Spaces of Algebra Structures and Cohomology of Operads

by

Charles W. Rezk

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Author ....
Department of Mathematics
May 3, 1996

Certified by ....
Michael J. Hopkins
Professor of Mathematics
Thesis Supervisor

Accepted by ....
David A. Vogan
Chairman, Departmental Committee on Graduate Students

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Abstract

The aim of this paper is two-fold. First, we compare two notions of a "space" of algebra structures over an operad A:

1. the **classification space**, which is the nerve of the category of weak equivalences of A-algebras, and

2. the **moduli space** $A\{X\}$, which is the space of maps from A to the endomorphism operad of an object X.

We show that under certain hypotheses the moduli space of A-algebra structures on X is the homotopy fiber of a map between classification spaces.

Second, we address the problem of computing the homotopy type of the moduli space $A\{X\}$. Because this is a mapping space, there is a spectral sequence computing its homotopy groups with E_2-term described by the **Quillen cohomology** of the operad A in coefficients which depend on X. We show that this Quillen cohomology is essentially the same, up to a dimension shift, as the **Hochschild cohomology** of A, and that the Hochschild cohomology may be computed using a "bar construction".

Thesis Supervisor: Michael J. Hopkins
Title: Professor of Mathematics

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Chapter 1

Introduction and statement of results

1.1 Introduction

1.1.1 Operads

Operads and algebras over operads may be viewed as generalizations of the familiar concepts of rings and modules. Thus, let $\mathcal{A}$ denote the category of abelian groups, and let $\otimes$ denote tensor product of abelian groups. Any abelian group A induces a functor $A\otimes - : \mathcal{A} \to \mathcal{A}$ sending an abelian group X to $A\otimes X$. We see that composition of such functors $A \otimes -$ and $B \otimes -$ corresponds to tensoring with the abelian group $A \otimes B$. An **associative ring** A is simply a monoid-object with respect to this tensor product. An A-**module** is an algebra over the induced triple $A \otimes -$ on $\mathcal{A}$.

We can proceed in an analogous fashion when we replace $\mathcal{A}$ with a category of Σ-*objects.* Let $(\mathbf{C}, \otimes)$ be a symmetric monoidal category, such as $(\mathcal{S}, \times)$, the category of of simplicial sets with monoidal product given by cartesian product, or $(\mathcal{M}_{\mathrm{R}}, \otimes_R)$, the category of simplicial R-modules with monoidal product given by tensor product over R. A Σ-**object** on $\mathbf{C}$ is essentially a collection of objects $\{A[n] \in \mathbf{C}\}$ where each $A[n]$ is equipped with an action of the symmetric group Σ_n; we let $\mathbf{\Sigma C}$ denote the category of Σ-objects on $\mathbf{C}$. Each Σ-object A induces a functor $A(-)\colon \mathbf{C} \to \mathbf{C}$ defined by the formula

$$A(X) \simeq \coprod_{n\geq 0} A[n] \otimes_{\Sigma_n} X^{\otimes n}.$$

There is a Σ-object I which induces the identity functor on $\mathbf{C}$.

It turns out that for $A, B \in \mathbf{\Sigma C}$ the composite functor $A(B(-))\colon \mathbf{C} \to \mathbf{C}$ is itself induced by a Σ-object, which we denote by $A\circ B$. The operation $-\circ-\colon \mathbf{\Sigma C} \times \mathbf{\Sigma C} \to \mathbf{\Sigma C}$ makes $\mathbf{\Sigma C}$ into a (non-symmetric) monoidal category with unit I.

An **operad** A on $\mathbf{C}$, is an object of $\mathbf{\Sigma C}$ equipped with maps $A \circ A \to A$ and $I \to A$ making the appropriate associativity and unit diagrams commute; i.e., an operad is simply a "monoid" in $\mathbf{\Sigma C}$. The category of operads on $\mathbf{C}$ is denoted by oper$\mathbf{C}$. Thus any operad A induces a corresponding **triple** (also called a **monad**) on

$\mathbf{C}$.

An A-**algebra** is an object $X \in \mathbf{C}$ equipped with a map $A(X) \to X$ making the appropriate associativity and unit diagrams commute; equivalently, an A-algebra is an algebra over the triple $A(-)$. We refer to such a map $A(X) \to X$ as an A-**algebra structure on** X. The category of A-algebras is denoted by $\mathbf{C}^A$.

1.1.2 Classifying objects for the category of algebras

The introduction of the concept of operads allows us to note that the structure of the category of A-algebras can be recovered entirely in terms of the operad A. We illustrate this first with the example of rings and modules. Given an abelian group M, the group $\mathrm{End}_M = \hom_{\mathcal{A}}(M, M)$ is naturally a ring, and given a ring R there is a natural bijective correspondence

$$\{R\text{-module structures on } M\} \longleftrightarrow \{\text{ring maps } R \to \mathrm{End}_M\}.$$

Likewise, given R-modules M and N, the set of abelian group homomorphisms $\hom_{\mathcal{A}}(M, N)$ has a natural structure as an R-bimodule, and there is a natural bijective correspondence

$$\{R\text{-module maps } M \to N\} \longleftrightarrow \{R\text{-bimodule maps } R \to \hom_{\mathcal{A}}(M, N)\}.$$

These observations have analogies when we replace $\mathcal{A}$ with a category of Σ-objects. Let $\mathbf{C}$ be a symmetric monoidal category as before. Given an object $X \in \mathbf{C}$, there is an **endomorphism operad** $\mathcal{E}_X$ defined by $\mathcal{E}_X[n] = \mathrm{map}_{\mathbf{C}}(X^{\otimes n}, X)$. It has the property that there is a natural bijective correspondence

$$\{A\text{-algebra structures on } X\} \longleftrightarrow \{\text{operad maps } A \to \mathcal{E}_X\},$$

(see 2.1.11 and 2.2.13).

Likewise, given two A algebras X, Y there is a Σ-object $\mathfrak{Hom}(X, Y)$ defined by $\mathfrak{Hom}(X, Y)[n] = \mathrm{map}_{\mathbf{C}}(X^{\otimes n}, Y)$. The A-algebra structures on X and Y induce on $\mathfrak{Hom}(X, Y)$ the structure of an A-**biobject**; i.e., there are maps $A \circ \mathfrak{Hom}(X, Y) \to \mathfrak{Hom}(X, Y)$ and $\mathfrak{Hom}(X, Y) \circ A \to \mathfrak{Hom}(X, Y)$ satisfying obvious properties. It has the property that there is a natural bijective correspondence

$$\{A\text{-algebra maps } X \to Y\} \longleftrightarrow \{A\text{-biobject maps } A \to \mathfrak{Hom}(X, Y)\}$$

(see 2.1.19 and 2.2.15).

The above shows that the category of A-algebras can be understood by looking at appropriate maps from A into certain "classifying objects"; in particular, we can get *derived functors* of these by resolving the operad A. In the case of maps of operads $A \to \mathcal{E}_X$, the relevant derived functor is the *moduli space of A-algebra structures* on X.

1.1.3 Moduli space of algebra structures

Let $\mathbf{C}$ denote either $\mathcal{S}$, the category of simplicial sets, or $\mathcal{M}_{\mathrm{R}}$, the category of simplicial R-modules. These are categories "enriched over simplicial sets"; in particular, for any pair of objects $X, Y \in \mathbf{C}$ there is a simplicial set $\mathrm{map}_{\mathbf{C}}(X, Y)$ having as 0-simplices the maps $X \to Y$ (see 2.3.19). Thus given an operad A over $\mathbf{C}$ we may consider the simplicial set of maps

$$A\{X\} = \mathrm{map}_{\mathbf{C}}(A, \mathcal{E}_X).$$

We call this space the **moduli space of A-algebra structures**.

1.1.4 Classification space

This is another kind of "moduli space" defined entirely in terms of a category $\mathbf{C}$ along with a subcategory of weak equivalences.

Let $\mathbf{C}$ be a category, and let $w\mathbf{C} \subset \mathbf{C}$ denote a subcategory containing all the objects of $\mathbf{C}$, the maps of which we will call "weak equivalences". One defines the **classification space** (1.2.4) to be the simplicial nerve of $w\mathbf{C}$. The classification space may be viewed as a "moduli space of objects of $\mathbf{C}$"; in particular, the components of $w\mathbf{C}$ are in one-to-one correspondence with weak equivalence classes of objects in $\mathbf{C}$. If $\mathbf{C}$ is in fact a simplicial closed model category (see 3.1.1), then the component of $w\mathbf{C}$ which contains an object $X \in \mathbf{C}$ has the homotopy type of the classifying space of the monoid of self-equivalences of X (see Proposition 1.2.6).

Clearly, if $\pi\colon \mathbf{C} \to \mathbf{D}$ is a functor between such categories which preserves all weak equivalences, then there is an induced map $w\mathbf{C} \to w\mathbf{D}$ of classification spaces.

We will introduce simplicial model category structures on the categories of A-algebras and the category of operads over simplicial sets or simplicial R-modules, allowing us to state and prove the following theorem.

1.1.5. Theorem. *Let $\mathbf{C}$ denote the category $\mathcal{S}$ or the category $\mathcal{M}_{\mathrm{R}}$. Let A be a cofibrant operad, and let X be a fibrant and cofibrant object of $\mathbf{C}$. Then the moduli space $A\{X\}$ is naturally weakly equivalent to the homotopy fiber over X of the map of classification spaces $N(w\mathbf{C}^A) \to N(w\mathbf{C})$.*

This result is discussed in more detail in Section 1.2.

1.1.6 Quillen cohomology of operads

Recall that the moduli space $A\{X\}$ of algebra structures is a space of maps between operads. If A is a cofibrant operad and $X \in \mathbf{C}$ is a fibrant and cofibrant object (whence $\mathcal{E}_X$ is a fibrant operad), and $f\colon A \to \mathcal{E}_X$ is a given map, then there is a second quadrant spectral sequence of the form

$$E_2^{st} \simeq H^s_{\mathrm{oper}}\big(A, \pi_t(\mathcal{E}_X; f)\big) \Rightarrow \pi_{t-s}(A\{X\}; f),$$

where $H^s_{\mathrm{oper}}(A, U)$ denotes the **Quillen cohomology of the operad** A with coefficients in an **abelian group object** U over A, and $\pi_t(\mathcal{E}_X; f)$ is the abelian group object whose fiber over a point $a \in A[n]$ is given by

$$\pi_t(\mathcal{E}_X; f)_a \simeq \pi_t\big(\mathrm{map}_{\mathbf{C}}(X^n, X); f(a)\big).$$

An instance of this kind of spectral sequence is described by Robinson [16].

Quillen cohomology of an operad A is defined by resolving A by a cofibrant operad. Such cofibrant resolutions of operads are difficult to compute with, and so it is useful to have another description of Quillen cohomology.

1.1.7 Hochschild cohomology of operads

Because operads are "monoids" in the monoidal category $\Sigma\mathbf{C}$, we define an **A-biobject** to be a Σ-object M with a two-sided action by the monoid A. We may consider the **Quillen cohomology of an A-biobject** M. A special example of an A-biobject is A itself. Thus we may define $\mathrm{Hoch}^s(A, K)$, the **Hochschild cohomology of an operad** A with coefficients in an abelian group object U over A, to be $H^s_{\mathsf{Abiobj}}(A, U)$, the Quillen cohomology of A as an A-biobject.

1.1.8. Theorem. *Let $\mathbf{C}$ denote the category $\mathcal{M}_{\mathrm{R}}$ (and hopefully the category $\mathcal{S}$). Let $A \in \mathrm{oper}\mathbf{C}$ be an operad with cofibrant underlying Σ-object, and let U be an abelian group object over A. Then there is a natural exact sequence*

$$0 \to \mathrm{Hoch}^0(A, U) \to \mathrm{Hoch}^0(I, \eta^* U) \to H^0_{\mathsf{oper}}(A, U) \to \mathrm{Hoch}^1(A, U) \to 0$$

and for $n \geq 1$ natural isomorphisms

$$\mathrm{Hoch}^{n+1}(A, U) \simeq H^n_{\mathsf{oper}}(A, U).$$

(Here I denotes the trivial operad, and $\eta^* U$ is the pull-back of the coefficient system U along the unit map $\eta\colon I \to A$.)

Furthermore, the cohomology of a Σ-cofibrant operad may be computed using a "bar complex"; the bar complex $\mathcal{B}(A, A, A)$ of an operad A is a simplicial object whose n-th degree term is the A-biobject $A^{\circ(n+2)}$; see 3.7.1. (This is not the same as the Ginzburg-Kapranov bar complex of an operad of [8].)

Finally, we note that the argument which proves Theorem 1.1.8 is largely formal. We call a monoidal category $\mathbf{M}$ **right-closed** (2.1.9) if the monoidal product $- \circ -$ has a "right adjoint on one side"; i.e., if for each $A \in \mathbf{M}$ the functor $B \mapsto B \circ A$ has a right adjoint. Given a monoid A in $\mathbf{M}$ and mild conditions on the category $\mathbf{M}$, we will show (Proposition 5.2.14) that there is a short exact sequence of abelian group objects

$$0 \to \mathcal{D}(A) \to \mathcal{A}_b(A \circ A) \to \mathcal{A}_b(A) \to 0,$$

where $\mathcal{D}(A)$ denotes the "abelianization" of the operad A and $\mathcal{A}_b(M)$ denotes the "abelianization" of an A-biobject M. This short exact sequence is a key step in the proof of Theorem 1.1.8.

1.1.9 Notation

We will use $\mathcal{S}$ to denote the category of simplicial sets, and will use $\mathcal{M}_R$ to denote the category of *simplicial* modules over a commutative ring R.

For a category $\mathbf{C}$ and objects $X, Y \in \mathbf{C}$ we write $\mathbf{C}[X, Y]$ to denote the set of maps from X to Y in $\mathbf{C}$.

Given categories $\mathbf{C}$, $\mathbf{D}$, we write $\mathbf{C}^{\mathbf{D}}$ for the *category of functors* $\mathbf{D} \to \mathbf{C}$.

Given a functor $X\colon \mathbf{D} \to \mathbf{C}$ from a small category $\mathbf{D}$ we write $\operatorname{colim}^{\mathbf{D}} X$ for the colimit of this functor. If $\mathbf{D} = G$ is a group, we write $X^G \in \mathbf{C}$ for the limit and $X_G \in \mathbf{C}$ for the colimit.

Given a functor $\pi\colon \mathbf{D} \to \mathbf{C}$ and an object $X \in \mathbf{C}$ we let $(\pi \downarrow X)$ denote the **over category**, having as objects pairs $(Y \in \mathbf{D}, f\colon \pi Y \to X \in \mathbf{C})$ and as maps $(Y, f) \to (Y', f')$ morphisms $g\colon Y \to Y' \in \mathbf{D}$ such that $f'g = f$. If $\mathbf{C} = \mathbf{D}$ and $\pi = 1_{\mathbf{C}}$ we write $(\mathbf{C} \downarrow X)$.

1.1.10 Looking ahead

In this paper we consider only operads acting on simplicial sets or on simplicial modules. It is desirable to extend this theory to a category of spectra; however, it seems some care is needed in order to carry this out. For example, the category of operads on a universe, as described by Lewis-May-Steinberger [11], does not admit an action by a category of Σ-objects; rather, it is Σ-objects over a certain fixed operad which act on these spectra. Once these differences are worked out, the theory proceeds similarly.

1.1.11 Organization of the paper

In Section 1.2 we describe Theorem 1.1.5 and some of its consequences. In Section 1.3 we do the same for Theorem 1.1.8.

In Sections 2.1 through 2.4 we establish the categories in which we work. In Section 2.1 we define the notion of a right-closed monoidal category. In Section 2.2 we define the notion of an operad, along with the related notions of algebras over an operad, Σ-objects, and biobjects over an operad; we also define certain enriched hom-objects. In Section 2.3 we show that various categories are complete and cocomplete. In Section 2.4 we discuss monoidal functors.

In Sections 3.1 through 3.7 we study the homotopy theory of the categories in question. In Section 3.1 we recall the notion of a closed model category. In Sections 3.2 and 3.3 we show that in the contexts of simplicial sets or simplicial modules, all of these categories can be given the structure of a Quillen simplicial closed model category. In Section 3.4 we show that monoidal structures and enriched hom-objects are compatible with the closed model category structures as best they can be. In Section 3.5 we examine when objects such as operads or A-algebras are cofibrant in some underlying category. In Section 3.6 we show that weakly equivalent operads which are cofibrant in an underlying category generate categories of algebras with equivalent homotopy theories. In Section 3.7 we show that the "standard resolutions" of an A-algebra, or of A as an A-biobject, are cofibrant.

In Sections 4.1 and 4.2 we describe the relationship between the moduli space of algebra structures and classification spaces. In Section 4.1 we define the moduli space of algebra structures of a diagram, and in Section 4.2 we prove Theorem 1.1.5.

In Sections 5.1 through 5.4 we define and compare two notions of cohomology of operads. In Section 5.1 we describe categories of abelian group objects and show that the relevant abelianization functors exist. In Section 5.2 we demonstrate an exact sequence relating the abelianizations of operads and biobjects. In Section 5.3 we show that these categories of abelian group objects admit a Quillen simplicial closed model category structure. In Section 5.4 we use the results of the previous sections to prove the equivalence of Quillen and Hochschild cohomologies of operads.

In Appendix A we discuss the construction of the free operad.

1.2 Moduli spaces of algebra structures

In this section we define for each simplicial set X and each operad A over simplicial sets a **structure space**. We then state a theorem relating this structure space, under certain hypotheses, to the **classification spaces** of the categories of simplicial sets and of A-algebras. We then state analogous results concerning operads over simplicial R-modules.

1.2.1 Operads over simplicial sets and their algebras

The category $\mathcal{S}$ of simplicial sets is a symmetric monoidal category via cartesian product (2.2.1). As described in Section 2.2, we can therefore define in $\mathcal{S}$ the notion of an operad. In brief, an operad over $\mathcal{S}$ is a collection of simplicial sets $\{A[n]\}$ such that $A[n]$ is equipped with a Σ_n-action, and such that certain other structure maps are present which make the functor

$$X \mapsto A(X) = \coprod_{n=0} A[n] \times_{\Sigma_n} X^n$$

into a triple on $\mathcal{S}$ (for the full definition, see 2.2.9). Any operad over $\mathcal{S}$ thus determines a category of algebras (2.2.13), denoted by $\mathcal{S}^A$.

Each such category $\mathcal{S}^A$ will be given the structure of a Quillen closed model category, in which the weak equivalences are maps $f\colon X \to Y$ of algebras which are weak equivalences on the underlying simplicial sets (3.2.4). Thus we obtain a homotopy category $\operatorname{Ho}\mathcal{S}^A$ of A-algebras by formally inverting the weak equivalences.

Any map of operads $f\colon A \to B$ determines a pair of adjoint functors

$$f_* : \mathcal{S}^A \rightleftarrows \mathcal{S}^B : f^*$$

where the right adjoint forgets structure, but preserves the underlying simplicial set

(2.3.10). These functors induce by Corollary 3.6.3 a pair of adjoint functors

$$\operatorname{Ho}\mathcal{S}^A \rightleftarrows \operatorname{Ho}\mathcal{S}^B$$

between the corresponding homotopy categories.

It seems appropriate to define a notion of weak equivalence of operads for which weakly equivalent operads would have equivalent homotopy categories of algebras. Thus we say a map of operads $f\colon A \to B$ is a **weak equivalence** if for each $n \geq 0$ the geometric realization $|f[n]|\colon |A[n]| \to |B[n]|$ is a Σ_n-equivariant homotopy equivalence (i.e., there is a Σ_n-equivariant map $g\colon |B[n]| \to |A[n]|$ and Σ_n-equivariant homotopies $|f[n]|g \sim 1$ and $g|f[n]| \sim 1$.) Then we obtain the following.

