8+14q^7+15q^6+11q^5+5q^4+q^3).\end{aligned}$$

Remark 4.6.5. *Note that the result above does not hold for all Rothe diagrams. There exist permutations w for which there does not exist any permutation v such that $\mathrm{mat}_q(n, R_w, r) = q^{\binom{n}{2}-\mathrm{inv}(w)}(q-1)^r P_v(q)$. For example, take $w = 31524$ (see*

w	1324	24153 or 31524
$\frac{\mathrm{mat}_q(n,R_w,n)}{(q-1)^n q^k}$	$q^6 + 3\,q^5 + 5\,q^4 + 5q^3 + 3\,q^2 + q$	$q^{10} + 4q^9 + 9q^8 + 12q^7 + 10q^6 + 5q^5 + q^4$
$P_w(q)$	$q^6 + 3q^5 + 5q^4 + 6q^3 + 4q^2 + q$	$q^{10} + 4q^9 + 9q^8 + 13q^7 + 11q^6 + 5q^5 + q^4$
$q^a R_n^{(\mathsf{SE})}(H_L(w))$	$q^6 + 3q^5 + 5q^4 + 6q^3 + 5q^2 + 3q + 1$	$q^{10} + 4q^9 + 9q^8 + 13q^7 + 12q^6 + 7q^5 + 2q^4$

w	426153
$\frac{\mathrm{mat}_q(n,R_w,n)}{(q-1)^n q^k}$	$q^{15} + 5q^{14} + 14q^{13} + 24q^{12} + 27q^{11} + 19q^{10} + 7q^9 + q^8$
$P_w(q)$	$q^{15} + 5q^{14} + 14q^{13} + 25q^{12} + 28q^{11} + 19q^{10} + 7q^9 + q^8$
$q^a R_n^{(\mathsf{SE})}(H_L(w))$	$q^{15} + 5q^{14} + 14q^{13} + 25q^{12} + 29q^{11} + 21q^{10} + 8q^9 + q^8$

Table 4.1: For the four special patterns w of Conjecture 4.6.6 we give $\mathrm{mat}_q(n, R_w, n)/((q-1)^n q^k)$ where $k = \binom{n}{2} - \mathrm{inv}(w)$, the Poincaré polynomials $P_w(q)$, and $q^a R_n^{(\mathsf{SE})}(H_L(w), q)$ where a is the size of the subtracted partition of the skew shape $H_L(w)$.

Figure 4-5 (iii) and Table 4.1). In this case

$$\mathrm{mat}_q(5, R_{31524}, 5) = q^6(q-1)^5(q^{10} + 4q^9 + 9q^8 + 12q^7 + 10q^6 + 5q^5 + q^4).$$

One can show (either by computer search or by a direct argument about the possible structure of the inversions) that there is no permutation v in $\mathfrak{S}_5$ such that $P_v(q) = q^{10} + 4q^9 + 9q^8 + 12q^7 + 10q^6 + 5q^5 + q^4$. ■

We have shown that for skew-vexillary permutations w, $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is equal (up to a power of q) to the Poincaré polynomial of some permutation v. Next we consider the problem of classifying permutations w such that $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is equal (up to a power of q) to the Poincaré polynomial of the same permutation.

4.6.2 Further relationships between $\mathrm{mat}_q(n, R_w, n)$ and Poincaré polynomials

Computational evidence for $n \leq 7$ [34] suggests the following conjecture.

Conjecture 4.6.6. *Fix a permutation w in $\mathfrak{S}_n$ and let R_w be its Rothe diagram. We have that $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is coefficient-wise less than or equal to $q^{\binom{n}{2}-\mathrm{inv}(w)} P_w(q)$. We have equality if and only if w avoids the patterns* 1324, 24153, 31524, *and* 426153.

Remark 4.6.7. *The patterns that appear in Conjecture 4.6.6 and in Theorem 4.6.2 are the same. Also, the reverses* 4231, 35142, 42513, *and* 351624 *of these patterns have appeared in related contexts in a conjecture of Postnikov [52] proved by Hultman-Linusson-Shareshian-Sjöstrand [30], and in work by Gasharov-Reiner [22]. This suggests further interesting connections.* ■

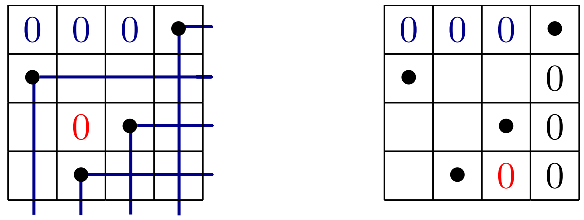


Figure 4-9: Example of Proposition 4.6.10. For the permutation $w = 4132$, the Rothe diagram is $R_w = \{(1,1),(1,2),(1,3),(3,2)\}$ and the left-hull $\overline{H_L(w)} = \{(1,1),(1,2),(1,3),(2,4),(3,4),(4,3),(4,4)\}$. The map φ is given by $(1,j) \mapsto (1,j)$ for $j = 1,2,3$ and $(3,2) \mapsto (4,3)$.