1.2.2. Proposition. *A weak equivalence $A \to B$ of operads over $\mathcal{S}$ induces an equivalence* $\operatorname{Ho}\mathcal{S}^A \simeq \operatorname{Ho}\mathcal{S}^B$.

Proof. This follows from Corollary 3.6.5 using Proposition 3.5.1. □

1.2.3 The structure space

The category of operads on $\mathcal{S}$ is **enriched over simplicial sets**; thus, for any $A, B \in \operatorname{oper}\mathcal{S}$ there is a simplicial set $\operatorname{map}_{\operatorname{oper}\mathcal{S}}(A, B)$, in which the 0-simplices correspond to the usual maps from A to B.

The set of A-algebra structures on a simplicial set X is in bijective correspondence with the set of operad maps $A \to \mathcal{E}_X$, where $\mathcal{E}_X$ denotes the endomorphism operad of X. Thus we define the **moduli space of algebra structures** by

$$A\{X\} = \operatorname{map}_{\operatorname{oper}\mathcal{S}}(A, \mathcal{E}_X).$$

Note that this construction defines a functor $\operatorname{oper}\mathcal{S}^{\mathrm{op}} \to \mathcal{S}$ sending $A \mapsto A\{X\}$; note that $A\{X\}$ *is not a functor of* X.

We note that an n-simplex of $A\{X\}$ corresponds precisely to a map of operads $\Delta[n] \otimes A \to \mathcal{E}_X$, where $K \otimes A$ is the "tensor product" of a simplicial set K with an operad A (see 2.3.19), and $\Delta[n]$ is the standard n-simplex. The map $\Delta[n] \to \Delta[0]$ of simplicial sets is a simplicial homotopy equivalence, and it follows that the map $\Delta[n] \otimes A \to \Delta[0] \otimes A \simeq A$ is a weak equivalence of operads.

1.2.4 The classification space

Let $\mathbf{C}$ be a simplicial model category (3.1.1). Both $\mathcal{S}$ and $\mathcal{S}^A$ are simplicial model categories, by Propositions 3.1.7 and 3.2.5. Let $w\mathbf{C} \subset \mathbf{C}$ denote the subcategory consisting of all objects and all weak-equivalences between them. We call the simplicial nerve of $w\mathbf{C}$ the **classification space** of $\mathbf{C}$. By abuse of notation we will write $w\mathbf{C}$ for its simplicial nerve; a t-simplex in $(w\mathbf{C})_t$ is a chain of t composable maps.

1.2.5. Remark. Note that since $\mathbf{C}$ is not a *small* category, $w\mathbf{C}$ is not a simplicial set, but merely a simplicial class. We take the position that $\mathbf{C}$ is a category defined in some Grothendieck universe of sets (see [10]) $\mathcal{U}$, and that we can regard $w\mathbf{C}$ as a simplicial set in some higher universe $\mathcal{U}'$.

Dwyer-Kan prove the following results:

1.2.6. Proposition. *[3, 4.6], [4, 2.3]. Given* $\mathbf{C}$ *as above we have that:*

1. *The components of* $w\mathbf{C}$ *are in one-to-one correspondence with the weak equivalence classes of objects in* $\mathbf{C}$.
2. *Each component of* $w\mathbf{C}$ *is* ***homotopically small****; i.e., has the homotopy type of a small simplicial set.*
3. *If* $X \in \mathbf{C}$ *is a fibrant and cofibrant object, then the component of* $w\mathbf{C}$ *containing* X *has the weak homotopy type of the classifying space* $B\,\mathrm{haut}(X)$ *of the simplicial monoid* $\mathrm{haut}(X) \subseteq \mathrm{map}_{\mathbf{C}}(X, X)$ *of self-homotopy equivalences of* X.

More succinctly, we may write

$$w\mathbf{C} \simeq \coprod_{\text{ho. types } [X]} B\,\mathrm{haut}_{\mathbf{C}}(X).$$

If $\pi\colon \mathbf{C} \to \mathbf{D}$ is a functor which preserves weak equivalences, then it induces a map $w\pi\colon w\mathbf{C} \to w\mathbf{D}$ of classification spaces. If π is part of a Quillen equivalence, then the induced map $w\pi$ is a weak equivalence. Thus, we have the following:

1.2.7. Proposition. *If* $f\colon A \to B$ *is a map of operads, there is an induced map* $w\mathcal{S}^B \to w\mathcal{S}^A$ *of classification spaces; furthermore, if* f *is a weak equivalence of operads, then the induced map on classification spaces is one as well.*

1.2.8 Comparison between structure space and classification space

We can relate $A\{X\}$ and $w\mathcal{S}^A$ as follows. Let $w\mathcal{S}^{\Delta[-]\otimes A}$ denote the bisimplicial set having $(w\mathcal{S}^{\Delta[t]\otimes A})_s$ as its (s,t)-simplices. There are maps $w\mathcal{S}^A \to w\mathcal{S}^{\Delta[t]\otimes A}$ induced by the map $\Delta[t] \otimes A \to \Delta[0] \otimes A \simeq A$ of operads, and maps $A\{X\} \to (w\mathcal{S}^{\Delta[-]\otimes A})_0$ which send an n-simplex of of $A\{X\}$, considered as a map $f\colon \Delta[t] \otimes A \to \mathcal{E}_X$, to the corresponding object (X, f) in $\mathcal{S}^{\Delta[t]\otimes A}$.

Thus we obtain a diagram of simplicial sets

$$\begin{tikzcd}
A\{X\} \arrow[r, "i"] \arrow[d] & \mathrm{diag}(w\mathcal{S}^{\Delta[-]\otimes A}) \arrow[d, "j"] & w\mathcal{S}^A \arrow[l, "k"', "\sim"] \arrow[d, "\pi"] \\
\mathrm{pt} \simeq \{X\} \arrow[r] & w\mathcal{S} \arrow[r, equal] & w\mathcal{S},
\end{tikzcd}$$

where k is a weak equivalence by Proposition 1.2.7. Furthermore, i is exactly inclusion of the fiber of j over the vertex in $w\mathcal{S}$ corresponding to the object X.

Since k is a weak equivalence the top row gives a map $A\{X\} \to w\mathcal{S}^A$ in the homotopy category of simplicial sets.

1.2.9 Main theorem

To state our theorem, we need another definition. An operad A over $\mathcal{S}$ is said to be **cofibrant** if it is a retract of one for which there exists a collection $S_t \subseteq A_t$ of Σ-objects over sets which are closed under degeneracy maps, and isomorphisms $FS_t \xrightarrow{\sim} A_t$, where FS_t denotes the **free operad** on S_t (see 2.3.6).

1.2.10. Theorem. *Let $A \in \operatorname{oper}\mathcal{S}$ be a cofibrant operad over $\mathcal{S}$, and let $X \in \mathcal{S}$ be a Kan complex. Then the square*

$$\begin{tikzcd} A\{X\} \arrow[r] \arrow[d] & w\mathcal{S}^A \arrow[d] \\ \mathrm{pt} \arrow[r] & w\mathcal{S} \end{tikzcd}$$

is a homotopy pull-back square of simplicial sets.

1.2.11. Remark. 1. By Proposition 1.2.6, if we restrict the maps in this fiber square to the component of $w\mathcal{S}$ containing X, we obtain a homotopy pull-back square

$$\begin{tikzcd} A\{X\} \arrow[r] \arrow[d] & \coprod_{[Y],\pi Y \simeq X} B\,\mathrm{haut}_{\mathcal{S}^A}(X) \arrow[d] \\ \mathrm{pt} \arrow[r] & B\,\mathrm{haut}_{\mathcal{S}}(X), \end{tikzcd}$$

where the coproduct in the upper-right corner is indexed over representatives of weak-equivalence-types of A-algebras whose underlying simplicial sets are weakly equivalent to X. We can think of this square as defining a $\mathrm{haut}_{\mathcal{S}}(X)$-action on $A\{X\}$ up to homotopy, and we can think of $\coprod B\,\mathrm{haut}_{\mathcal{S}^A}(X)$ as the homotopy quotient by this action.

2. By Proposition 3.2.11 operads over $\mathcal{S}$ are a closed model category having cofibrant objects as above, and weak equivalences as described in 1.2.1. Thus for any $A \in \operatorname{oper}\mathcal{S}$ there exists a weak equivalence $B \to A \in \operatorname{oper}\mathcal{S}$ from a cofibrant operad, whence $\operatorname{Ho}\mathcal{S}^B \simeq \operatorname{Ho}\mathcal{S}^A$. Thus, even if A is not cofibrant, there is a homotopy pull-back square

$$\begin{tikzcd} B\{X\} \arrow[r] \arrow[d] & w\mathcal{S}^A \arrow[d] \\ \mathrm{pt} \arrow[r] & w\mathcal{S}. \end{tikzcd}$$

There is a "relative" version of Theorem 1.2.10:

1.2.12. Corollary. *Let $A \to B \in \operatorname{oper}\mathcal{S}$ be a map between cofibrant of operads over*

$\mathcal{S}$, *and let* $X \in \mathcal{S}$ *be a Kan complex. Then the square*

$$\begin{array}{ccc} B\{X\} & \longrightarrow & w\mathcal{S}^B \\ \downarrow & & \downarrow \\ A\{X\} & \longrightarrow & w\mathcal{S}^A \end{array}$$

is a homotopy pull-back square in the homotopy category of simplicial sets.

1.2.13 Corresponding results over simplicial R-modules

Let R be a commutative ring, and let $\mathcal{M}_{\mathrm{R}}$ denote the category of simplicial R-modules. As before, we can define the category of **operads** over $\mathcal{M}_{\mathrm{R}}$, denoted by $\mathrm{oper}\mathcal{M}_{\mathrm{R}}$, and their categories of algebras, denoted by ${\mathcal{M}_{\mathrm{R}}}^A$ for $A \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$. We can obtain similar results in this setting, but we need to be more careful.

We say an A-algebra X over $\mathcal{M}_{\mathrm{R}}$ is **$\mathcal{M}_{\mathrm{R}}$-cofibrant** if the underlying simplicial R-module of X is projective in each dimension. We say an operad A over $\mathcal{M}_{\mathrm{R}}$ is **Σ-cofibrant** if in each degree $A[n]$ is a retract of some RS, a free R-module on a Σ_n-set S. Clearly, if R is a field, then every A-algebra is $\mathcal{M}_{\mathrm{R}}$-cofibrant, and if R is a field of characteristic zero, then every operad A is Σ-cofibrant.

The statements of Sections 1.2.1, 1.2.3, 1.2.8, and 1.2.9 hold with $\mathcal{S}$ replaced by $\mathcal{M}_{\mathrm{R}}$, with the following exceptions:

We say that a map $A \to B \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ is a **weak equivalence** if for each subgroup $H \subseteq \Sigma_n$ the map of fixed point sets $A[n]^H \to B[n]^H$ is a weak equivalence of simplicial sets. Then we obtain

1.2.14. Proposition. *A weak equivalence* $A \to B \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ *of operads over* $\mathcal{M}_{\mathrm{R}}$ *which are* Σ*-cofibrant induces an equivalence* $\mathrm{Ho}\,{\mathcal{M}_{\mathrm{R}}}^A \simeq \mathrm{Ho}\,{\mathcal{M}_{\mathrm{R}}}^B$.

Proof. This follows from Corollary 3.6.5 when we take $\mathbf{C} = \mathcal{M}_{\mathrm{R}}$. □

Thus the corresponding statement of the main theorem is

1.2.15. Theorem. *Let* $A \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ *be a cofibrant operad over* $\mathcal{M}_{\mathrm{R}}$*, and let* $X \in \mathcal{M}_{\mathrm{R}}$ *be a degree-wise projective object. Then the square*

$$\begin{array}{ccc} A\{X\} & \longrightarrow & w\mathcal{S}^A \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & w\mathcal{S} \end{array}$$

is a homotopy pull-back square in the homotopy category of simplicial sets.

1.3 Cohomology of operads

In this section we define the notions of Quillen cohomology of operads and Hochschild cohomology of operads in the category of simplicial R-modules. We then state a theorem which states that, under mild hypotheses, these cohomology theories are essentially the same up to a dimension shift.

1.3.1 Coefficient systems

Let A be an operad in the category of R-modules. A **coefficient system** over A is an abelian group object in the category of operads over A. Such an object consists of maps of operads $\pi \colon B \to A$, $0 \colon A \to B$, $- \colon B \to B$ and $+ \colon B \times_A B \to B$ satisfying an obvious set of relations. In particular $\pi 0 = 1_A$, whence B has the form $B[n] = A[n] \oplus U[n]$ for some Σ-object U; this U is a square-zero ideal of the operad B. We shall use the symbol U to denote the coefficient system.

Given such a coefficient system U, the ***t*-th Eilenberg-MacLane object** $K(U,t)$ is an operad in the category of simplicial R-modules whose n-th space is defined by

$$K(U,t)[n] = A[n] \oplus \bar{W}^t U[n],$$

where $\bar{W}$ is the construction of Eilenberg-Mac Lane (see [12, IV]).

More generally, if A is an operad in the category of *simplicial* R-modules, then given a coefficient system U over $\pi_0(A)$ we obtain Eilenberg-MacLane objects $K(U,t)$ over A by pulling back.

1.3.2 Quillen cohomology of operads

The ***t*-th Quillen cohomology group** $H^t_{\text{oper}}(A,U)$ **of an operad** A with coefficients in a coefficient system is defined to be the abelian group of maps $A \to K(U,t)$ in the homotopy category $\operatorname{Ho}(\text{oper}\mathcal{M}_{\text{R}} \downarrow A)$ of operads over A. This is just a special case of the general formulation of Quillen [13].

1.3.3 Biobjects and a bar complex

Recall that an operad in $\mathcal{M}_{\text{R}}$ is a monoid in the monoidal category $\Sigma\mathcal{M}_{\text{R}}$ (2.2.2). A **biobject** M of A is a Σ-object equipped with a two-sided action by A (2.2.15). A map of operads $f \colon A \to B$ gives B the structure of an A-biobject. Thus in particular, A is itself an A-biobject, and furthermore via the "zero-section" $A \to K(U,t)$ every Eilenberg-MacLane object becomes an A-biobject.

The **bar complex** $\mathcal{B}(A,A,A)$ of an operad A is a simplicial object in the category of A-biobjects, with

$$\mathcal{B}(A,A,A)_n = A^{\circ n}.$$

The simplicial structure maps are the obvious ones induced by the unit $I \to A$ and multiplication $A \circ A \to A$ maps.

1.3.4 Quillen cohomology of biobjects

Let A be an operad in $\mathcal{M}_{\text{R}}$. The ***t*-th Quillen cohomology group** $H^t_{A\text{-biobj}}(M,U)$ **of an** A**-biobject** M **with coefficients in** U is defined to be the abelian group of maps $M \to K(U,t)$ in the homotopy category $\mathrm{Ho}\,(A\text{-biobj}\mathcal{M}_{\mathrm{R}} \downarrow M)$ of A-biobjects over M. Here $K(U,t)$ is an Eilenberg-MacLane object, i.e., an A-biobject in the category of simplicial R-modules whose n-th space is defined by

$$K(U,t)[n] = M[n] \oplus \bar{W}^t U[n],$$

where $\bar{W}$ is the construction of Eilenberg-Mac Lane (see [12, IV]).

1.3.5 Hochschild cohomology of operads

Let A be an operad, and U a coefficient system over A as in 1.3.1. As noted in 1.3.3, A has the structure of an A-bimodule, as do the Eilenberg-MacLane objects $K(U,t)$. Thus we define the **t-th Hochschild cohomology group** $\mathrm{Hoch}^t(A,U)$ **of an operad A with coefficients in U** by

$$\mathrm{Hoch}^t(A,U) = H^t_{A\text{-biobj}}(A,U).$$

If A is a Σ-cofibrant operad, then its Hochschild cohomology can be computed by means of a bar construction.

1.3.6. Proposition. *Let A be a Σ-cofibrant operad over $\mathcal{M}_{\mathrm{R}}$. Then*

$$\mathrm{Hoch}^t(A,U) \simeq H^t\big(A\text{-biobj}\mathcal{M}_{\mathrm{R}}(A^{\circ *+2}, A \oplus U), \partial\big),$$

the cohomology of a cochain complex with differential

$$\partial\colon A\text{-biobj}\mathcal{M}_{\mathrm{R}}(A^{\circ s}, A \oplus U) \to A\text{-biobj}\mathcal{M}_{\mathrm{R}}(A^{\circ s+1}, A \oplus U)$$

induced by the alternating sum of the face maps of $\mathcal{B}(A,A,A)$.

Proof. This is an immediate consequence of the definition of Hochschild cohomology and Proposition 5.4.3. □

1.3.7 Equivalence of Quillen and Hochschild cohomology

We will prove the following theorem in Section 1.3.

1.3.8. Theorem. *Let A be an operad in the category of R-modules which is Σ-cofibrant. Let U be a coefficient system on A. Then there are natural isomorphisms*

$$\mathrm{Hoch}^{t+1}(A,U) \simeq H^t_{\mathrm{oper}}(A,U)$$

for $t \geq 1$, and a natural exact sequence

$$0 \to \mathrm{Hoch}^0(A,U) \to U[1] \to H^0_{\mathrm{oper}}(A,U) \to \mathrm{Hoch}^1(A,U) \to 0.$$

Chapter 2

Operads and related categories

2.1 Monoidal categories

In this section we discuss the notion of a *right-closed monoidal* category, and the related notion of a *right-closed action.* These notions provide a convenient language for our discussion of Σ-objects and operads in Sections 2.2 and 2.3.

2.1.1 Monoidal categories

A **monoidal category** $\mathbf{M}$, as defined in [10, VII], is a category equipped with functors

$$\eta\colon \mathbf{1} \to \mathbf{M}$$

and

$$\mu\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$$

along with natural isomorphisms

$$a\colon \mu(1_{\mathbf{M}} \times \mu) \xrightarrow{\sim} \mu(\mu \times 1_{\mathbf{M}})\colon \mathbf{M} \times \mathbf{M} \times \mathbf{M} \to \mathbf{M},$$

$$\ell\colon \mu(\eta \times 1_{\mathbf{M}}) \xrightarrow{\sim} 1_{\mathbf{M}}\colon \mathbf{M} = \mathbf{1} \times \mathbf{M} \to \mathbf{M},$$

and

$$r\colon \mu(1_{\mathbf{M}} \times \eta) \xrightarrow{\sim} 1_{\mathbf{M}}\colon \mathbf{M} = \mathbf{M} \times \mathbf{1} \to \mathbf{M},$$

such that the following diagrams commute,

$$\begin{tikzcd}
 & \mu(1 \times \mu(1 \times \mu)) \arrow[r, "\mu(1\times a)"] \arrow[dl, "a(1\times 1\times\mu)"'] & \mu(1 \times \mu(\mu \times 1)) \arrow[dd, "a(1\times\mu\times 1)"] \\
\mu(\mu \times \mu) \arrow[dr, "a(\mu\times 1\times 1)"'] & & \\
 & \mu(\mu(\mu \times 1) \times 1) & \mu(\mu(1 \times \mu) \times 1) \arrow[l, "\mu(a\times 1)"]
\end{tikzcd},$$

$$
\begin{tikzcd}
\mu(1\times\mu(\eta\times 1)) \arrow[rr, "a(1\times\eta\times 1)"] \arrow[dr, "\mu(1\times\ell)"'] & & \mu(\mu(1\times\eta)\times 1) \arrow[dl, "\mu(r\times 1)"] \\
& \mu, &
\end{tikzcd}
$$

and such that

$$\ell\eta = r\eta\colon \mathbf{1} = \mathbf{1}\times\mathbf{1} \to \mathbf{M}.$$

We will generally write $I \in \mathbf{M}$ for the object which is the image of the functor η, and will write $A \circ B$ for $\mu(A, B)$. We usually suppress mention of the structural isomorphisms a, ℓ, and r; thus we write $A \circ B \circ C$ for the object naturally isomorphic to $A \circ (B \circ C)$.

2.1.2 Actions by monoidal categories

An **action** of a monoidal category $\mathbf{M}$ on a category $\mathbf{C}$ is a functor

$$\psi\colon \mathbf{M}\times\mathbf{C} \to \mathbf{C}$$

along with natural isomorphisms

$$b\colon \psi(\mu\times 1_{\mathbf{C}}) \xrightarrow{\sim} \psi(1_{\mathbf{M}}\times\psi)\colon \mathbf{M}\times\mathbf{M}\times\mathbf{C} \to \mathbf{C}$$

and

$$u\colon \psi(\eta\times 1_{\mathbf{C}}) \xrightarrow{\sim} 1_{\mathbf{C}}\colon \mathbf{C} = \mathbf{1}\times\mathbf{C} \to \mathbf{C}$$

making the following diagrams commute:

$$
\begin{tikzcd}
& \psi(1\times\psi(1\times\psi)) \arrow[r, "\mu(1\times b)"] \arrow[dl, "b(1\times 1\times\psi)"'] & \psi(1\times\psi(\mu\times 1)) \arrow[dd, "b(1\times\mu\times 1)"] \\
\psi(\mu\times\psi) \arrow[dr, "b(\mu\times 1\times 1)"'] & & \\
& \psi(\mu(\mu\times 1)\times 1) & \psi(\mu(1\times\mu)\times 1), \arrow[l, "\psi(a\times 1)"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\psi(1\times\psi(\eta\times 1)) \arrow[rr, "b(1\times\eta\times 1)"] \arrow[dr, "\psi(1\times u)"'] & & \psi(\mu(1\times\eta)\times 1) \arrow[dl, "\psi(r\times 1)"] \\
& \psi. &
\end{tikzcd}
$$

We will generally write $A \circ (X)$ or even $A(X)$ for $\psi(A, X)$. We usually suppress mention of the structure isomorphisms b and u; thus we write $A \circ B \circ (X)$ or $AB(X)$ for the object naturally isomorphic to $A \circ \big(B \circ (X)\big)$.

2.1.3. Example. Let $\mathbf{C}$ be a category, and let $\mathbf{M} = \mathbf{C}^{\mathbf{C}}$ the category of endofunctors on $\mathbf{C}$. Then $\mathbf{M}$ is a monoidal category under composition of functors, and there is an action of $\mathbf{M}$ on $\mathbf{C}$ induced by evaluation of functors. In particular, given $A\colon \mathbf{C} \to \mathbf{C}$ and $X \in \mathbf{C}$, we see that $A \circ X$ is just evaluation of the functor A at X.

2.1.4. Remark. It is not hard to show that an action of $\mathbf{M}$ on $\mathbf{C}$ is equivalent to giving a monoidal functor (see Section 2.4) $\mathbf{M} \to \mathbf{C}^{\mathbf{C}}$ where $\mathbf{C}^{\mathbf{C}}$ denotes the category of functors from $\mathbf{C}$ to $\mathbf{C}$.

2.1.5 Right-closed action

A **right-closed action** of a monoidal category $\mathbf{M}$ on a category $\mathbf{C}$ is an action such that for each $X \in \mathbf{C}$ the functor $- \circ X\colon \mathbf{M} \to \mathbf{C}$ has a right adjoint $\mathfrak{Hom}(X, -)\colon \mathbf{C} \to \mathbf{M}$. These right adjoints fit together to give a functor

$$\mathfrak{Hom}(-,-)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{M}$$

inducing for $X, Y \in \mathbf{C}$ and $A \in \mathbf{M}$ a natural isomorphism

$$\mathbf{C}[A \circ X, Y] \simeq \mathbf{M}[A, \mathfrak{Hom}(X, Y)]. \tag{2.1.6}$$

Furthermore, for $X, Y, Z \in \mathbf{C}$ there is a natural "unit" map

$$I \to \mathfrak{Hom}(X, X)$$

which is adjoint to the identity on X, and a natural "composition" map

$$\mathfrak{Hom}(Y, Z) \circ \mathfrak{Hom}(X, Y) \to \mathfrak{Hom}(X, Z)$$

which is induced by "evaluation" maps $\mathfrak{Hom}(X, Y) \circ X \to Y$ and $\mathfrak{Hom}(Y, Z) \circ Y \to Z$ which are adjoint to the identity on $\mathfrak{Hom}(X, Y)$ and $\mathfrak{Hom}(Y, Z)$ respectively.

2.1.7. Proposition. *Given a right-closed action of* $\mathbf{M}$ *on* $\mathbf{C}$ *and an object* $X \in \mathbf{C}$, *the functor* $- \circ X\colon \mathbf{M} \to \mathbf{C}$ *preserves colimits whenever they exist, and the functor* $\mathfrak{Hom}(X, -)\colon \mathbf{C} \to \mathbf{M}$ *preserves limits whenever they exist.*

Proof. Straightforward. $\square$

2.1.8. Remark. Note, however, that for $X \in \mathbf{C}$ the functor $X \circ -$ need not preserve colimits, and the functor $\mathfrak{Hom}(-, X)$ need not take colimits to limits.

2.1.9 Right-closed monoidal categories

A **right-closed monoidal category** is a monoidal category $\mathbf{M}$ such that the action of $\mathbf{M}$ on itself induced by the monoidal structure is right-closed. This means that there is a functor

$$\mathcal{F}(-,-)\colon \mathbf{M}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{M}$$

inducing for $A, B, C \in \mathbf{M}$ a natural isomorphism

$$\mathbf{M}[A \circ B, C] \simeq \mathbf{M}[A, \mathcal{F}(B, C)].$$

2.1.10. Remark. There are corresponding notions of **left-closed actions** and **left-closed monoidal categories.**

2.1.11 Monoids and algebras

A **monoid** in a monoidal category $\mathbf{M}$ is an object $A \in \mathbf{M}$ along with maps $\mu\colon A \circ A \to A \in \mathbf{M}$ and $\eta\colon I \to A \in \mathbf{M}$ making the following diagrams commute.

$$\begin{tikzcd} A \circ A \circ A \arrow[r, "\mu \circ 1"] \arrow[d, "1 \circ \mu"'] & A \circ A \arrow[d, "\mu"] \\ A \circ A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} I \circ A \arrow[r, "\eta \circ 1"] \arrow[dr, equal] & A \circ A \arrow[d, "\mu"] & A \circ I \arrow[l, "1 \circ \eta"'] \arrow[dl, equal] \\ & A. & \end{tikzcd}$$

Let $\mathrm{mon}\mathbf{M}$ denote the category of monoids in $\mathbf{M}$.

Suppose $\mathbf{M}$ acts on $\mathbf{C}$. Then an **algebra** over the monoid A is an object $X \in \mathbf{C}$ equipped with a map $\psi\colon A \circ X \to X \in \mathbf{C}$ making the following diagrams commute.

$$\begin{tikzcd} A \circ A \circ X \arrow[r, "\mu \circ 1"] \arrow[d, "1 \circ \psi"'] & A \circ X \arrow[d, "\psi"] & I \circ X \arrow[l, "\eta \circ 1"'] \arrow[dl, equal] \\ A \circ X \arrow[r, "\psi"] & X. & \end{tikzcd}$$

Such a map ψ is called an **A-algebra structure** on X. The category of A-algebras is denoted by $\mathbf{C}^A$. We can characterize the set of A-algebra structures on X as the equalizer of the pair of maps

$$\mathbf{C}[A(X), X] \rightrightarrows \mathbf{C}[A^2(X), X] \times \mathbf{C}[X, X] \tag{2.1.12}$$

where the top map takes $\psi\colon A(X) \to X$ to the pair $(\psi\mu, \psi\eta)$ and the bottom map takes ψ to the pair $(\psi A(\psi), 1_X)$.

Now suppose that the action of $\mathbf{M}$ on $\mathbf{C}$ is right-closed. Then for any object $X \in \mathbf{C}$ there is a monoid $\mathcal{E}_X$ in $\mathbf{M}$ defined by $\mathcal{E}_X = \mathfrak{Hom}(X, X)$ with structure maps given by the unit and composition maps $e\colon I \to \mathcal{E}_X$ and $m\colon \mathcal{E}_X \circ \mathcal{E}_X \to \mathcal{E}_X$ as described in 2.1.9. This $\mathcal{E}_X$ is called the **endomorphism monoid** of X. It classifies A-algebra structures on X as described in the following proposition.

2.1.13. Proposition. *If $A \in \mathrm{mon}\mathbf{M}$ and $X \in \mathbf{C}$, then $\mathrm{mon}\mathbf{M}[A, \mathcal{E}_X]$ is naturally isomorphic to the the equalizer of the pair of maps in 2.1.12.*

Proof. The set of monoid maps from A to $\mathcal{E}_X$ is exactly the equalizer of the pair of maps

$$\mathbf{M}[A, \mathcal{E}_X] \rightrightarrows \mathbf{M}[A \circ A, \mathcal{E}_X] \times \mathbf{M}[I, \mathcal{E}_X], \tag{2.1.14}$$

where the top map sends α to $(\alpha\mu, \alpha\eta)$ and the bottom map sends α to $(m(\alpha \circ \alpha), e)$. Thus it suffices to show that the adjunction isomorphism of 2.1.6 carries the diagram 2.1.12 to the diagram 2.1.14, which is a routine verification. □

2.1.15. Example. Let $\mathbf{M}$ denote any closed symmetric monoidal category. Then $\mathbf{M}$ is clearly a right-closed monoidal category. Examples of such $\mathbf{M}$ include the category of simplicial sets $\mathcal{S}$, in which case mon$\mathbf{M}$ is the category of simplicial monoids, and the category of simplicial R-modules $\mathcal{M}_\mathrm{R}$, in which case mon$\mathbf{M}$ is the category of associative R-algebras.

*2.1.16. Example (**O-categories**).* Let O be a set, and let $\mathbf{C} = \mathrm{Set}^O$ be the category of sets indexed by O; thus, an object $X \in \mathrm{Set}^O$ is a collection of sets $\{X(x)\}_{x\in O}$. The category $\mathbf{M} = \mathrm{Set}^{O\times O}$ is equivalent to the category of directed graphs with vertex set O. There is a right-closed monoidal structure on $\mathbf{M}$ given by

$$(A \circ B)(x, y) = \coprod_{z\in O} A(z, y) \times B(x, z)$$

and

$$\mathcal{F}(B, C)(y, z) = \prod_{x\in O} \mathrm{Set}[B(x, y), C(x, z)],$$

and a right-closed action of $\mathbf{M}$ on $\mathbf{C}$ given by

$$(A \circ X)(x) = \coprod_{z\in O} A(z, x) \times X(z)$$

and

$$\mathfrak{Hom}(X, Y)(x, y) = \mathrm{Set}[X(x), Y(y)].$$

Then a monoid in $\mathbf{M}$ is an O-**category**; i.e., a category with a fixed set of objects O, and maps between such monoids are functors which are the identity on the set of objects. The category of algebras over an O-category C is equivalent to the category of C-diagrams.

It is not hard to see that there is an analogous left-closed action of $\mathbf{M}$ on $\mathbf{C}$.

*2.1.17. Example (**Triples**).* Let $\mathbf{C}$ be a complete category, and let $\mathbf{M} = \mathbf{C}^\mathbf{C}$ the category of endofunctors on $\mathbf{C}$. Then $\mathbf{C}^\mathbf{C}$ is a monoidal category under composition of functors, and there is a right-closed action of $\mathbf{C}^\mathbf{C}$ on $\mathbf{C}$ induced by evaluation of functors. In fact, objects $X, Y \in \mathbf{C}$ can be identified with functors $X, Y : \mathbf{1} \to \mathbf{C}$; then $\mathfrak{Hom}(X, Y) : \mathbf{C} \to \mathbf{C}$ is the "right Kan extension" (see [10, X]) of X along Y. Explicitly, this is the functor defined at $C \in \mathbf{C}$ by

$$\mathfrak{Hom}(X, Y)(C) = \prod_{\mathbf{C}[C,X]} Y.$$

A monoid in $\mathbf{C}^\mathbf{C}$ is exactly a **triple** on $\mathbf{C}$ (also known as a **monad**), and an algebra over a monoid is just an algebra over the triple. The triple $\mathcal{E}_X : \mathbf{C} \to \mathbf{C}$ may be called the **endomorphism triple**; it classifies triple algebra structures on $X \in \mathbf{C}$.

This category $\mathbf{C}^\mathbf{C}$ does *not* seem in general to be a right-closed monoidal category: $\mathcal{F}(X, Y)$ would necessarily be defined as the right Kan extension of $X : \mathbf{C} \to \mathbf{C}$ along $Y : \mathbf{C} \to \mathbf{C}$; one expects this to exist only if $\mathbf{C}$ is a small complete category, which is rare (see [10, V.2]).

2.1.18. Example ***(Operads)***. Another example of a right-closed monoidal category and right-closed action is that of Σ-objects acting on a closed symmetric monoidal category. This example is described in detail in Section 2.2.

2.1.19 Biobjects

Let $\mathbf{M}$ be a monoidal category. Suppose $A \in \operatorname{mon}\mathbf{M}$, and let $X \in \mathbf{M}$. A **left-A-action** of A on X is a map $\psi\colon A \circ X \to X$ making the diagram

$$\begin{tikzcd} A \circ A \circ X \arrow[r, "\psi \circ X"] \arrow[d, "A \circ \psi"'] & A \circ X \arrow[d, "\psi"] & X \arrow[l, "\eta \circ X"'] \arrow[dl, equal] \\ A \circ X \arrow[r, "\psi"] & X & \end{tikzcd}$$

commute; in other words, an algebra over the triple $\mathbf{M} \to \mathbf{M}$ defined by $X \mapsto A \circ X$. The category of algebras over this triple is denoted by ${}_A\mathbf{M}$. Likewise given $B \in \operatorname{mon}\mathbf{M}$, a **right-$B$-action** of B on X is a map $\phi\colon X \circ B \to X$ making the diagram

$$\begin{tikzcd} X \circ B \circ B \arrow[r, "X \circ \phi"] \arrow[d, "\phi \circ B"'] & X \circ B \arrow[d, "\phi"] & X \arrow[l, "X \circ \eta"'] \arrow[dl, equal] \\ X \circ B \arrow[r, "\phi"] & X & \end{tikzcd}$$

commute; in other words, an algebra over the triple $\mathbf{M} \to \mathbf{M}$ defined by $X \mapsto X \circ B$. The category of algebras over this triple is denoted by $\mathbf{M}_B$.

More generally given $A, B \in \operatorname{mon}\mathbf{M}$ and $X \in \mathbf{M}$ an **A, B-action** on X consists of a left-A-action and a right-B-action which commute; in other words, an algebra over the triple $\mathbf{M} \to \mathbf{M}$ defined by $X \mapsto A \circ X \circ B$. The category of algebras over this triple is denoted by ${}_A\mathbf{M}_B$, which we refer to as the category of **A, B-biobjects**.

We are particular interested in the case when A and B are the same monoid. We shall refer to an $X \in \mathbf{M}$ with an A, A-action as an **A-biobject**; the category of A-biobjects is thus denoted by ${}_A\mathbf{M}_A$.

Suppose $\mathbf{M}$ is a right-closed monoidal category with a right closed action on $\mathbf{C}$. Then if $X \in \mathbf{C}^A$ and $Y \in \mathbf{C}^B$ then $\mathfrak{Hom}(X, Y)$ has a natural B, A-action, with action map given by

$$B \circ \mathfrak{Hom}(X, Y) \circ A \to \mathfrak{Hom}(Y, Y) \circ \mathfrak{Hom}(X, Y) \circ \mathfrak{Hom}(X, X) \to \mathfrak{Hom}(X, Y),$$

where $A \to \mathfrak{Hom}(X, X)$ and $B \to \mathfrak{Hom}(Y, Y)$ are maps of monoids classifying the algebra structures on X and Y respectively. Furthermore, note that $A^{\circ n} = A \circ \cdots \circ A$ for $n \geq 1$ has a natural structure as an A-biobject. The following proposition says that maps between A-algebras are represented by an A-biobject.

2.1.20. Proposition. *If $A \in \operatorname{mon}\mathbf{M}$ and $X, Y \in \mathbf{C}^A$, then there is a natural isomorphism*

$${}_A\mathbf{M}_A[A, \mathfrak{Hom}(X, Y)] \simeq \mathbf{C}^A[X, Y].$$

Proof. A map of A-biobjects is exactly a map which is both a map of left-A-objects and a map of right-A-objects. Thus ${}_A\mathbf{M}_A[A, \mathfrak{Hom}(X, Y)]$ is the equalizer of

$$\mathbf{M}_A[A, \mathfrak{Hom}(X, Y)] \rightrightarrows \mathbf{M}_A[A \circ A, \mathfrak{Hom}(X, Y)],$$

where the top arrow sends f to $f\mu$ and the bottom arrow sends f to $\ell(A \circ f)$, where $\ell\colon A \circ \mathfrak{Hom}(X, Y) \to \mathfrak{Hom}(X, Y)$ is the left-A-action. A straightforward argument shows that this diagram is isomorphic to

$$\mathbf{M}[I, \mathfrak{Hom}(X, Y)] \rightrightarrows \mathbf{M}[A, \mathfrak{Hom}(X, Y)],$$

where the top arrow sends g to $r(g \circ A)$ and the bottom arrow sends g to $\ell(A \circ g)$. Here $r\colon \mathfrak{Hom}(X, Y) \circ A \to \mathfrak{Hom}(X, Y)$ denotes the right-A-action. A straightforward argument shows that by the adjunction isomorphism of 2.1.6 this diagram is the same as

$$\mathbf{C}[X, Y] \rightrightarrows \mathbf{C}[A(X), Y],$$

where the top arrow sends h to $h\psi_X$ and the bottom arrow sends h to $\psi_Y h$, where $\psi_X\colon A(X) \to X$ and $\psi_Y\colon A(Y) \to Y$ are the algebra structures. But the equalizer of this diagram is exactly $\mathbf{C}^A[X, Y]$ □

2.1.21 Monoidal structures for over-categories

In this section, we note that right-closed monoidal structures and right-closed actions naturally induce similar structures on over-categories. We will make use of this fact in Section 5.2.

2.1.22. Proposition. *Let* $\mathbf{M}$ *be a monoidal category, and let* A *be a monoid in* $\mathbf{M}$. *Then there is a natural monoidal structure on* $(\mathbf{M} \downarrow A)$.

Proof. The monoidal structure is defined as follows. Given objects $(B \xrightarrow{b} A), (C \xrightarrow{c} A) \in (\mathbf{M} \downarrow A)$, we define

$$(B \xrightarrow{b} A) \circ (C \xrightarrow{c} A) = B \circ C \xrightarrow{b \circ c} A \circ A \xrightarrow{\mu} A.$$

The unit of this monoidal structure is the unit map $\eta\colon I \to A$ of A. □

2.1.23. Proposition. *Let* $\mathbf{M}$ *be a complete category with a right-closed action on* $\mathbf{C}$. *Let* A *be a monoid in* $\mathbf{M}$ *and let* X *be an* A*-algebra. Then there is a canonical right-closed action*

$$(\mathbf{M} \downarrow A) \times (\mathbf{C} \downarrow X) \to (\mathbf{C} \downarrow X).$$

Proof. The right-closed action of $(\mathbf{M} \downarrow A)$ on $(\mathbf{C} \downarrow X)$ is defined as follows. Given objects $(B \xrightarrow{b} A) \in (\mathbf{M} \downarrow A)$, and $(Y \xrightarrow{y} X), (Z \xrightarrow{z} X) \in (\mathbf{C} \downarrow X)$, we define

$$(B \xrightarrow{b} A) \circ (Y \xrightarrow{y} X) = B \circ Y \xrightarrow{b \circ y} A \circ X \xrightarrow{\psi} X$$

and
$$\mathfrak{Hom}_{A,X}(Y \xrightarrow{y} X, Z \xrightarrow{z} X) = \mathfrak{Hom}(Y, Z) \times_{\mathfrak{Hom}(Y,X)} A \to A.$$

$\square$

2.1.24. Corollary. *Let* $\mathbf{M}$ *be a complete right-closed monoidal category. Let* A *be a monoid in* $\mathbf{M}$*. Then there is a canonical right-closed monoidal structure on* $(\mathbf{M} \downarrow A)$.