The values of the three polynomials $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$, $P_w(q)$, and $R_n^{(\mathsf{SE})}(H_L(w), q)$ when w is equal to the four patterns of Conjecture 4.6.6 are shown in Table 4.1. In these cases the three polynomials are all different. By Theorem 4.6.2 and Theorem 4.4.2, Conjecture 4.6.6 is equivalent to the following:

Conjecture 4.6.8. *Fix a permutation w in $\mathfrak{S}_n$, let R_w be its Rothe diagram and let $a_w = n^2 - \#H_L(w) - \mathrm{inv}(w)$. We have that $\mathrm{mat}_q(n, R_w, n)/(q-1)^n$ is coefficient-wise less than or equal to $q^{a_w}\,\mathrm{mat}_q(n, \overline{H_L(w)}, n)/(q-1)^n$. We have equality if and only if w avoids the patterns 1324, 24153, 31524, and 426153.*

Remark 4.6.9. *If Conjecture 4.6.8 holds then by Theorem 4.4.2 and [39, Prop. 5.1] it follows that whenever w avoids the four patterns, the shapes $\overline{R_w}$ and $H_L(w)$ have the same number of placements of n non-attacking rooks. Computer experiments for $n \leq 7$ [34] suggest that the converse is also true, i.e., if w contains any of the four patterns, the shapes have different numbers of rook placements. This apparent equivalence of necessary and sufficient conditions between the "q case" and the "$q = 1$ case" does not necessarily hold in similar settings (see [50, Thm. 7] and [30, Thm. 3.4] for an example).* ∎

We end by giving a very preliminary step in proving these conjectures. We show that if w avoids 1324 then the complement of the left hull has at least as many entries as the Rothe diagram.

Proposition 4.6.10. *If w is a 1324-avoiding permutation then the complement of $H_L(w)$ has at least as many entries as the Rothe diagram R_w of w.*

Proof. Let w be a 1324-avoiding permutation. We give a one-to-one map φ between the entries of the Rothe diagram R_w and the complement of the left hull $H_L(w)$.

Given an entry (i,j) in R_w we have two possibilities: either there is or there is not an entry (k, w_k) of w such that $k < i$ and $w_k < j$ (i.e., an entry of w NW of (i,j)). Let A_w be the set of entries of R_w of the first type and let B_w be the set of entries of the second type. If $(i,j) \in A_w$ then define $\varphi(i,j) = (i,j)$. If instead $(i,j) \in B_w$ then define $\varphi(i,j) = (w_j^{-1}, w_i)$. See Figure 4-9 for an illustration of φ. We show that φ is well-defined and injective.

Choose (i, j) in R_w. There are no entries of w above (i, j) in the same column or to its left in the same row. If in addition (i, j) is in A_w then by definition of the left hull the entry (i, j) is not in $H_L(w)$. In this case $\varphi(i, j) = (i, j) \in \overline{H_L(w)}$ as desired.

On the other hand, if (i, j) is in B_w then there is some entry (k, w_k) of w with $k < i$ and $w_k < j$. Since w is 1324-avoiding, there can be no entry (ℓ, w_ℓ) of w such that $\ell \geq w_j^{-1}$ and $w_\ell \geq w_i$. Thus, $\varphi(i, j) = (w_j^{-1}, w_i) \in \overline{H_L(w)}$. This completes the proof that the map φ is well-defined.