2.1.25 Enriched categories

In this section, we note that all the constructions described above can be carried out when $\mathbf{C}$ and $\mathbf{M}$ are replaced with categories *enriched* over some symmetric closed monoidal category $\mathbf{V}$ [9], with the definitions suitably modified to take into account the enriched structure. If $\mathbf{C}$ is enriched over $\mathbf{V}$ we write $\mathrm{map}_{\mathbf{C}}(X, Y) \in \mathbf{V}$ for the enriched hom-object.

Thus, if $\mathbf{C}$ and $\mathbf{M}$ are categories enriched over $\mathbf{V}$, and if $\mathbf{M}$ is is right-closed monoidal category with a right-closed action on $\mathbf{C}$ for which the structure functors $\eta \colon \mathbf{1} \to \mathbf{M}$, $\mu \colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$, and $\psi \colon \mathbf{M} \times \mathbf{C}$ are $\mathbf{V}$-functors, then all the constructions of this section apply with *hom sets* $\mathbf{C}[-,-]$ and $\mathbf{M}[-,-]$ replaced by *hom objects* $\mathrm{map}_{\mathbf{C}}(-,-)$ and $\mathrm{map}_{\mathbf{M}}(-,-)$.

In particular, the discussion of 2.1.11 implies that for $X \in \mathbf{C}$ and $A \in \mathrm{mon}\mathbf{M}$ there exists a $\mathbf{V}$*-object*
$$\mathrm{map}_{\mathrm{mon}\mathbf{M}}(A, \mathcal{E}_X) \in \mathbf{V}$$
of A-algebra structures on X. In the sequel we will consider the case when $\mathbf{V} = \mathcal{S}$, the category of simplicial sets.

2.2 Operads and related categories

In this section we define (following Getzler-Jones [7]) the notion of a Σ-object in a symmetric monoidal category (Getzler and Jones call them S-objects). A Σ-object in a category $\mathbf{C}$ induces a functor from $\mathbf{C}$ to itself. We will define a monoidal structure on Σ-objects corresponding to composition of the induced functors; this monoidal structure will turn out to be right-closed, and have a right-closed action on $\mathbf{C}$. Using this construction we will be able to define *operads*, *algebras over an operad*, and *biobjects over an operad* over the category $\mathbf{C}$.

2.2.1 Closed symmetric monoidal categories

In this section, we let $\mathbf{C}$ denote a closed symmetric monoidal category [10, VII, 7] with monoidal structure $- \otimes -$ and unit object k, which has internal hom-objects $\mathrm{hom}(-,-)$, so that
$$\mathbf{C}[X \otimes Y, Z] \simeq \mathbf{C}[X, \mathrm{hom}(Y, Z)].$$
Furthermore, we assume that $\mathbf{C}$ has all small limits and colimits. Note that the functors $X \otimes -$ and $- \otimes Y$ must commute with all colimits.

The key examples of symmetric monoidal categories we need are:

1. The category Set (resp. $\mathcal{S}$) of (simplicial) sets, with cartesian product as the monoidal product,

2. The category $\mathcal{T}$ of compactly generated weak-Hausdorff topological spaces, with the categorical product as the monoidal product,

3. The category $\mathcal{M}_\mathrm{R}$ (resp. $\mathcal{M}_\mathrm{R}$) of (simplicial) modules over a commutative ring R, with tensor product over R as the monoidal product. (This monoidal structure will be symmetric via the natural isomorphism $X \otimes Y \xrightarrow{\sim} Y \otimes X$ sending $x \otimes y \mapsto y \otimes x$.)

4. The category $\mathcal{M}_\mathrm{R}$ (resp. $\mathcal{M}_\mathrm{R}$) of graded (simplicial) modules over a graded commutative ring R, with tensor product over R as the monoidal product. (This monoidal structure will be symmetric via the natural isomorphism $X \otimes Y \xrightarrow{\sim} Y \otimes X$ sending $x \otimes y \mapsto (-1)^{pq} y \otimes x$, where x and y are homogeneous elements of degree p and q respectively.)

Each of the categories $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{M}_\mathrm{R}$ is enriched over simplicial sets, and the monoidal structure is likewise enriched, in the sense that the monoidal product and the hom functor are simplicial functors.

2.2.2 Σ-objects

Let Π denote the category whose objects are finite totally ordered sets, and whose maps are all functions between sets, which need *not* preserve the orderings; in other words, the objects of Π are finite sets which are "secretly" equipped with an ordering. Note that if S is an object of Π then any subset of T of S with the induced ordering is also an object of Π. Let Σ denote the subcategory of Π consisting of the objects and all isomorphisms (again, these maps need not preserve the ordering). Let Σ_S denote the group of automorphisms of the set S.

If $S = \{s_1, \dots, s_n\} \in \Sigma$ with $s_1 < \cdots < s_n$, and if we have a collection of objects $\{X_s \in \mathbf{C}\}_{s \in S}$ in some symmetric monoidal category $\mathbf{C}$, then we define

$$\bigotimes_{s \in S} X_s = X_{s_1} \otimes \cdots \otimes X_{s_n}.$$

Thus given a map $f \colon T \to S \in \Sigma$ there is an obvious induced map

$$\bigotimes_{t \in T} X_{f(t)} \to \bigotimes_{s \in S} X_s.$$

In particular, if $Y \in \mathbf{C}$, then the notation $Y^{\otimes S}$ for the tensor product of copies of Y indexed by S is non-ambiguous, and has a natural Σ_S-action.

Let $\mathbf{\Sigma C}$ denote the category of functors from Σ^{op} to $\mathbf{C}$. We refer to $\mathbf{\Sigma C}$ as the category of **Σ-objects** in $\mathbf{C}$. Let $\mathbf{k} = \{1, \dots, k\}$ denote a distinguished ordered set with k elements, and let Σ' denote the full subcategory of Σ consisting of the objects $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$. Then the inclusion $\Sigma' \to \Sigma$ is clearly an equivalence of categories. Thus there is an equivalence of the categories $\mathbf{\Sigma C}$ and the functor category $\mathbf{C}^{\Sigma'^{\mathrm{op}}}$. Since an object in $\mathbf{\Sigma C}$ is thus determined by its restriction to the subcategory Σ'^{op}, we will often regard an $X \in \mathbf{\Sigma C}$ as a sequence of objects $X[\mathbf{0}], X[\mathbf{1}], X[\mathbf{2}], \ldots \in \mathbf{C}$ where each $X[\mathbf{n}]$ is equipped with a right action by Σ_n.

2.2.3 Monoidal structure in the category of Σ-objects

To any Σ-object A is associated an endofunctor $A(-)$ on the category $\mathbf{C}$. It is defined to be the coequalizer in

$$\coprod_{T \to T' \in \Sigma} A[T'] \otimes X^{\otimes T} \rightrightarrows \coprod_{T \in \Sigma} A[T] \otimes X^{\otimes T} \to A(X).$$

This defines a functor $\mathbf{\Sigma C} \times \mathbf{C} \to \mathbf{C}$. Using the equivalence $\Sigma' \to \Sigma$, we see that

$$A(X) \simeq \coprod_{s \geq 0} A[\mathbf{s}] \otimes_{\Sigma_s} X^{\otimes s}.$$

(This construction may also be described as the *coend* [10] of the functor $\Sigma \to \mathbf{C}$ defined by $T \mapsto X^{\otimes T}$ and the functor $A \colon \Sigma^{\mathrm{op}} \to \mathbf{C}$.)

We will define a monoidal structure $-\circ-$ on $\mathbf{\Sigma C}$ such that $(A \circ B)(X) \simeq A(B(X))$. For this purpose we need to describe the functor $X \mapsto A(X)^T$ in terms of a Σ-object, to be denoted by $A[-, T]$.

To make this construction, we note that one can define a *symmetric* monoidal category structure on $\mathbf{\Sigma C}$ as follows. For $A, B \in \mathbf{\Sigma C}$ define

$$(A \otimes B)[\mathbf{n}] = \coprod_{r+s=n} A[\mathbf{r}] \otimes B[\mathbf{s}] \otimes_{\Sigma_r \times \Sigma_s} \Sigma_n.$$

Thus $A \otimes B \in \mathbf{\Sigma C}$, and $- \otimes - \colon \mathbf{\Sigma C} \times \mathbf{\Sigma C} \to \mathbf{\Sigma C}$ is a functor. It is not hard to show that this defines a symmetric monoidal product on $\mathbf{\Sigma C}$ with the following property.

2.2.4. Lemma. *There are natural and coherent isomorphisms*

$$(A \otimes B)(X) \simeq A(X) \otimes B(X)$$

for $A, B \in \mathbf{\Sigma C}$ *and* $X \in \mathbf{C}$.

Proof. The isomorphism is defined by

$$\begin{aligned}
(A \otimes B)(X) &\simeq \coprod_{n \geq 0} (A \otimes B)[\mathbf{n}] \otimes_{\Sigma_n} X^{\otimes \mathbf{n}} \\
&\simeq \coprod_{n} \coprod_{r+s=n} (A[\mathbf{r}] \otimes B[\mathbf{s}]) \otimes_{\Sigma_r \times \Sigma_s} X^{\otimes \mathbf{n}} \\
&\simeq \coprod_{r \geq 0, s \geq 0} (A[\mathbf{r}] \otimes_{\Sigma_r} X^{\otimes \mathbf{r}}) \otimes (B[\mathbf{s}] \otimes_{\Sigma_s} X^{\otimes \mathbf{s}}) \\
&\simeq A(X) \otimes B(X).
\end{aligned}$$

□

It immediately follows that $A(X)^{\otimes T} \simeq A^{\otimes T}(X)$. We define a functor

$$A[-,-]\colon \Sigma^{\mathrm{op}} \times \Sigma \to \mathbf{C}$$

by $A[S,T] = A^{\otimes T}[S]$. It is not hard to show that

$$A[S,T] \simeq \coprod_{\pi \in \Pi(S,T)} \bigotimes_{t \in T} A[\pi^{-1}(t)],$$

Given A and B in $\Sigma\mathbf{C}$, define $A \circ B \in \Sigma\mathbf{C}$ by the following coequalizer diagram in $\mathbf{C}$

$$\coprod_{T \to T' \in \Sigma} A[T'] \otimes B[S,T] \rightrightarrows \coprod_{T \in \Sigma} A[T] \otimes B[S,T] \to (A \circ B)[S].$$

Using the equivalence of categories $\Sigma' \to \Sigma$, we obtain the following description of this construction:

$$(A \circ B)[\mathbf{s}] \simeq \coprod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} B[\mathbf{s},\mathbf{t}] \simeq \coprod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} B^{\otimes \mathbf{t}}.$$

(Equivalently, $(A \circ B)[S]$ is the coend of functors $B[S,-]\colon \Sigma \to \mathbf{C}$ and $A\colon \Sigma^{\mathrm{op}} \to \mathbf{C}$.) Also define an object I in $\Sigma\mathbf{C}$ by

$$I[S] = \begin{cases} k & \text{if } |S| = 1, \\ \varnothing & \text{otherwise,} \end{cases}$$

where $\varnothing$ is the initial object in $\mathbf{C}$.

2.2.5. Lemma. *There are natural and coherent isomorphisms*

$$(A \otimes A') \circ B \simeq (A \circ B) \otimes (A' \circ B)$$

for $A, A', B \in \Sigma\mathbf{C}$.

Proof. Similar to proof of Lemma 2.2.4. □

2.2.6. Proposition. *With the given structure $\Sigma\mathbf{C}$ becomes a monoidal category with an action on $\mathbf{C}$. In particular, there are natural isomorphisms*

$$(A \circ B)(X) \simeq A\big(B(X)\big),$$

$$(X \circ Y) \circ Z \simeq X \circ (Y \circ Z),$$

$$X \circ I \simeq X, \qquad I \circ X \simeq X,$$

and

$$I(X) \simeq X.$$

Proof. Straightforward, using Lemmas 2.2.4 and 2.2.5. For example the first equation follows from

$$\begin{aligned} A(B(X)) &\simeq \coprod_{n\geq 0} A[\mathbf{n}] \otimes_{\Sigma_n} B(X)^{\otimes \mathbf{n}} \\ &\simeq \coprod_{n\geq 0} (A[\mathbf{n}] \otimes_{\Sigma_n} B^{\otimes \mathbf{n}})(X) \\ &\simeq (A \circ B)(X). \end{aligned}$$

□

Given two objects X and Y in $\mathbf{C}$, we define an object $\mathfrak{Hom}(X, Y)$ in $\Sigma\mathbf{C}$ by

$$\mathfrak{Hom}(X, Y)[S] = \hom[X^{\otimes S}, Y].$$

This gives a functor $\mathfrak{Hom}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \Sigma\mathbf{C}$.

2.2.7. Proposition. *This structure defines a right-closed action of* $\Sigma\mathbf{C}$ *on* $\mathbf{C}$. *In particular, there is a natural isomorphism*

$$\mathbf{C}[A(X), Y] \simeq \Sigma\mathbf{C}[A, \mathfrak{Hom}(X, Y)].$$

Proof. Straightforward. □

Given two objects B and C in $\Sigma\mathbf{C}$, we define $\mathcal{F}(B, C) \in \Sigma\mathbf{C}$ by the following equalizer in $\mathbf{C}$

$$\mathcal{F}(B, C)[T] \to \prod_{S\in\Sigma} \hom(B[S, T], C[S]) \rightrightarrows \prod_{S'\to S\in\Sigma} \hom(B[S, T], C[S']).$$

Using the equivalence $\Sigma' \to \Sigma$ we obtain the description

$$\mathcal{F}(Y, Z)[\mathbf{t}] \simeq \prod_{s\geq 0} \hom(Y[\mathbf{s}, \mathbf{t}], Z[\mathbf{s}])^{\Sigma_s}.$$

2.2.8. Proposition. *With the given structure,* $\Sigma\mathbf{C}$ *becomes a right-closed monoidal category. In particular, there is a natural isomorphism*

$$\Sigma\mathbf{C}[X \circ Y, Z] \simeq \Sigma\mathbf{C}[X, \mathcal{F}(Y, Z)].$$

Proof. Straightforward. □

2.2.9 Operads

An **operad** is defined to be a monoid in the monoidal category $\Sigma\mathbf{C}$. We will refer to the category of operads by oper$\mathbf{C}$.

2.2.10. Example. Let $\mathbf{C} = \mathcal{S}$ be the category of simplicial sets. The Σ-object P defined by

$$P[S] = \begin{cases} \mathrm{pt} & \text{if } |S| = 0, 1, \\ \varnothing & \text{otherwise,} \end{cases}$$

admits a unique structure as an operad. We have that

$$P(X) = \mathrm{pt} \amalg X.$$

2.2.11. Example. Again, let $\mathbf{C}$ be the category of simplicial sets. The Σ-object N defined by

$$N[S] = \mathrm{pt}$$

for all S admits a unique structure as an operad, and is referred to as the **symmetric operad**. We have that

$$N(X) = \coprod_{s \geq 0} (X^{\times s})_{\Sigma_s}.$$

2.2.12. Example. Again, let $\mathbf{C}$ be the category of simplicial sets. The Σ-object M defined by

$$M[S] = \Sigma_S,$$

with the obvious Σ action admits the structure of an operad, and is referred to as the **associative operad**. We have that

$$M(X) = \coprod_{s \geq 0} X^{\times s}.$$

2.2.13 Algebras over an operad

Let A be an operad. The category of A-**algebras** is the category of algebras over the triple $A(-)$ on $\mathbf{C}$, and is denoted by $\mathbf{C}^A$.

2.2.14. Example. The category of algebras over P is equivalent to the category of pointed simplicial sets. The category of algebras over N is equivalent to the category of simplicial commutative monoids. The category of algebras over M is equivalent to the category of simplicial associative monoids.

Given X in $\mathbf{C}$, the **endomorphism operad** is the operad $\mathcal{E}_X = \mathfrak{Hom}(X, X)$. By Proposition 2.1.13 it has the property that the set of operad maps $A \to \mathcal{E}_X$ is in natural one-to-one correspondence with the set of A-algebra structures on X.

2.2.15 Biobjects over an operad

Let A and B be operads. An A, B-**biobject** is an object X in $\Sigma\mathbf{C}$ equipped with a map $A \circ X \circ B \to X$ which satisfies the obvious unital and associativity properties; i.e., it is an object with a left-A-action and a right-B-action as in 2.1.19. We denote the category of A, B-biobjects by $(A, B)\text{-biobj}\mathbf{C}$. It is not hard to see that $(A, B)\text{-biobj}\mathbf{C}$ is equivalent to the category of algebras over a triple on $\Sigma\mathbf{C}$ which takes X to $A \circ X \circ B$.

As noted in 2.1.19, if X is an A-algebra and Y is a B-algebras, then $\mathfrak{Hom}(X, Y)$ has a natural structure of a B, A-biobject, defined by

$$B \circ \mathfrak{Hom}(X, Y) \circ A \to \mathfrak{Hom}(Y, Y) \circ \mathfrak{Hom}(X, Y) \circ \mathfrak{Hom}(X, X) \to \mathfrak{Hom}(X, Y),$$

where $A \to \mathcal{E}_X \simeq \mathfrak{Hom}(X, X)$ and $B \to \mathcal{E}_Y \simeq \mathfrak{Hom}(Y, Y)$ are maps of operads which classify the algebra structures on X and Y respectively.

2.2.16. Example. An operad A has the structure of an A-biobject in a natural way; the structure map $A \circ A \circ A \to A$ is just multiplication in the operad. As shown in 2.1.19, there is a natural isomorphism

$$\mathbf{C}^A[X, Y] \simeq A\text{-biobj}\mathbf{C}[A, \mathfrak{Hom}(X, Y)].$$

2.3 Constructions involving Σ-objects

In this section we show the existence of limits and colimits in Σ-objects, operads, and biobjects, and describe a "relative" circle-product construction for biobjects, which will lead in particular to adjoint functor pairs

$$f_* : \mathbf{C}^A \rightleftarrows \mathbf{C}^B : f^*$$

for any map $f\colon A \to B$ of operads.

2.3.1 Limits and colimits

In this section we show the existence of limits and colimits in the categories of operads, and of algebras and biobjects over an operad. We shall assume in this section that $(\mathbf{C}, \otimes)$ denotes a symmetric monoidal category with all small limits and colimits.

A **reflexive pair** in any category $\mathbf{C}$ is a pair of maps $f, g\colon X \rightrightarrows Y$ along with a section $s\colon Y \to X$ such that $fs = 1_Y = gs$. The **coequalizer** of such a reflexive pair is a map $k\colon Y \to Z \in \mathbf{C}$ which is initial among all maps such that $kf = kg$. The significance of reflexive coequalizers comes from the following lemma; the usefulness of reflexive coequalizers in these sorts of arguments was noticed by Mike Hopkins.

2.3.2. Lemma. *Let $\mathbf{C}$ be as in 2.3.1.*

1. *Let $f_i, g_i\colon X_i \rightrightarrows Y_i \xrightarrow{k_i} Z_i$, $s_i\colon Y_i \to X_i$ for $i = 1, \ldots, n$ be reflexive coequalizers in $\mathbf{C}$. Then the induced diagram*

$$X_1 \otimes \cdots \otimes X_n \rightrightarrows Y_1 \otimes \cdots \otimes Y_n \to Z_1 \otimes \cdots \otimes Z_n$$

is a reflexive coequalizer in $\mathbf{C}$.