Finally, we show that φ is one-to-one. Since φ is defined piecewise it is enough to show that φ is one-to-one on A_w and B_w and that $\varphi(A_w)$ and $\varphi(B_w)$ are disjoint. The injectivity on A_w is trivial. The injectivity on B_w follows since w is a permutation and so (w_j^{-1}, w_i) uniquely defines (i, j). Moreover, $\overline{H_L(w)}$ has two components; $\varphi(A_w)$ is the NW component while $\varphi(B_w)$ is contained in the SE component, so the images are disjoint. This completes the proof that φ is one-to-one. $\square$

Appendix A

Computations to prove Corollary 2.4.2

Here we finish the computations from Section 2.4 to complete the proof of Corollary 2.4.2: If $\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}$ are compositions of n with p_1, p_2, p_3 parts then the number $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$ of colored factorizations is

$$c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}} = (n-p_1)!(n-p_2)!(n-p_3)! \times \sum_{a\geq 0} \frac{(n-a-2)! \cdot \Theta}{a!(p_3-1-a)!(p_2-1-a)!(p_1-1)!(n-p_1-a)!(n+2-p_2-p_3+a)!},$$

where

$$\Theta = (n+2-p_2-p_3+a)\left((n-a-1)(p_3-a)+(p_1-1)(n-p_3)\right)+ \\ +(n-a_1-p_1)\left((n+1-p_2-p_3+a)(n+2-p_2-p_3+a)+(n+1-p_2)(p_2-1-a)\right).$$

Recall that $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}} = \sum_{a_1,a_2,a_3\geq 0} c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1,a_2,a_3)$, where $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)})}(a_1,a_2,a_3)$ be the number of tree-rooted constellations of vertex-compositions $(\gamma^{(1)},\gamma^{(2)},\gamma^{(3)})$ where $\gamma^{(t)} = (n+1-p_t,1,1,\ldots,1)$ for $t=1,2,3$ where the type t vertex labelled 1 is incident to a_t 3-gons whose other two vertices are of hyperdegree 1. From (2.4.6) we have that

$$c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1,a_2,a_3) = n\cdot\prod_{t=1}^{3}(p_t-1)!\cdot(N_1+N_2+\cdots+N_8).$$

The formulae for the numbers N_i were given in Proposition 2.4.7:

$$N_1 = \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\binom{n-1}{a_1,a_2,a_3,a_{12},a_{23},a_{13},a_{123}}, \qquad \text{(A.0.1)}$$

and for $i = 2, 3, \ldots, 8$,

$$N_i = \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-2)!}\binom{n-2}{a_1, a_2, a_3, a_{12}, a_{23}, a_{13}, a_{123}}, \tag{A.0.2}$$

where a_{23}, a_{13}, a_{12}, and a_{123} for each case are given in Figure 2-9.

First we group some of the N_is using manipulations with multinomial coefficients.

$$\begin{aligned}
N_1 &= \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\binom{n-1}{a_1, a_2, a_3, p_1-1-a_2-a_3, p_2-1-a_1-a_3, p_3-1-a_1-a_2}, \\
\frac{N_2+N_4}{n+2-p_1-p_2+a_3} &= \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\binom{n-1}{a_1, a_2, a_3, p_1-1-a_2-a_3, p_2-2-a_1-a_3, p_3-1-a_1-a_2}, \\
\frac{N_3+N_6}{n+2-p_1-p_3+a_2} &= \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\binom{n-1}{a_1, a_2, a_3, p_1-2-a_2-a_3, p_2-1-a_1-a_3, p_3-1-a_1-a_2}, \\
\frac{N_5+N_7}{n+2-p_2-p_3+a_1} &= \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\binom{n-1}{a_1, a_2, a_3, p_1-1-a_2-a_3, p_2-1-a_1-a_3, p_3-2-a_1-a_2}, \\
N_8 &= \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-2)!}\binom{n-2}{a_1, a_2, a_3, p_1-1-a_2-a_3, p_2-1-a_1-a_3, p_3-1-a_1-a_2}.
\end{aligned}$$

Next we use Chu-Vandermonde identity repeatedly to turn the three parameter sums $\sum_{a_1,a_2,a_3\geq 0} N_i$ into sums involving just the parameter a_1. We illustrate this for N_1.

Proposition A.0.3.

$$\sum_{a_1,a_2,a_3\geq 0} N_1 = \frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\sum_{a_1\geq 0}\binom{n-1}{a_1}\binom{n-a_1-1}{p_3-1-a_1}\binom{n-p_3}{p_2-1-a_1}\binom{n-a_1-1}{p_1-1}.$$

Proof.