2. *Let $\mathbf{D}$ be a filtered category, and let $X_i\colon \mathbf{D} \to \mathbf{C}$ for $i = 1, \ldots, n$ be diagrams. Then there is a canonical isomorphism*

$$\operatorname{colim}^{\mathbf{D}}(X_1 \otimes \cdots \otimes X_n) \simeq (\operatorname{colim}^{\mathbf{D}} X_1) \otimes \cdots \otimes (\operatorname{colim}^{\mathbf{D}} X_n).$$

Proof. The lemma follows from the fact that $\otimes$ distributes over colimits and the easily checked fact that in both cases (1) and (2) the functor $\mathbf{D} \to \mathbf{D} \times \cdots \times \mathbf{D}$ is *final* in the sense of [10, IX.3]. (For case (1) take $\mathbf{D}$ to be the category which indexes a reflexive pair of maps.) $\square$

This leads immediately to the following statements for the action of $\Sigma\mathbf{C}$ on $\mathbf{C}$, and for the monoidal structure on $\Sigma\mathbf{C}$.

2.3.3. Lemma. *Let* $\mathbf{C}$ *be as in 2.3.1.*

1. *Let* $f, g\colon X \rightrightarrows Y \xrightarrow{k} Z$, $s\colon Y \to X$ *be a reflexive coequalizer in* $\mathbf{C}$. *Then for* $A \in \Sigma\mathbf{C}$ *the induced diagram*

$$A(X) \rightrightarrows A(Y) \to A(Z)$$

   is a reflexive coequalizer in $\mathbf{C}$.

2. *Let* $\mathbf{D}$ *be a filtered category, and let* $X\colon \mathbf{D} \to \mathbf{C}$ *be a diagram. Then for* $A \in \Sigma\mathbf{C}$ *there is a canonical isomorphism*

$$A(\operatorname{colim}^{\mathbf{D}} X) \simeq \operatorname{colim}^{\mathbf{D}} A(X).$$

Proof. This follows immediately from Lemma 2.3.2 and the fact that $A(X)$ is constructed from A and X using only colimits and the monoidal product in $\mathbf{C}$. $\square$

2.3.4. Lemma. *Let* $\mathbf{C}$ *be as in 2.3.1.*

1. *Let* $f_i, g_i\colon A_i \rightrightarrows B_i \xrightarrow{k_i} C_i$, $s_i\colon B_i \to A_i$ *for* $i = 1, \ldots, n$ *be reflexive coequalizers in* $\Sigma\mathbf{C}$. *Then the induced diagram*

$$A_1 \circ \cdots \circ A_n \rightrightarrows B_1 \circ \cdots \circ B_n \to C_1 \circ \cdots \circ C_n$$

   is a reflexive coequalizer in $\mathbf{C}$.

2. *Let* $\mathbf{D}$ *be a filtered category, and let* $A_i\colon \mathbf{D} \to \Sigma\mathbf{C}$ *for* $i = 1, \ldots, n$ *be diagrams. Then there is a canonical isomorphism*

$$\operatorname{colim}^{\mathbf{D}}(A_1 \circ \cdots \circ A_n) \simeq (\operatorname{colim}^{\mathbf{D}} A_1) \circ \cdots \circ (\operatorname{colim}^{\mathbf{D}} A_n).$$

Proof. This follows immediately from Lemma 2.3.2 and the fact that $A \circ B$ is constructed from A and B using only colimits and the monoidal product in $\mathbf{C}$. $\square$

Lemmas 2.3.3 and 2.3.4 allow us to show that categories of algebras over an operad (respectively, categories of biobjects over an operad) are complete and cocomplete.

2.3.5. Proposition. *Let* A *and* B *be operads over* $\mathbf{C}$.

1. *The category* $\mathbf{C}^A$ *of* A*-algebras is complete and cocomplete; furthermore, the forgetful functor* $\mathbf{C}^A \to \mathbf{C}$ *preserves limits, reflexive coequalizers, and filtered colimits.*

2. *The category* $(A, B)\text{-biobj}\mathbf{C}$ *of* A, B*-biobjects is complete and cocomplete; furthermore, the forgetful functor* $(A, B)\text{-biobj}\mathbf{C} \to \Sigma\mathbf{C}$ *preserves limits, reflexive coequalizers, and filtered colimits.*

Proof. We prove part 1; the proof of part 2 is similar.

It is straightforward to show that $\mathbf{C}^A$ has all limits, and that these limits are preserved by the forgetful functor.

We now show that $\mathbf{C}^A$ has all reflexive coequalizers, and that these are preserved by the forgetful functor. Let $f, g \colon X \rightrightarrows Y, s \colon Y \to X \in \mathbf{C}^A$ be a reflexive pair. Let $k \colon Y \to Z \in \mathbf{C}$ be the coequalizer of f and g in the underlying category $\mathbf{C}$. We claim that Z admits a unique A-algebra structure making k the coequalizer in $\mathbf{C}^A$ of f and g. Consider the following diagram.

$$\begin{tikzcd}
A^2X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & A^2Y \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & A^2Z \arrow[d, dotted, shift left] \arrow[d, dotted, shift right] \\
AX \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\psi_X"] & AY \arrow[r] \arrow[d, "\psi_Y"] & AZ \arrow[d, dotted, "\psi_Z"] \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[u, bend left, "\eta_X"] & Y \arrow[r, "k"] \arrow[u, bend left, "\eta_Y"] & Z. \arrow[u, dotted, bend left]
\end{tikzcd}$$

The rows are coequalizers in $\mathbf{C}$ by Proposition 2.3.3. Thus there exist *unique* dotted arrows in $\mathbf{C}$ making the appropriate squares commute. Let $\psi_Z \colon A(Z) \to Z$ be one of the induced arrows. It is then easy to check that, by uniqueness, the other dotted arrows must be $\mu, A(\psi_Z) \colon A^2Z \to AZ$ and $\eta_Z \colon Z \to AZ$, and thus that ψ_Z induces an A-algebra structure on Z. It is now easy to check that Z is the coequalizer of f and g in the category of A-algebras as desired.

A similar argument shows that one may also construct filtered colimits in $\mathbf{C}^A$ by first constructing them in $\mathbf{C}$.

We can now construct arbitrary small colimits in $\mathbf{C}^A$. Let $X \colon \mathbf{D} \to \mathbf{C}^A$ be a functor from a small category. We consider a pair of parallel maps

$$f, g \colon A\big(\operatorname{colim}^{\mathbf{C}} A(X)\big) \rightrightarrows A(\operatorname{colim}^{\mathbf{C}} X),$$

where $\operatorname{colim}^{\mathbf{C}}$ denotes the colimit of a diagram in the underlying category $\mathbf{C}$, defined as follows. Applying the functor $\mathbf{C}^A[-, W]$ for any object $W \in \mathbf{C}$ to the parallel maps and using standard adjunctions, we get a pair of parallel maps

$$\lim \mathbf{C}[A(X), W] \rightrightarrows \lim \mathbf{C}[X, W]$$

where in each factor of the limit the top arrow sends $f \colon A(X) \to W$ to $\psi_W A(f)$, and the bottom arrow sends f to $f\psi_X$. It is easy to see that if the coequalizer of f and g in $\mathbf{C}^A$ exists then it is exactly the colimit of X in $\mathbf{C}^A$. The natural transformation $\eta \colon I \to A$ induces a map $t \colon \operatorname{colim}^{\mathbf{C}} X \to \operatorname{colim}^{\mathbf{C}} A(X)$; let $s = A(t)$. It is not hard to see that s is a section of f and g; thus, by the first part of the proof the coequalizer of f and g exists. $\square$

2.3.6 Free operads

Given a Σ-object A, the **free operad** on A is an operad FA together with a map of Σ-objects $A \to FA$ inducing a bijection

$$\mathrm{oper}\mathbf{C}[FA, B] \xrightarrow{\sim} \Sigma\mathbf{C}[A, B]$$

for all operads B. If the free operad exists for every $A \in \Sigma\mathbf{C}$, we call the resulting functor $F\colon \Sigma\mathbf{C} \to \mathrm{oper}\mathbf{C}$ the **free operad functor**; it is left adjoint to the forgetful functor $\mathrm{oper}\mathbf{C} \to \Sigma\mathbf{C}$.

2.3.7. Proposition. *Let* $\mathbf{C}$ *be as in 2.3.1. Then the free operad functor* $F\colon \Sigma\mathbf{C} \to \mathrm{oper}\mathbf{C}$ *exists.*

Proof. A construction of the free operad is given in Appendix A. □

An argument similar to those given above shows that the category of operads on $\mathbf{C}$ is complete and cocomplete.

2.3.8. Lemma. *Let* $\mathbf{C}$ *be as in 2.3.1.*

1. *Let* $f, g\colon A \rightrightarrows B \xrightarrow{k} C$, $s\colon Y \to X$ *be a reflexive coequalizer in* $\Sigma\mathbf{C}$. *Then the induced diagram*
$$FA \rightrightarrows FB \to FC$$
is a reflexive coequalizer in $\mathbf{C}$.

2. *Let* $\mathbf{D}$ *be a filtered category, and let* $A\colon \mathbf{D} \to \mathbf{C}$ *be a diagram. Then there is a canonical isomorphism*
$$F(\mathrm{colim}^{\mathbf{D}} A) \simeq \mathrm{colim}^{\mathbf{D}} FA.$$

Proof. This follows from Lemma 2.3.3 and from the construction of FA in Appendix A: there FA is defined as a colimit $FA = \mathrm{colim}_n F_n A$ in $\Sigma\mathbf{C}$ where $F_0 A = I$ and $F_n A = I \amalg A \circ F_{n-1} A$ for $n > 0$. Thus FA is constructed from A using only colimits and the circle product in $\Sigma\mathbf{C}$. □

2.3.9. Proposition. *The category* $\mathrm{oper}\mathbf{C}$ *of operads on* $\mathbf{C}$ *is complete and cocomplete; furthermore, the forgetful functor* $\mathrm{oper}\mathbf{C} \to \Sigma\mathbf{C}$ *preserves limits, reflexive coequalizers, and filtered colimits.*

Proof. The proof is similar to that of Proposition 2.3.5. □

2.3.10 The "circle-over" construction

As before, let $(\mathbf{C}, \otimes)$ be a complete and cocomplete symmetric monoidal category. Given an operad A over $\mathbf{C}$, a right A-object M, and an A-algebra X, we define $M \circ_A (X) \in \mathbf{C}$ to be the coequalizer in the diagram

$$(M \circ A)(X) \rightrightarrows M(X) \to M \circ_A (X),$$

where the two left hand maps are induced by the action maps of M and X respectively. This is a reflexive coequalizer, with reflection $M(X) \to (M \circ A)(X)$ induced by the unit map $I \to A$ of the operad.

Similarly, if M is a right A-object and N a left A-object we may define $M \circ_A N \in \Sigma\mathbf{C}$ to be the coequalizer in

$$M \circ A \circ N \rightrightarrows M \circ N \to M \circ_A N$$

where the two left hand maps are the action maps of M and N respectively. This is a reflexive coequalizer, with reflection induced by the unit map of the operad.

2.3.11. Proposition. *Let A be an operad over $\mathbf{C}$, and let M be a right A-object, N a left A-object, and X an A-algebra. Then there are natural isomorphisms*

$$M \circ_A A \simeq M, \qquad A \circ_A N \simeq N, \qquad A \circ_A (X) \simeq X.$$

2.3.12. Proposition. *Let M be an A, B-biobject, let N be a B, C-biobject, and let X be a B-algebra.*

1. *The object $M \circ_B (X) \in \mathbf{C}$ has a natural structure as an A-algebra.*
2. *The object $M \circ_B N \in \Sigma\mathbf{C}$ has a natural structure as an A, C-biobject.*

Proof. This follows immediately from Lemmas 2.3.3 and 2.3.4. □

2.3.13. Proposition. *Let M be a right A-object, let N be an A, B-biobject, let P be a left B-object, and let X be a B-algebra.*

1. *There is a unique isomorphism*

   $$(M \circ_A N) \circ_B (X) \simeq M \circ_A (N \circ_B (X))$$

   commuting with the obvious maps $(M \circ N)(X) \to (M \circ_A N) \circ_B (X)$ and $M \circ \big(N(X)\big) \to M \circ_A \big(N \circ_B (X)\big)$.

2. *There is a unique isomorphism*

   $$(M \circ_A N) \circ_B P \simeq M \circ_A (N \circ_B P)$$

   commuting with the obvious maps $(M \circ N) \circ P \to (M \circ_A N) \circ_B P$ and $M \circ (N \circ P) \to M \circ_A (N \circ_B P)$.

Proof. This follows immediately from Lemmas 2.3.3 and 2.3.4. □

2.3.14. Corollary. *The category of A-biobjects is a monoidal category with monoidal product*

$$- \circ_A -\colon A\text{-biobj}\mathbf{C} \times A\text{-biobj}\mathbf{C} \to A\text{-biobj}\mathbf{C}$$

and unit object $A \in A\text{-biobj}\mathbf{C}$, and there is an action

$$- \circ_A (-)\colon A\text{-biobj}\mathbf{C} \times \mathbf{C}^A \to \mathbf{C}^A.$$

2.3.15. Example. Let $\mathbf{C} = \mathcal{S}$ and let P and M be operads as in Example 2.2.14. There is a map of operads $P \to M$. Given a P-algebra X, i.e., a pointed simplicial set, the object $M \circ_P (X)$ is the free associative monoid on X such that the basepoint is the identity element. In other words, $M \circ_P (X)$ is the James construction on X.

2.3.16. Remark. Although we will not need it, one can show that with the structure of Corollary 2.3.14 the category A-biobj$\mathbf{C}$ is a *right-closed monoidal category*, and its action on $\mathbf{C}^A$ is a *right-closed action*. In fact, for $X \in \mathbf{C}^B$ there is a pair of adjoint functors

$$- \circ_B X : (A, B)\text{-biobj}\mathbf{C} \rightleftarrows \mathbf{C}^A : \mathfrak{Hom}(X, -),$$

where $\mathfrak{Hom}$ coincides with the one defined on $\mathbf{C}$, and for $N \in (B, C)$-biobj$\mathbf{C}$ there is a pair of adjoint functors

$$- \circ_B N : (A, B)\text{-biobj}\mathbf{C} \rightleftarrows (A, C)\text{-biobj}\mathbf{C} : \mathcal{F}^C(N, -),$$

where $\mathcal{F}^C(N, P)$ is defined to be the equalizer of

$$\mathcal{F}^C(N, P) \to \mathcal{F}(N, P) \rightrightarrows \mathcal{F}(N \circ C, P),$$

where the top arrow on the right is induced by $N \circ C \to N$, and the bottom arrow on the right is adjoint to the map $\mathcal{F}(N, P) \circ N \circ C \to P \circ C \to P$.

2.3.17 Adjoint functors

Finally, we note that associated to maps of operads $f\colon A \to B$ and $g\colon A' \to B'$ there are obvious forgetful functors $f^*\colon \mathbf{C}^B \to \mathbf{C}^A$ and $(f, g)^*\colon (B, B')\text{-biobj}\mathbf{C} \to (A, A')\text{-biobj}\mathbf{C}$.

2.3.18. Proposition. *The functors*

$$f_* = B \circ_A -\colon \mathbf{C}^A \to \mathbf{C}^B$$

and

$$(f, g)_* = B \circ_A - \circ_{A'} B'\colon (A, A')\text{-biobj}\mathbf{C} \to (B, B')\text{-biobj}\mathbf{C}$$

are left adjoint to f^* and $(f, g)^*$ respectively.

2.3.19 Simplicial constructions

If $\mathbf{C}$ is enriched over simplicial sets (2.1.25), and if the structure functor $- \otimes -\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a *simplicial functor*, then it is not hard to see that $\Sigma\mathbf{C}$, mon$\mathbf{C}$, $\mathbf{C}^A$, and (A, B)-biobj$\mathbf{C}$ are enriched over simplicial sets, and that structure functors of the right-closed monoidal structure on $\Sigma\mathbf{C}$ and the right-closed action of $\Sigma\mathbf{C}$ on $\mathbf{C}$ are simplicial functors.

Examples of such categories $\mathbf{C}$ are simplicial sets and simplicial R-modules. Thus, the functors $\times\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ and $\otimes_R\colon \mathcal{M}_R \times \mathcal{M}_R \to \mathcal{M}_R$ extend to simplicial functors.

Moreover, for these categories there exist functors

$$- \otimes - \colon \mathcal{S} \times \mathbf{C} \to \mathbf{C}$$

(not to be confused with the monoidal structure $\otimes$ in $\mathbf{C}$) and

$$(-)^{(-)} \colon \mathcal{S}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$$

such that for $X, Y \in \mathbf{C}$ and $K \in \mathcal{S}$ there are natural isomorphisms

$$\mathbf{C}[K \otimes X, Y] \simeq \mathcal{S}[K, \mathrm{map}_{\mathbf{C}}(X, Y)] \simeq \mathbf{C}[X, Y^K].$$

These remarks in turn imply that the categories of sigma objects, operads, algebras over an operad, and biobjects over an operad, over either $\mathcal{S}$ or $\mathcal{M}_{\mathrm{R}}$ are enriched over simplicial sets.

2.4 Monoidal functors

In this section we define monoidal functors and show how they induce corresponding functors on categories of operads and biobjects.

2.4.1 Monoidal functors

Let $(\mathbf{S}, \otimes, I_S)$ and $(\mathbf{T}, \otimes, I_T)$ be monoidal categories. A **monoidal functor** $L \colon \mathbf{S} \to \mathbf{T}$ is a functor equipped with natural isomorphisms

$$b \colon LX \otimes LY \xrightarrow{\sim} L(X \otimes Y)$$

and

$$u \colon I_{\mathbf{T}} \xrightarrow{\sim} LI_{\mathbf{S}}$$

making the following diagrams commute:

$$\begin{tikzcd}
L(X \otimes (Y \otimes Z)) \arrow[r, "La"] \arrow[d, "b"'] & L((X \otimes Y) \otimes Z) \arrow[d, "b"] \\
LX \otimes L(Y \otimes Z) \arrow[d, "1\otimes b"'] & L(X \otimes Y) \otimes LZ \arrow[d, "b\otimes 1"] \\
LX \otimes (LY \otimes LZ) \arrow[r, "a"] & (LX \otimes LY) \otimes LZ,
\end{tikzcd}$$

and

$$\begin{tikzcd} I_{\mathbf{T}} \circ LX \arrow[r, "\ell"] \arrow[d, "u \circ 1"'] & LX \arrow[d, "L\ell"] \\ LI_{\mathbf{S}} \circ LX \arrow[r, "b"] & L(I_{\mathbf{S}} \circ X), \end{tikzcd} \qquad \begin{tikzcd} LX \circ I_{\mathbf{T}} \arrow[r, "r"] \arrow[d, "1 \circ u"'] & LX \arrow[d, "Lr"] \\ LX \circ LI_{\mathbf{S}} \arrow[r, "b"] & L(X \circ I_{\mathbf{S}}). \end{tikzcd}$$

If $L\colon \mathbf{S} \to \mathbf{T}$ is a monoidal functor between *symmetric* monoidal categories such that the diagram

$$\begin{tikzcd} LX \circ LY \arrow[r, "b"] \arrow[d, "\tau"'] & L(X \circ Y) \arrow[d, "\tau"] \\ LY \circ LX \arrow[r, "b"] & L(Y \circ X) \end{tikzcd}$$

commutes, where $\tau\colon X \circ Y \xrightarrow{\sim} Y \circ X$ denotes the symmetry isomorphism in a symmetric monoidal category, then we say that L is a **symmetric monoidal functor**.

2.4.2. Proposition. *If $L\colon \mathbf{S} \to \mathbf{T}$ is a monoidal functor which has a right adjoint $R\colon \mathbf{T} \to \mathbf{S}$, then there is a natural map*

$$c\colon RX \otimes RY \to R(X \otimes Y).$$

Furthermore, if $\mathbf{S}$ and $\mathbf{T}$ are right-closed, then there is a natural isomorphism

$$d\colon \hom_{\mathbf{T}}(Y, RZ) \simeq R\hom_{\mathbf{S}}(LY, Z).$$

If furthermore L is a symmetric monoidal functor between symmetric monoidal categories then the natural map c commutes with the symmetry operators τ.

The main examples of symmetric monoidal functors which we will be interested in are

$$|-| : \mathcal{S} \rightleftarrows \mathcal{T} : \mathrm{Sing}$$

where $|-|$ is **geometric realization** and Sing is the **singular complex**, and

$$R(-) : \mathcal{S} \rightleftarrows \mathcal{M}_{\mathrm{R}} : \mathcal{U}$$

where $R(-)$ is the **free R-module** functor and $\mathcal{U}$ is the **underlying set** functor. All of these are examples of simplicial functors.

2.4.3. Proposition. *If $L : \mathbf{S} \rightleftarrows \mathbf{T} : R$ are an adjoint pair of functors between closed symmetric monoidal categories, and L is a monoidal functor, then there are naturally induced functors*

$$L' : \Sigma\mathbf{S} \rightleftarrows \Sigma\mathbf{T} : R'.$$

Furthermore L' is a monoidal functor, and there are natural isomorphisms

$$L\big(A(X)\big) \simeq (L'A)(LX),$$

for $X \in \mathbf{S}$ and $A \in \Sigma\mathbf{S}$.

Proof. The functors L' and R' are defined "object-wise", so that for $A \in \mathbf{\Sigma S}$ and $B \in \mathbf{\Sigma T}$ we have that $(L'A)[S] = L(A[S])$ and $(R'B)[S] = R(B[S])$. The adjunction is clear. To show that L' is monoidal, it suffices to note that $(L'A)[S,T] \simeq L(A[S,T])$ and that L is monoidal and preserves colimits. The proof of the last part of the proposition is similar. $\square$

2.4.4. Proposition. *There are induced adjoint functors*

$$L' : \text{oper}\mathbf{S} \rightleftarrows \text{oper}\mathbf{T} : R'$$

which coincide with $L' : \mathbf{\Sigma S} \rightleftarrows \mathbf{\Sigma T} : R'$ *on underlying* Σ*-objects, and which induce a natural equivalence*

$$F(L'A) \simeq L'(FA),$$

where $A \in \mathbf{\Sigma S}$ *and* F *denotes the free operad functor.*

Proof. The existence of the induced functors follows from the fact that L' is monoidal, and from Proposition 2.4.2. The rest of the argument is straightforward from the construction of the free operad in Appendix A. $\square$

2.4.5. Corollary. *Given* $A \in \text{oper}\mathbf{S}$*, there are induced adjoint functors*

$$L : \mathbf{S}^A \rightleftarrows \mathbf{T}^{L'A} : R,$$

which coincide with $L : \mathbf{S} \rightleftarrows \mathbf{T} : R$ *on the underlying categories.*

2.4.6. Corollary. *Given* $A, B \in \text{oper}\mathbf{S}$*, there are induced adjoint functors*

$$L' : (A,B)\text{-biobj}\mathbf{S} \rightleftarrows (L'A, L'B)\text{-biobj}\mathbf{T} : R',$$

which coincide with $L' : \mathbf{\Sigma S} \rightleftarrows \mathbf{\Sigma T} : R'$ *on underlying* Σ*-objects.*

Chapter 3

Homotopy theory of operads

3.1 Closed model categories

In this section we recall the notion of a simplicial closed model category and give a principle for recognizing such structures in the cases that interest us. We also recall certain examples of simplicial closed model categories which we will need.

3.1.1 Simplicial closed model categories

A **closed model category** $\mathbf{M}$ is a category equipped with three distinguished classes of maps, called **fibrations**, **cofibrations**, and **weak equivalences**, satisfying axioms (1)-(5) of [14]. The properties of closed model categories are elaborated in [13]; in particular, we note that for any closed model category $\mathbf{M}$ the **homotopy category** $\operatorname{Ho}\mathbf{M}$, obtained by formally inverting the weak equivalences of $\mathbf{M}$, exists.

A **simplicial closed model category** is a closed model category $\mathbf{M}$ which is enriched over simplicial sets and has enriched limits and colimits over simplicial sets (2.3.19), and which satisfies one of the following three equivalent conditions:

1. Let $i: K \to L \in \mathcal{S}$ be a cofibration and $j: X \to Y \in \mathbf{M}$ be a cofibration. Then the induced map

$$f: K \otimes Y \amalg_{K\otimes X} L \otimes X \to L \otimes Y \in \mathbf{M}$$

is a cofibration. If furthermore either i or j is a weak equivalence, then so is f.

2. Let $i: K \to L \in \mathcal{S}$ be a cofibration and $p: Z \to W \in \mathbf{M}$ be a fibration. Then the induced map

$$g: Z^L \to Z^K \times_{W^K} W^L \in \mathbf{M}$$

is a fibration. If furthermore either i or p is a weak equivalence, then so is g.

3. Let $j: X \to Y \in \mathbf{M}$ be a cofibration and $p: Z \to W \in \mathbf{M}$ be a fibration. Then the induced map

$$h: \operatorname{map}_{\mathbf{M}}[Y, Z] \to \operatorname{map}_{\mathbf{M}}[X, Z] \times_{\operatorname{map}_{\mathbf{M}}[X,W]} \operatorname{map}_{\mathbf{M}}[Y, W] \in \mathcal{S}$$

is a fibration. If furthermore either j or p is a weak equivalence, then so is h.

3.1.2 Cofibrantly generated model categories

If $\mathbf{M}$ is a closed model category, we say that $\mathbf{M}$ is **cofibrantly generated** if there exist sets of maps $I = \{i_\alpha : S_\alpha \to D_\alpha\}$ and $J = \{j_\beta : \Lambda_\beta \to \Delta_\beta\}$ in $\mathbf{M}$ such that

1. a map in $\mathbf{M}$ is a trivial fibration if and only if it has the right lifting property with respect to every $i_\alpha \in I$,

2. a map in $\mathbf{M}$ is a fibration if and only if it has the right lifting property with respect to every $j_\beta \in J$,

3. the domains S_α and Λ_β of the maps in I and J are small; an object $S \in \mathbf{M}$ is said to be **small** if for any countable directed sequence X_i in $\mathbf{M}$ the natural map
$$\operatorname{colim}_i \mathbf{M}[S, X_i] \to \mathbf{M}[S, \operatorname{colim}_i X_i]$$
is an isomorphism.

It follows from this definition that the maps in I are cofibrations, and the maps in J are trivial cofibrations. We say that the set I (resp. J) is a set of **generators** of the class of cofibrations (resp. trivial cofibrations).

3.1.3. Remark. The formulation of the notion of cofibrantly generated closed model category given in 3.1.2 is not the most general form possible; our definition is in fact a very restricted case of a definition of Dwyer and Kan [2].

Note that if $\mathbf{M}$ is a cofibrantly generated closed model category, and $X \in \mathbf{M}$ is any object, then $(\mathbf{M} \downarrow X)$ is also a cofibrantly generated closed model category, with generators
$$I' = \{S_\alpha \to D_\alpha \to X\}$$
and
$$J' = \{\Lambda_\beta \to \Delta_\beta \to X\}.$$

3.1.4 Recognizing model categories

The following proposition will be useful in showing that various categories are simplicial closed model categories.

3.1.5. Proposition. *Let $\mathbf{M}$ and $\mathbf{N}$ be categories enriched over simplicial sets, having all enriched limits and colimits, and let*
$$L : \mathbf{M} \rightleftarrows \mathbf{N} : R$$
be a pair of adjoint simplicial functors. Suppose that $\mathbf{M}$ is a simplicial closed model category with sets of maps $I = \{S_\alpha \to D_\alpha \in \mathbf{M}\}$ and $J = \{\Lambda_\beta \to \Delta_\beta \in \mathbf{M}\}$ which generate cofibrations and trivial cofibrations respectively. If

1. *the objects LS_α and $L\Lambda_\beta$ are small in* $\mathbf{N}$, *and*

2. *there is a simplicial functor*
$$E\colon \mathbf{N} \to \mathbf{N}$$
and a natural transformation
$$\epsilon\colon 1 \to E\colon \mathbf{N} \to \mathbf{N}$$
such that for each object $X \in \mathbf{N}$ the object $REX \in \mathbf{M}$ is fibrant and the map $R\epsilon_X\colon RX \to REX \in \mathbf{M}$ is a weak equivalence,

then $\mathbf{N}$ *is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y \in \mathbf{N}$ is a* ***weak equivalence*** *(resp. a* ***fibration****) if and only if the map*
$$Rf\colon RX \to RY \in \mathbf{M}$$
is a weak equivalence (resp. a fibration) in $\mathbf{M}$.

2. *The sets of maps*
$$I' = \{LS_\alpha \to LD_\alpha\}$$
and
$$J' = \{L\Lambda_\beta \to L\Delta_\beta\}$$
are generators for the cofibrations and trivial cofibrations of $\mathbf{N}$ *respectively.*

3. *There exist functorial factorizations of maps in* $\mathbf{N}$ *into a cofibration followed by a trivial fibration, or respectively a trivial cofibration followed by a fibration.*

Proof. The proof is a variation of an argument of Quillen. To show that $\mathbf{N}$ is a simplicial closed model category, we carry out the small object argument [13, Ch.II,4], using the sets of maps I' and J' to construct the factorizations. The only tricky part of the argument is to show that if a map $f\colon X \to Y$ in $\mathbf{N}$ has the left lifting property with respect to maps $p \in \mathbf{N}$ for which Rp is a fibration, then Rf is a weak equivalence. To show this, we use the map $\epsilon_X\colon X \to EX$. Thus there is a lifting g in

$$\begin{tikzcd} X \arrow[r, "\epsilon_X"] \arrow[d, "f"'] & EX \\ Y. \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Since $R\epsilon_X$ is a weak equivalence, it suffices to show that Rg is also. To prove this, we show that $Ef \cdot g$ is simplicially homotopic to ϵ_Y; this follows from the lifting in the diagram

$$\begin{tikzcd} X \arrow[r, "\epsilon_Y f = Ef\cdot\epsilon_X"] \arrow[d, "f"'] & EY \arrow[r] & EY^{\Delta[1]} \arrow[d] \\ Y \arrow[rr, "{(\epsilon_Y, Ef\cdot g)}"] \arrow[urr, dashed] & & EY \times EY. \end{tikzcd}$$

□

3.1.6 Model category structure for simplicial sets

We recall that the category $\mathcal{S}$ of simplicial sets forms a simplicial closed model category.

3.1.7. Proposition. *[13] The category $\mathcal{S}$ of simplicial sets is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y \in \mathcal{S}$ is a **weak equivalence** if and only if its geometric realization $|f|\colon |X| \to |Y|$ is a homotopy equivalence of spaces.*

2. *A map $f\colon X \to Y \in \mathcal{S}$ is a **fibration** if and only if it is a Kan fibration.*

3. *A map $f\colon X \to Y \in \mathcal{S}$ is a **cofibration** if and only if it is an inclusion.*

4. *The sets of maps*
$$I = \{\dot{\Delta}[n] \to \Delta[n], \quad n \geq 0\}$$
and
$$J = \{\Lambda^k[n] \to \Delta[n], \quad n \geq 1, \quad 0 \leq k \leq n\}$$
are generators for the cofibrations and trivial cofibrations respectively.

3.1.8 Model category structure for G-simplicial sets

Let G be a discrete group. The category of G-simplicial sets, denoted by $G\mathcal{S}$, is the category with objects simplicial sets with an action of G on the right, and with morphisms G-equivariant maps.

3.1.9. Proposition. *[5] The category $G\mathcal{S}$ of G-simplicial sets is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y \in G\mathcal{S}$ is a **weak equivalence** (resp. a **fibration**) if and only if for each subgroup $H \subseteq G$ the induced map*
$$f^H\colon X^H \to Y^H \in \mathcal{S}$$
on fixed points is a weak equivalence (resp. a fibration) of simplicial sets.

2. *A map $f\colon X \to Y \in G\mathcal{S}$ is a **cofibration** if and only if it is an inclusion.*

3. *The sets of maps*
$$I = \{\dot{\Delta}[n] \times H\backslash G \to \Delta[n] \times H\backslash G, \quad n \geq 0, \quad H \subseteq G\}$$
and
$$J = \{\Lambda^k[n] \times H\backslash G \to \Delta[n] \times H\backslash G, \quad n \geq 0, \quad 0 \leq k \leq n, H \subseteq G\}$$
are generators for the cofibrations and trivial cofibrations respectively.

There is another characterization of weak equivalences in $G\mathcal{S}$.

3.1.10. Proposition. *A map $f\colon X \to Y \in G\mathcal{S}$ is a weak equivalence if and only if the induced map*

$$|f|\colon |X| \to |Y|$$

of geometric realizations is a G-equivariant homotopy equivalence; i.e., there exists a G-equivariant map $g\colon |Y| \to |X|$ and G-equivariant homotopies $|f|g \sim 1$ and $g|f| \sim 1$.

To prove Proposition 3.1.10 we need several results about G-spaces. The category $G\mathcal{T}$ of G-spaces is the category having as objects topological spaces with a G-action, and having as maps G-equivariant maps.

3.1.11. Proposition. *[5] The category $G\mathcal{T}$ of G-spaces is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y \in G\mathcal{T}$ is a **weak equivalence** (resp. a **fibration**) if and only if for each subgroup $H \subseteq G$ the induced map*

   $$f^H\colon X^H \to Y^H \in \mathcal{T}$$

   on fixed points is a weak equivalence (resp. a Serre fibration) of spaces.

2. *The sets of maps*

   $$I = \{|\dot{\Delta}[n]| \times H\backslash G \to |\Delta[n]| \times H\backslash G\}$$

   and

   $$J = \{|\Lambda^k[n]| \times H\backslash G \to |\Delta[n]| \times H\backslash G\}$$

   are generators for the cofibrations and trivial cofibrations respectively.

3.1.12. Proposition. *For $X \in G\mathcal{S}$, the adjunction map $\epsilon_X\colon X \to \mathrm{Sing}|X|$ is a weak equivalence in $G\mathcal{S}$.*

Proof. We note the easily proved fact that geometric realization of simplicial sets commutes with taking the fixed point set of a finite group action. (In fact, geometric realization preserves all finite limits; see [6, Ch.III,3].) Thus for $H \subseteq G$, we have

$$X^H \to \mathrm{Sing}|X|^H \simeq \mathrm{Sing}|X^H|,$$

which is a weak equivalence. □

Proof of Proposition 3.1.10. Since every object in $G\mathcal{T}$ is fibrant, a map between cofibrant objects is a weak equivalence if and only if it is a simplicial homotopy equivalence. Thus, a map f in $G\mathcal{S}$ is a weak equivalence if and only if its realization $|f|$ in $G\mathcal{T}$ is a simplicial homotopy equivalence, and thus a G-equivariant homotopy equivalence. □

3.1.13 Model category structure for simplicial R-modules

We recall that the category $\mathcal{M}_{\mathrm{R}}$ of simplicial R-modules forms a simplicial closed model category.

3.1.14. Proposition. *[13] The category $\mathcal{M}_{\mathrm{R}}$ of simplicial R-modules is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y \in \mathcal{M}_{\mathrm{R}}$ is a **weak equivalence** (resp. a **fibration**) if and only if it is a weak equivalence (resp. a fibration) of the underlying simplicial sets.*

2. *A map $f\colon X \to Y \in \mathcal{M}_{\mathrm{R}}$ is a **cofibration** if and only if in each simplicial degree $q \geq 0$ the maps $f_q\colon X_q \to Y_q$ are injective and the cokernel of f_q is a projective R-module.*

3. *The sets of maps*
$$I = \{R\dot{\Delta}[n] \to R\Delta[n]\}$$
and
$$J = \{R\Lambda^k[n] \to R\Delta[n]\}$$
are generators for the cofibrations and trivial cofibrations respectively.

Note that every object in $\mathcal{M}_{\mathrm{R}}$ is fibrant as a simplicial set by [12], and hence is fibrant in $\mathcal{M}_{\mathrm{R}}$.

3.1.15 Model category structure for G-equivariant simplicial R-modules

Let G be a discrete group. The category of G-equivariant simplicial R-modules, denoted by $G\mathcal{M}_{\mathrm{R}}$, is the category with objects simplicial R-modules with an action of G on the right, and with morphisms G-equivariant maps. Equivalently, $G\mathcal{M}_{\mathrm{R}}$ is the category of simplicial $R[G]$-modules, where $R[G]$ is the group ring of G.

3.1.16. Proposition. *The category $G\mathcal{M}_{\mathrm{R}}$ of G-equivariant simplicial R-modules is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y \in G\mathcal{M}_{\mathrm{R}}$ is a **weak equivalence** (resp. a **fibration**) if and only if for each subgroup $H \subseteq G$ the induced map*
$$f^H\colon X^H \to Y^H \in \mathcal{M}_{\mathrm{R}}$$
on invariants is a weak equivalence (resp. a fibration) of underlying simplicial sets.

2. *A map $f\colon X \to Y \in G\mathcal{M}_{\mathrm{R}}$ is a **cofibration** if and only in each simplicial degree $q \geq 0$ the maps $f_q\colon X_q \to Y_q$ are inclusions and the cokernel of f_q is a retract of a $R[G]$-module of the form RS, where S is a G-set.*

3. *The sets of maps*

$$I = \{R(\dot{\Delta}[n] \times H\backslash G) \to R(\Delta[n] \times H\backslash G)\}$$

and

$$J = \{R(\Lambda^k[n] \times H\backslash G) \to R(\Delta[n] \times H\backslash G)\}$$

are generators for the cofibrations and trivial cofibrations respectively.

Proof. Everything except part 2 follows from Proposition 3.1.5 applied to the adjunction

$$R : G\mathcal{S} \rightleftarrows G\mathcal{M}_{\mathrm{R}} : \mathcal{U},$$

where we take E to be the identity functor on $\mathcal{M}_{\mathrm{R}}$.

The characterization of cofibrations in part 2 follows from the small object argument. □

Note that every object in $G\mathcal{M}_{\mathrm{R}}$ is fibrant.

3.1.17. Corollary. *An object $X \in G\mathcal{M}_{\mathrm{R}}$ is cofibrant if and only if in each simplicial degree $q \geq 0$ the module X_q is a retract of a $R[G]$-module of the form RS, where S is a G-set.*

3.2 Model categories over simplicial sets

In this section we describe model category structures for Σ-objects, operads, algebras over an operad, and biobjects over an operad, when defined over the category of simplicial sets.

3.2.1 A model category structure for Σ-objects over simplicial sets

Given a subgroup $H \subseteq \Sigma_m$, we define $Z_H \in \mathbf{\Sigma}\mathcal{S}$ by

$$Z_H[k] = \begin{cases} H\backslash\Sigma_m & \text{if } k = m, \\ \varnothing & \text{otherwise.} \end{cases}$$

3.2.2. Proposition. *The category $\mathbf{\Sigma}\mathcal{S}$ of Σ-objects over simplicial sets is a simplicial closed model category with the following structure.*

1. *A map $f \colon X \to Y \in \mathbf{\Sigma}\mathcal{S}$ is a **weak equivalence** (resp. a **fibration**) if and only if for each $m \geq 0$ and each subgroup $H \subseteq \Sigma_m$ the induced map*

$$f[m]^H \colon X[m]^H \to Y[m]^H \in \mathcal{S}$$

on fixed points is a weak equivalence (resp. a fibration) of simplicial sets.