$$\begin{aligned}
&\sum_{a_1,a_2,a_3\geq 0}\binom{n-1}{a_1, a_2, a_3, p_1-1-a_2-a_3, p_2-1-a_1-a_3, p_3-1-a_1-a_2} \\
&\qquad \sum_{a_1,a_2,a_3\geq 0}\binom{n-1}{a_1}\binom{n-a_1-1}{a_2}\binom{n-1-a_1-a_2}{p_3-1-a_2-a_3}\binom{n-p_3}{a_3}\times \\
&\qquad \times\binom{n-p_3-a_3}{p_2-1-a_1-a_3}\binom{n-p_2-p_3+a_1+1}{p_1-1-a_2-a_3}, \\
&= \sum_{a_1,a_2\geq 0}\binom{n-1}{a_1}\binom{n-a_1-1}{a_2}\binom{n-1-a_1-a_2}{p_3-1-a_2-a_3}\binom{n-p_3}{p_2-1-a_1}\times \\
&\qquad \times\sum_{a_3\geq 0}\binom{p_2-1-a_1}{a_3}\binom{n-p_2-p_3+a_1+1}{p_1-1-a_2-a_3},
\end{aligned}$$

by using the Chu-Vandermonde identity on the sum over a_3 we obtain

$$= \sum_{a_1,a_2\geq 0} \binom{n-1}{a_1}\binom{n-a_1-1}{a_2}\binom{n-1-a_1-a_2}{p_3-1-a_2-a_3}\binom{n-p_3}{p_2-1-a_1}\binom{n-p_3}{p_1-1-a_2},$$

$$= \sum_{a_1\geq 0} \binom{n-1}{a_1}\binom{n-a_1-1}{p_3-1-a_1}\binom{n-p_3}{p_2-1-a_1}\sum_{a_2\geq 0}\binom{p_3-1-a_1}{a_2}\binom{n-p_3}{p_1-1-a_2},$$

again using the Chu-Vandermonde identity on the sum over a_2 we obtain,

$$\sum_{a_1,a_2,a_3\geq 0} \binom{n-1}{a_1,a_2,a_3,p_1-1-a_2-a_3,p_2-1-a_1-a_3,p_3-1-a_1-a_2} =$$

$$\sum_{a_1\geq 0} \binom{n-1}{a_1}\binom{n-a_1-1}{p_3-1-a_1}\binom{n-p_3}{p_2-1-a_1}\binom{n-a_1-1}{p_1-1},$$

as desired. $\square$

By similar simplifications as the ones above for the proof of Proposition A.0.3, we can also show the following:

Proposition A.0.4.

$$\sum_{a_1,a_2,a_3\geq 0} N_2 + N_4 =$$

$$\frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\sum_{a_1\geq 0}\binom{n-1}{a_1}\binom{n-a_1-1}{p_3-1-a_1}\binom{n-p_3}{p_2-2-a_1}\binom{n-a_1-2}{p_1-1}(n+1-p_2),$$

$$\sum_{a_1,a_2,a_3\geq 0} N_3 + N_6 =$$

$$\frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\sum_{a_1\geq 0}\binom{n-1}{a_1}\binom{n-a_1-1}{p_3-1-a_1}\binom{n-p_3}{p_2-1-a_1}\binom{n-a_1-2}{p_1-2}(n-p_3),$$

$$\sum_{a_1,a_2,a_3\geq 0} N_5 + N_7 =$$

$$\frac{\prod_{t=1}^{3}(n-p_t)!}{(n-1)!}\sum_{a_1\geq 0}\binom{n-1}{a_1}\binom{n-a_1-1}{p_3-2-a_1}\binom{n-p_3}{p_2-1-a_1}\binom{n-a_1-1}{p_1-1}(n+1-p_3),$$

$$\sum_{a_1,a_2,a_3\geq 0} N_8 =$$

$$\frac{\prod_{t=1}^{3}(n-p_t)!}{(n-2)!}\sum_{a_1\geq 0}\binom{n-2}{a_1}\binom{n-a_1-2}{p_3-1-a_1}\binom{n-p_3-1}{p_2-1-a_1}\binom{n-a_1-2}{p_1-1}.$$

Now we put everything together to compute $c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}$:

$$c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}} = \sum_{a_1,a_2,a_3\geq 0} c_{\gamma^{(1)},\gamma^{(2)},\gamma^{(3)}}(a_1,a_2,a_3), \tag{A.0.5}$$

$$= n \cdot \prod_{t=1}^{3}(p_t - 1)! \sum_{a_1,a_2,a_3\geq 0} \cdot (N_1 + N_2 + \cdots + N_8). \tag{A.0.6}$$

Using the simplifications from Propositions A.0.3 and A.0.4 we turn the three parameter sum in (A.0.6) into the sum of one parameter $a_1 = a$ in (2.4.3). This completes the proof of Corollary 2.4.2.

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