2. *A map $f\colon X \to Y \in \mathbf{\Sigma}\mathcal{S}$ is a* ***cofibration*** *if and only if each $f[m]\colon X \to Y$ is an inclusion of simplicial sets.*

3. *The sets of maps*
$$I = \{\dot{\Delta}[n] \otimes Z_H \to \Delta[n] \otimes Z_H\}$$
and
$$J = \{\Lambda^k[n] \otimes Z_H \to \Delta[n] \otimes Z_H\}$$
are generators for the cofibrations and trivial cofibrations respectively.

Proof. This follows from Proposition 3.1.9, since $\mathbf{\Sigma}\mathcal{S}$ is equivalent to a product of categories $\Sigma_m\mathcal{S}$. □

3.2.3. Proposition. *1. A map $f\colon X \to Y \in \mathbf{\Sigma}\mathcal{S}$ is a weak equivalence if and only if for each $m \geq 0$ the induced maps*
$$|f[m]|\colon |X[m]| \to |Y[m]|$$
of geometric realizations are Σ_m-equivariant homotopy equivalences.

2. *For $X \in \mathbf{\Sigma}\mathcal{S}$, the adjunction map $\epsilon_X\colon X \to \mathrm{Sing}|X|$ is a weak equivalence in $\mathbf{\Sigma}\mathcal{S}$.*

Proof. Immediate from Proposition 3.1.10. □

Note that the Σ-object I is cofibrant in $\mathbf{\Sigma}\mathcal{S}$.

3.2.4 A model category structure for A-algebras over simplicial sets

3.2.5. Proposition. *Let $A \in \mathrm{oper}\mathcal{S}$ be a simplicial operad. The category $\mathcal{S}^A$ of algebras over A is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y$ of A-algebras is a* ***weak equivalence*** *(resp. a* ***fibration****) if it is a weak equivalence (resp. a fibration) of underlying simplicial sets.*

2. *The sets of maps*
$$I = \{A(\dot{\Delta}[n]) \to A(\Delta[n])\}$$
and
$$J = \{A(\Lambda^k[n]) \to A(\Delta[n])\}$$
generate the cofibrations and trivial cofibrations respectively.

Proof. This follows from Proposition 3.1.5 applied to the adjunction
$$\mathcal{S} \rightleftarrows \mathcal{S}^A,$$
where we take $E = \mathrm{Sing}|-|$ and ϵ to be the adjunction map; by Corollary 2.4.5 the functor E is well defined, and by Proposition 3.1.7 the adjunction ϵ is always a weak equivalence in $\mathcal{S}$. □

3.2.6. Remark. Following Quillen [13] we note that a map $X \to Y \in \mathcal{S}^A$ is a cofibration if and only if it is a retract of a free map; a map $i\colon X \to Y \in \mathcal{S}^A$ is **free** if Y is the diagonal of a simplicial object in $\mathcal{S}^A$ with qth degree $X \amalg A(S_q)$, where the S_q are a collection of discrete simplicial sets which are closed under the degeneracy operators, and i is induced by inclusion of the constant simplicial object X.

3.2.7 A model category structure for A, B-biobjects over simplicial sets

3.2.8. Proposition. *Let $A, B \in$ oper$\mathcal{S}$ be operads. The category (A, B)-biobj$\mathcal{S}$ of A, B-biobjects is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y$ of A, B-biobjects is a **weak equivalence** (resp. a **fibration**) if it is a weak equivalence (resp. a fibration) in $\Sigma\mathcal{S}$.*
2. *The sets of maps*

$$I = \{A \circ (\dot{\Delta}[n] \otimes Z_H) \circ B \to A \circ (\Delta[n] \otimes Z_H) \circ B\}$$

and

$$J = \{A \circ (\Lambda^k[n] \otimes Z_H) \circ B \to A \circ (\Delta[n] \otimes Z_H) \circ B\}$$

generate the cofibrations and the trivial cofibrations respectively.

Proof. This follows from Proposition 3.1.5 applied to the adjunction

$$\Sigma\mathcal{S} \rightleftarrows (A, B)\text{-biobj}\mathcal{S},$$

where we take $E = \mathrm{Sing}|-|$ and ϵ to be the adjunction map; by Corollary 2.4.6 the functor E is well defined, and by Proposition 3.2.3 the adjunction ϵ is always a weak equivalence in $\Sigma\mathcal{S}$. □

3.2.9. Remark. Following Quillen [13] we note that a map $X \to Y \in (A, B)$-biobj$\mathcal{S}$ is a cofibration if and only if it is a retract of a free map; a map $i\colon X \to Y \in (A, B)$-biobj$\mathcal{S}$ is **free** if Y is the diagonal of a simplicial object in (A, B)-biobj$\mathcal{S}$ with qth degree $X \amalg A \circ S_q \circ B$, where the S_q are a collection of discrete Σ-objects which are closed under the degeneracy operators, and i is induced by inclusion of the constant simplicial object X.

3.2.10 A model category structure for operads over simplicial sets

3.2.11. Proposition. *The category* oper$\mathcal{S}$ *of operads is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y$ of operads is a **weak equivalence** (resp. a **fibration**) if it is a weak equivalence (resp. a fibration) in $\Sigma\mathcal{S}$.*

2. *The sets of maps*

$$I = \{F(\dot{\Delta}[n] \otimes Z_H) \to F(\Delta[n] \otimes Z_H)\}$$

and

$$J = \{F(\Lambda^k[n] \otimes Z_H) \to F(\Delta[n] \otimes Z_H)\}$$

generate the cofibrations and trivial cofibrations respectively.

Proof. This follows from Proposition 3.1.5 applied to the adjunction

$$\Sigma\mathcal{S} \rightleftarrows \mathrm{oper}\mathcal{S},$$

where we take $E = \mathrm{Sing}|-|$ and ϵ to be the adjunction map; by Corollary 2.4.4 the functor E is well defined, and by Proposition 3.2.3 the adjunction ϵ is always a weak equivalence in $\mathcal{S}$.

□

3.2.12. Remark. Following Quillen [13] we note that a map $X \to Y \in \mathrm{oper}\mathcal{S}$ is a cofibration if and only if it is a retract of a free map; a map $i\colon X \to Y \in \mathrm{oper}\mathcal{S}$ is **free** if Y is the diagonal of a simplicial object in $\mathrm{oper}\mathcal{S}$ with qth degree $X \amalg F(S_q)$, where the S_q are a collection of discrete Σ-objects which are closed under the degeneracy operators, and i is induced by inclusion of the constant simplicial object X.

3.3 Model category structures over simplicial modules

In this section we describe model category structures for Σ-objects, operads, algebras over an operad, and biobjects over an operad when defined the category of simplicial R-modules.

3.3.1 A model category structure for Σ-objects over simplicial modules

Given a subgroup $H \subseteq \Sigma_m$, let $RZ_H \in \Sigma\mathcal{M}_{\mathrm{R}}$ be defined by

$$RZ_H[k] = \begin{cases} R(\Sigma_m/H) & \text{if } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

3.3.2. Proposition. *The category $\Sigma\mathcal{M}_{\mathrm{R}}$ is a simplicial closed model category, with the following structure.*

1. *A map $f\colon X \to Y$ is a* ***weak equivalence*** *(resp. a* ***fibration****) if for every $H \subseteq \Sigma_m$ the map $f[m]^H\colon X[m]^H \to Y[m]^H$ is a weak equivalence (resp. a fibration) of simplicial sets.*

2. *The sets of maps*
$$I = \{\dot{\Delta}[n] \otimes RZ_H \to \Delta[n] \otimes RZ_H\}$$
and
$$J = \{\Lambda^k[n] \otimes RZ_H \to \Delta[n] \otimes RZ_H\}$$
generate the cofibrations and trivial cofibrations respectively.

Proof. This follows from Proposition 3.1.16, since $\Sigma\mathcal{M}_{\mathrm{R}}$ is equivalent to a product of categories $\Sigma_m\mathcal{M}_{\mathrm{R}}$. □

3.3.3. Proposition. *Let $f\colon X \to Y \in \Sigma\mathcal{M}_{\mathrm{R}}$ be a map. The following are equivalent.*

1. *The map f is a cofibration.*

2. *The map f is injective and* $\mathrm{Cok}(f)$ *is cofibrant.*

3. *In each simplicial degree $q \geq 0$ the maps $f_q[m]\colon X_q[m] \to Y_q[m]$ are injective and the cokernel of $f_q[m]$ is a retract of a Σ_m-module of the form RS, where S is a Σ_m-set.*

Proof. This follows immediately from Proposition 3.1.16 □

3.3.4. Corollary. *An object $X \in \Sigma\mathcal{M}_{\mathrm{R}}$ is cofibrant if and only if in each simplicial degree $q \geq 0$ the module $X_q[m]$ is a retract of a Σ_m-module of the form RS, where S is a Σ_m-set.*

3.3.5 A model category structure for A-algebras over simplicial modules

3.3.6. Proposition. *Let $A \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ be an operad. The category ${\mathcal{M}_{\mathrm{R}}}^A$ of algebras over A is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y$ of A-algebras is a **weak equivalence** (resp. a **fibration**) if it is a weak equivalence (resp. a fibration)of underlying simplicial sets.*

2. *The sets of maps*
$$I = \{A(R\dot{\Delta}[n]) \to A(R\Delta[n])\}$$
and
$$J = \{A(R\Lambda^k[n]) \to A(R\Delta[n])\}$$
generate the cofibrations and trivial cofibrations respectively.

Proof. This follows from Proposition 3.1.5 applied to the adjunction
$$\mathcal{M}_{\mathrm{R}} \rightleftarrows {\mathcal{M}_{\mathrm{R}}}^A,$$
where we take E to be the identity functor, since every object in ${\mathcal{M}_{\mathrm{R}}}^A$ is fibrant as a simplicial set. □

3.3.7. Remark. Following Quillen [13] we note that a map $X \to Y \in \mathcal{M}_{\mathrm{R}}{}^{A}$ is a cofibration if and only if it is a retract of a free map; a map $i\colon X \to Y \in \mathcal{M}_{\mathrm{R}}{}^{A}$ is **free** if Y is the diagonal of a simplicial object in $\mathcal{M}_{\mathrm{R}}{}^{A}$ with qth degree $X \amalg A(RS_q)$, where the S_q are a collection of discrete simplicial sets which are closed under the degeneracy operators, and i is induced by inclusion of the constant simplicial object X.

3.3.8 A model category structure for A, B-biobjects on simplicial modules

3.3.9. Proposition. *Let $A, B \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ be operads. The category (A, B)-$\mathrm{biobj}\mathcal{M}_{\mathrm{R}}$ of simplicial A, B-biobjects is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y$ of A, B-biobjects is a **weak equivalence** (resp. a **fibration**) if it is a weak equivalence (resp. a fibration) in $\Sigma\mathcal{M}_{\mathrm{R}}$.*
2. *The sets of maps*
$$I = \{A \circ (\dot{\Delta}[n] \otimes RZ_H) \circ B \to A \circ (\Delta[n] \otimes RZ_H) \circ B\}$$
and
$$J = \{A \circ (\Lambda^k[n] \otimes RZ_H) \circ B \to A \circ (\Delta[n] \otimes RZ_H) \circ B\}$$
generate the cofibrations and trivial cofibrations respectively.

Proof. This follows from Proposition 3.1.5 applied to the adjunction
$$\Sigma\mathcal{M}_{\mathrm{R}} \rightleftarrows (A, B)\text{-}\mathrm{biobj}\mathcal{M}_{\mathrm{R}},$$
where we take E to be the identity functor, since every object in (A, B)-$\mathrm{biobj}\mathcal{M}_{\mathrm{R}}$ is fibrant in $\Sigma\mathcal{M}_{\mathrm{R}}$. □

3.3.10. Remark. Following Quillen [13] we note that a map $X \to Y \in (A, B)$-$\mathrm{biobj}\mathcal{M}_{\mathrm{R}}$ is a cofibration if and only if it is a retract of a free map; a map $i\colon X \to Y \in (A, B)$-$\mathrm{biobj}\mathcal{M}_{\mathrm{R}}$ is **free** if Y is the diagonal of a simplicial object in (A, B)-$\mathrm{biobj}\mathcal{M}_{\mathrm{R}}$ with qth degree $X \amalg A \circ RS_q \circ B$, where the S_q are a collection of discrete objects in $\Sigma\mathcal{S}$ which are closed under the degeneracy operators, and i is induced by inclusion of the constant simplicial object X.

3.3.11 A model category structure for operads on simplicial modules

3.3.12. Proposition. *The category $\mathrm{oper}\mathcal{M}_{\mathrm{R}}$ of operads is a simplicial closed model category with the following structure.*

1. *A map $f\colon X \to Y$ of operads is a **weak equivalence** (resp. a **fibration**) if it is a weak equivalence (resp. a fibration) in $\Sigma\mathcal{M}_{\mathrm{R}}$.*

2. *The sets of maps*

$$I = \{F(\dot{\Delta}[n] \otimes RZ_H) \to F(\Delta[n] \otimes RZ_H)\}$$

and

$$J = \{F(\Lambda^k[n] \otimes RZ_H) \to F(\Delta[n] \otimes RZ_H)\}$$

generate the cofibrations and trivial cofibrations respectively.

Proof. This follows from Proposition 3.1.5 applied to the adjunction

$$\Sigma\mathcal{M}_{\mathrm{R}} \rightleftarrows \mathrm{oper}\mathcal{M}_{\mathrm{R}},$$

where we take E to be the identity functor, since every object in $\Sigma\mathcal{M}_{\mathrm{R}}$ is fibrant. □

3.3.13. Remark. Following Quillen [13] we note that a map $X \to Y \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ is a cofibration if and only if it is a retract of a free map; a map $i\colon X \to Y \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ is **free** if Y is the diagonal of a simplicial object in $\mathrm{oper}\mathcal{M}_{\mathrm{R}}$ with qth degree $X \amalg F(RS_q)$, where the S_q are a collection of discrete objects in $\Sigma\mathcal{S}$ which are closed under the degeneracy operators, and i is induced by inclusion of the constant simplicial object X.

3.4 Compatibility of actions with model category structure

In this section we investigate how well the actions of $\Sigma\mathcal{S}$ on $\mathcal{S}$ and $\Sigma\mathcal{S}$, and the actions of $\Sigma\mathcal{M}_{\mathrm{R}}$ on $\mathcal{M}_{\mathrm{R}}$ and $\Sigma\mathcal{M}_{\mathrm{R}}$ interact with the closed model category structures. In particular, we show that these actions satisfy axioms which are analogous to, but weaker than, those which characterize a simplicial closed model category structure (3.1.1).

3.4.1 Compatibility axioms

Let $\mathbf{M}$ be a monoidal category with a right-closed action on $\mathbf{C}$, and suppose that both $\mathbf{M}$ and $\mathbf{C}$ are closed model categories. We say that the action of $\mathbf{M}$ on $\mathbf{C}$ is **compatible with the closed model category structure** if either of the following equivalent statements are true.

(C1) Let $i\colon X \to Y \in \mathbf{C}$ be a cofibration between cofibrant objects, and let $j\colon F \to G \in \mathbf{M}$ be a cofibration. Then the induced maps

$$f\colon F(\varnothing) \to G(\varnothing)$$

and

$$g\colon F(Y) \amalg_{F(X)} G(X) \to G(Y)$$

are cofibrations in $\mathbf{C}$. If i is also a weak equivalence, then g is a trivial cofibration. If j is also a weak equivalence, then f and g are trivial cofibrations.

(C2) Let $i\colon X \to Y \in \mathbf{C}$ be a cofibration between cofibrant objects and $p\colon Z \to W \in \mathbf{C}$ be a fibration. Then the induced maps

$$f\colon \mathfrak{Hom}(\varnothing, Z) \to \mathfrak{Hom}(\varnothing, W)$$

and

$$g\colon \mathfrak{Hom}(Y, Z) \to \mathfrak{Hom}(X, Z) \times_{\mathfrak{Hom}(X,W)} \mathfrak{Hom}(Y, W)$$

are fibrations in $\mathbf{M}$. If i is also a weak equivalence, then g is a trivial fibration. If p is also a weak equivalence, then f and g are trivial fibrations.

If $\mathbf{M}$ is a right-closed monoidal category which is also a closed model category, we say that the monoidal structure on $\mathbf{M}$ is **compatible with the closed model category structure** if the induced right-closed action of $\mathbf{M}$ on itself is compatible with the closed model category structure.

Note that if $\mathbf{C}$ is a category enriched over simplicial sets having all enriched limits and colimits, then there is a right-closed action of $\mathcal{S}$ on $\mathbf{C}$. If $\mathbf{C}$ is a simplicial closed model category then we see that, in particular, the action of $\mathcal{S}$ on $\mathbf{C}$ is compatible with the model category structure.

3.4.2. Proposition. *Let $\mathbf{M}$ and $\mathbf{C}$ be closed model categories equipped with a right-closed action of $\mathbf{M}$ on $\mathbf{C}$ which is compatible with the model category structure.*

1. *If $X \in \mathbf{C}$ and $A \in \mathbf{M}$ are cofibrant objects, then $A \circ X \in \mathbf{C}$ is cofibrant.*
2. *If $X \to Y \in \mathbf{C}$ is a cofibration and $A \in \mathbf{M}$ is a cofibrant object, then $A \circ X \to A \circ Y \in \mathbf{C}$ is a cofibration.*

Proof. Statement 2 follows immediately from **(C1)**, and statement 1 follows from statement 2 and the fact that $A \circ \varnothing$ is cofibrant by **(C1)**. □

3.4.3 Compatibility of actions of Σ-objects over $\mathcal{S}$ and $\mathcal{M}_{\mathrm{R}}$

3.4.4. Proposition. *The action of $\Sigma\mathcal{S}$ on $\mathcal{S}$ and the action of $\Sigma\mathcal{M}_{\mathrm{R}}$ on $\mathcal{M}_{\mathrm{R}}$ are compatible with the closed model category structure.*

3.4.5. Proposition. *The monoidal structure of $\Sigma\mathcal{S}$ and the monoidal structure of $\Sigma\mathcal{M}_{\mathrm{R}}$ are compatible with the closed model category structure.*

Before we prove Proposition 3.4.4, we will need lemmas on equivariant homotopy. Let $\mathbf{C}$ denote either $\mathcal{S}$ or $\mathcal{M}_{\mathrm{R}}$. Note that if X and Y are in $\mathbf{C}$, then for $H \subseteq \Sigma_n$ a subgroup we have that.

$$\mathfrak{Hom}(X, Y)[n]^H = \mathrm{map}_{\mathbf{C}}(X^{\otimes n}, Y)^H = \mathrm{map}_{\mathbf{C}}((X^{\otimes n})_H, Y).$$

3.4.6. Lemma. *Let $i\colon X \to Y$ be a cofibration (resp. a trivial cofibration) of simplicial sets, and let $H \subseteq \Sigma_n$ be a subgroup of the symmetric group. Then the induced map $j\colon (X^{\times n})_H \to (Y^{\times n})_H$ is a cofibration (resp. a trivial cofibration) of simplicial sets.*

Proof. To see that j is a cofibration, it suffices to note that since i is an inclusion, then j is also. Furthermore, if i is a trivial cofibration, then $|i|: |X| \to |Y|$ has a homotopy inverse in $\mathcal{T}$. Hence $|i|^{\times n}$ has a Σ_n-equivariant homotopy inverse, and thus $|j|$ has a homotopy inverse in $\mathcal{T}$, and thus in particular j is a weak equivalence. $\square$

3.4.7. Lemma. *Let $i: X \to Y$ be a cofibration (resp. a trivial cofibration) of cofibrant simplicial R-modules, and let $H \subseteq \Sigma_n$ be a subgroup of the symmetric group. Then the induced map $j: (X^{\otimes n})_H \to (Y^{\otimes n})_H$ is a cofibration (resp. a trivial cofibration) of simplicial R-modules.*

Proof. Because X is cofibrant and i is a cofibration, we may assume without loss of generality that X and $\mathrm{Cok}(i)$ are degree-wise free R-modules. In other words, $X_q \simeq R(K_q)$ and $Y_q \simeq R(K_q) \oplus R(L_q)$ for some sets K_q and L_q. To show that $j: (X^{\otimes n})_H \to (Y^{\otimes n})_H$ is a cofibration it suffices to show that the induced maps

$$\left(R(K_q)^{\otimes n}\right)_H \to \left(R(K_q) \oplus R(L_q)^{\otimes n}\right)_H$$

are inclusions with projective cokernel. This follows easily from the fact that

$$\left(K_q^{\times n}\right)_H \to \left((K_q \amalg L_q)^{\times n}\right)_H$$

is an inclusion of sets.

If i is a trivial cofibration, then since all objects in $\mathcal{M}_{\mathrm{R}}$ are fibrant there exists a simplicial homotopy inverse to i, and hence j has a simplicial homotopy inverse. $\square$

Proof of Proposition 3.4.4. We prove **(C2)**. Let $\mathbf{C}$ denote either $\mathcal{S}$ or $\mathcal{M}_{\mathrm{R}}$.

To prove the first statement in **(C2)**, it suffices to note that $\mathfrak{Hom}(\varnothing, Z) \simeq Z[0]$.

Suppose $H \subseteq \Sigma_n$ is a subgroup. Then we see that the induced map $g[n]^H$ may be written as

$$g[n]^H: \operatorname{map}_{\mathbf{C}}\left((Y^{\otimes n})_H, Z\right) \to \operatorname{map}_{\mathbf{C}}\left((X^{\otimes n})_H, Z\right) \times_{\operatorname{map}_{\mathbf{C}}\left((X^{\otimes n})_H, W\right)} \operatorname{map}_{\mathbf{C}}\left((Y^{\otimes n})_H, W\right).$$

By Lemma 3.4.6 or 3.4.7 we have that $i_H: (X^{\otimes n})_H \to (Y^{\otimes n})_H$ is a cofibration (resp. a trivial cofibration) in $\mathbf{C}$ whenever i is one, and thus the result follows from the simplicial closed model category structure on $\mathbf{C}$. $\square$

We need two preliminary lemmas before we can prove Proposition 3.4.5. Recall that if A and B are in $\Sigma\mathbf{C}$, then for $H \subseteq \Sigma_n$ a subgroup we have that

$$\mathcal{F}(A, B)[n]^H = \Big(\prod_s \operatorname{map}_{\mathbf{C}}(A[s,n], B[s])^{\Sigma_s}\Big)^H \simeq \prod_s \operatorname{map}_{\mathbf{C}}(A[s,n]_H, B[s])^{\Sigma_s}.$$

3.4.8. Lemma. *Let $i: A \to B$ be a cofibration (resp. a trivial cofibration) in $\Sigma\mathcal{S}$, and let $H \subseteq \Sigma_n$ be a subgroup of the symmetric group. Then the induced map $j: A[s,n]_H \to B[s,n]_H$ is a cofibration (resp. a trivial cofibration) of Σ_s-equivariant simplicial sets.*

Proof. To see that j is a cofibration, it suffices to note that since i is an inclusion, then so is j. Furthermore, if i is a trivial cofibration, then the realizations $|i[n]|\colon |A[n]| \to |B[n]|$ have a Σ_n-equivariant homotopy inverse in $\mathcal{T}$. Thus the induced map $|A[s,n]| \to |B[s,n]|$ has a $\Sigma_s \times \Sigma_n$-equivariant homotopy inverse in $\mathcal{T}$, whence $|j|$ has a Σ_s-equivariant homotopy inverse in $\mathcal{T}$. Thus, in particular, j is a weak equivalence. $\square$

3.4.9. Lemma. *Let $i\colon A \to B$ be a cofibration (resp. a trivial cofibration) between cofibrant objects in $\Sigma\mathcal{M}_{\mathrm{R}}$, and let $H \subseteq \Sigma_n$ be a subgroup of the symmetric group. Then the induced map $j\colon A[s,n]_H \to B[s,n]_H$ is a cofibration (resp. a trivial cofibration) of Σ_s-equivariant simplicial R-modules.*

Proof. Because A is cofibrant and i is a cofibration, we may assume without loss of generality using Proposition 3.3.3 that A and $\mathrm{Cok}(i)$ are degree-wise free over R. In other words, $A_q \simeq R(K_q)$ and $B_q \simeq R(K_q) \oplus R(L_q)$ for some Σ-objects K_q and L_q on the category of sets. To show that $j\colon A[s,n]_H \to B[s,n]_H$ is a cofibration it suffices to show that the induced maps

$$R(K_q)[s,n]_H \to R(K_q \amalg L_q)[s,n]_H$$

are inclusions such that the cokernel is a free R-module on a Σ_s-set. This follows easily from the fact that

$$K_q[s,n]_H \to (K_q \amalg L_q)[s,n]_H$$

is an inclusion of Σ_s-sets.

If i is a trivial cofibration, then since all objects in $\Sigma\mathcal{M}_{\mathrm{R}}$ are fibrant there exists a simplicial homotopy inverse to i in $\Sigma\mathcal{M}_{\mathrm{R}}$, and hence j has a simplicial homotopy inverse in $\Sigma_s\mathcal{M}_{\mathrm{R}}$. $\square$

Proof of Proposition 3.4.5. We prove **(C2)**. Let $\mathbf{C}$ denote either $\mathcal{S}$ or $\mathcal{M}_{\mathrm{R}}$.

To prove the first statement it suffices to note that

$$\mathcal{F}(\varnothing, Z) \simeq \prod_{s \geq 0} Z[s]^{\Sigma_s}.$$

Suppose $H \subseteq \Sigma_n$. Then we see that the induced map $g[n]^H$ may be written as

$$\prod_s \mathrm{map}_{\mathbf{C}}(G[s,n]_H, Z[s]) \xrightarrow{g[n]^H}$$
$$\prod_s \mathrm{map}_{\mathbf{C}}(F[s,n]_H, Z[s]) \times_{\mathrm{map}_{\mathbf{C}}(F[s,n]_H, W[s])} \mathrm{map}_{\mathbf{C}}(G[s,n]_H, W[s]).$$

Since by Lemma 3.4.8 or 3.4.9 we have that $F[s,n]_H \to G[s,n]_H$ is a cofibration (resp. a trivial cofibration) of Σ_s-equivariant objects whenever i is one in $\Sigma\mathbf{C}$, the result follows from the simplicial closed model category structure of $\mathbf{C}$. $\square$

3.5 Cofibrant objects and forgetful functors

In this section we study how well forgetful functors preserve cofibrancy. For any $A \in \text{oper}\mathbf{C}$ we say that an A-algebra $X \in \mathbf{C}^A$ is **C-cofibrant** if it is cofibrant in the underlying category $\mathbf{C}$. We say that an A-biobject is **Σ-cofibrant** if it is cofibrant in the underlying category of Σ-objects. We say that an operad A is **Σ-cofibrant** if the unit map $\eta\colon I \to A$ is a cofibration in the underlying category of Σ-objects.

Over the category of simplicial sets, cofibrancy is preserved by these forgetful functors for trivial reasons.

3.5.1. Proposition. *Every operad in* $\text{oper}\mathcal{S}$ *is* Σ*-cofibrant, and for any* $A \in \text{oper}\mathcal{S}$ *every object in* $\mathcal{S}^A$ *(resp.* $A\text{-biobj}\mathcal{S}$*) is* $\mathcal{S}$*-cofibrant (resp.* Σ*-cofibrant).*

Proof. Every object in $\mathcal{S}$ and $\Sigma\mathcal{S}$ is cofibrant, and for any operad A the unit map $I \to A$ is an inclusion, and thus is a cofibration from a cofibrant object. □

Over the category of simplicial R-modules we need to be more careful. First note that over a field of characteristic zero things become simple.

3.5.2. Proposition.

1. *If R is a field, then for any* $A \in \text{oper}\mathcal{M}_{\mathrm{R}}$ *every object in* ${\mathcal{M}_{\mathrm{R}}}^A$ *is* Σ*-cofibrant.*

2. *If R is a field of characteristic zero, then for an operad* $A \in \text{oper}\mathcal{M}_{\mathrm{R}}$ *every object in* $A\text{-biobj}\mathcal{M}_{\mathrm{R}}$ *is* $\Sigma\mathcal{M}_{\mathrm{R}}$*-cofibrant, and if* $A \neq 0$ *then the operad A is* $\Sigma\mathcal{M}_{\mathrm{R}}$*-cofibrant.*

Proof. If R is a field, then every R-module is projective, and thus every object in $\mathcal{M}_{\mathrm{R}}$ is cofibrant. If R is a field of characteristic zero then the group ring $R[\Sigma_n]$ is semi-simple, and thus every $R[\Sigma_n]$-module is a retract of a free one, and thus every object in $\Sigma\mathcal{M}_{\mathrm{R}}$ is cofibrant. If A is a non-zero operad, then the map $I \to A$ is necessarily an inclusion, and thus A is Σ-cofibrant. □

More generally, we have the following.

3.5.3. Proposition. *Every cofibrant operad in* $\Sigma\mathcal{M}_{\mathrm{R}}$ *is* Σ*-cofibrant.*

We begin by proving the following lemma.

3.5.4. Lemma. *If* $A \in \Sigma\mathcal{M}_{\mathrm{R}}$ *is cofibrant, then the free operad FA is* Σ*-cofibrant.*

Proof. We use the construction of the free operad given in Section A. Thus, FA is the colimit of a sequence of maps $i_n\colon F_nA \to F_{n+1}A$, where $F_{n+1}A = I \amalg A \circ F_nA$ and where $i_0\colon I \to I \amalg A$ and $i_n = I \amalg A \circ i_{n-1}$ for $n > 0$. Thus it suffices to show that each i_n is a cofibration of Σ-objects.

Since A and I are cofibrant, i_0 is a cofibration between cofibrant objects. Now suppose we have shown that i_n is a cofibration between cofibrant objects. Then by Propositions 3.4.2 and 3.4.5 we see that $A \circ i_n$ is a cofibration between cofibrant objects, and it follows that i_{n+1} is a cofibration between cofibrant objects, as desired. □

Proof of 3.5.3. By the construction of the model category structure on operads, a cofibrant operad is a retract of one which is obtained as the diagonal of a simplicial object in oper$\mathcal{M}_{\mathrm{R}}$ which in degree n has the form FA_n, where $A_n \in \Sigma\mathcal{M}_{\mathrm{R}}$ is a cofibrant Σ-object. The result now follows by Lemma 3.5.4 and Proposition 3.3.3. □

3.5.5. Proposition. *If $A \in$ oper$\mathcal{M}_{\mathrm{R}}$ is a Σ-cofibrant operad, then every cofibrant A-algebra is $\mathcal{M}_{\mathrm{R}}$-cofibrant.*

Proof. Since A is Σ-cofibrant it follows from Proposition 3.4.4 that $A(RS)$ is also Σ-cofibrant, where $S \in \mathcal{S}$ is a simplicial set. Thus by Proposition 3.1.14 and Remark 3.3.7, we see that a cofibrant A-algebra is $\mathcal{M}_{\mathrm{R}}$-cofibrant. □

3.5.6. Proposition. *If $A, B \in$ oper$\mathcal{M}_{\mathrm{R}}$ are Σ-cofibrant operads, then every cofibrant A, B-biobject is Σ-cofibrant.*

Proof. Since A and B are Σ-cofibrant it follows from Proposition 3.4.5 that $A \circ (RS) \circ B$ is also Σ-cofibrant, where $S \in \Sigma\mathcal{S}$ is a Σ-object over simplicial sets. Thus by Corollary 3.3.4 and Remark 3.3.10, we see that a cofibrant A, B-biobject is Σ-cofibrant. □

3.6 Relations between homotopy categories

In this section we show that under good conditions weakly equivalent operads induce equivalent homotopy categories of algebras and biobjects. Let $\mathbf{C}$ denote *either* $\mathcal{S}$ or $\mathcal{M}_{\mathrm{R}}$.

3.6.1 Homotopy categories of algebras

Let $f: A \to B$ be a map of operads on $\mathbf{C}$. Then as noted in 2.3.17, f induces a pair of adjoint functors

$$f_* : \mathbf{C}^A \rightleftarrows \mathbf{C}^B : f^*,$$

where f^* denotes the forgetful functor and $f_*X = B \circ_A (X)$.

3.6.2. Proposition.

1. *A map j in $\mathbf{C}^B$ is a weak equivalence (resp. a fibration) in $\mathbf{C}^B$ if and only if f^*j is a weak equivalence (resp. a fibration) in $\mathbf{C}^A$.*

2. *The functor f_* takes cofibrations (resp. trivial cofibrations) in $\mathbf{C}^A$ to cofibrations (resp. trivial cofibrations) in $\mathbf{C}^B$.*

3. *The functor f_* takes weak equivalences between cofibrant objects in $\mathbf{C}^A$ to weak equivalences between cofibrant objects in $\mathbf{C}^B$.*

Proof. Part (1) is immediate. Parts (2) and (3) follow from (1) and the characterization of weak equivalences and fibrations in $\mathcal{S}$ and $\mathcal{M}_{\mathrm{R}}$. □

3.6.3. Corollary. *[13] If $f\colon A \to B \in \mathrm{oper}\mathbf{C}$, then the adjoint functors f_*, f^* induce a pair of adjoint functors on homotopy categories*

$$\mathrm{Ho}\,\mathbf{C}^A \rightleftarrows \mathrm{Ho}\,\mathbf{C}^B.$$

3.6.4. Proposition. *If X is a cofibrant A-algebra, and $f\colon A \to B$ is a weak equivalence of Σ-cofibrant operads, then the adjunction map $i\colon X \to f^*f_*X$ is a weak equivalence of A-algebras.*

Proof. Because X is cofibrant in $\mathcal{S}^A$ (resp. $\mathcal{M}_R{}^A$, by Remarks 3.2.6 (resp. 3.3.7) we may without loss of generality assume that X is a retract of an A-algebra of the form $\mathrm{diag}\, A(S_\bullet)$, where $A(S_\bullet)$ is a simplicial object in $\mathcal{S}^A$ (resp. $\mathcal{M}_R{}^A$) such that each S_q is a discrete set (resp. a free R-module on a discrete set) and the degeneracy operators are induced by maps between the S_q's. Thus it suffices to show that the map

$$\mathrm{diag}\, A(S_\bullet) \to f^*f_*\, \mathrm{diag}\, A(S_\bullet) \simeq B(S_\bullet)$$

is a weak equivalence, but this is clear because each map $A(S_q) \to B(S_q)$ is a weak equivalence by Proposition 3.4.4. □

3.6.5. Corollary. *Suppose $f\colon A \to B$ is a weak equivalence of Σ-cofibrant operads. Then the adjoint pair of functors f_* and f^* induces an equivalence of homotopy theories in the sense of Quillen between $\mathbf{C}^A$ and $\mathbf{C}^B$. In particular, there is an induced equivalence of homotopy categories*

$$\mathrm{Ho}\,\mathbf{C}^A \simeq \mathrm{Ho}\,\mathbf{C}^B.$$

Proof. It suffices to prove Quillen's criterion [13]: that for cofibrant $X \in \mathbf{C}^A$ and fibrant $Y \in \mathbf{C}^B$, a map $j\colon f_*X \to Y$ is a weak equivalence if and only if its adjoint $k\colon X \to f^*Y$ is a weak equivalence. In fact, k factors into

$$k\colon X \xrightarrow{i} f^*f_*X \xrightarrow{f^*j} f^*Y,$$

where the unit i of the adjunction is a weak equivalence by Proposition 3.6.4, since X is cofibrant. By Proposition 3.6.2 (1) the map f^*j is a weak equivalence if and only if j is, and the Corollary follows. □

3.6.6 Homotopy categories of biobjects

Let $f\colon A \to B$ and $g\colon A' \to B'$ be maps of operads on $\mathbf{C}$. Then as noted in 2.3.17, f and g induce a pair of adjoint functors

$$(f,g)_* : (A, A')\text{-biobj}\mathbf{C} \rightleftarrows (B, B')\text{-biobj}\mathbf{C} : (f,g)^*,$$

where $(f,g)^*$ denotes the forgetful functor and $(f,g)_*M = B \circ_A M \circ_{A'} B'$.

3.6.7. Proposition.

*1. A map j in B-*biobj$\mathbf{C}$ *is a weak equivalence (resp. a fibration) in B-*biobj$\mathbf{C}$ *if and only if f^*j is a weak equivalence (resp. a fibration) in A-*biobj$\mathbf{C}$.

*2. The functor f_* takes cofibrations (resp. trivial cofibrations) in A-*biobj$\mathbf{C}$ *to cofibrations (resp. trivial cofibrations) in B-*biobj$\mathbf{C}$.

*3. The functor f_* takes weak equivalences between cofibrant objects in A-*biobj$\mathbf{C}$ *to weak equivalences between cofibrant objects in B-*biobj$\mathbf{C}$.

Proof. Part (1) is immediate. Parts (2) and (3) follow from (1) and the characterization of weak equivalences and fibrations in $\Sigma\mathcal{S}$ and $\Sigma\mathcal{M}_{\mathrm{R}}$. □

3.6.8. Proposition. *If M is a cofibrant (A, A')-biobject, and $f \colon A \to B$ and $g \colon A' \to B'$ are weak equivalences of Σ-cofibrant operads, then the adjunction map $i \colon M \to (f, g)^*(f, g)_*M$ is a weak equivalence of (A, A')-biobjects.*

Proof. Because M is cofibrant in (A, A')-biobj$\mathcal{S}$ (resp. (A, A')-biobj$\mathcal{M}_{\mathrm{R}}$, by Remarks 3.2.9 (resp. 3.3.10) we may without loss of generality assume that M is a retract of an A, A'-algebra of the form $\operatorname{diag} A \circ S_\bullet \circ A'$, where $A \circ S_\bullet \circ A'$ is a simplicial object in (A, A')-biobj$\mathcal{S}$ (resp. (A, A')-biobj$\mathcal{M}_{\mathrm{R}}$) such that each S_q is a Σ-object such that each $S_q[n]$ is a discrete Σ_n-set (resp. a free R-module on a discrete Σ_n-set) and the degeneracy operators are induced by maps between the S_q's. Thus it suffices to show that the map

$$\operatorname{diag} A \circ S_\bullet \circ A' \to f^* f_* \operatorname{diag} A \circ S_\bullet \circ A' \simeq B \circ S_\bullet \circ B'$$

is a weak equivalence, but this is clear because each map $A \circ S_q \circ A' \to B \circ S_q \circ B'$ is a weak equivalence by Proposition 3.4.5. □

3.6.9. Corollary. *Suppose $f \colon A \to B$ and $g \colon A' \to B'$ are weak equivalences of Σ-cofibrant operads. Then the adjoint pair of functors $(f, g)_*$ and $(f, g)^*$ induces an equivalence of homotopy theories in the sense of Quillen between (A, A')-*biobj$\mathbf{C}$ *and (B, B')-*biobj$\mathbf{C}$. *In particular, there is an induced equivalence of homotopy categories*

$$\mathrm{Ho}(A, A')\text{-biobj}\mathbf{C} \simeq \mathrm{Ho}(B, B')\text{-biobj}\mathbf{C}.$$

Proof. This is similar to the proof of Corollary 3.6.5. □

3.7 Standard resolutions

In this section we define notions of bar complexes involving operads, and show that under mild hypotheses they can be used to produce cofibrant resolutions of algebras or biobjects.

3.7.1 Bar complexes

Let A and B be operads on some simplicial symmetric monoidal category $\mathbf{C}$. Given $X \in \mathbf{C}^A$ and $M \in (B, A)$-biobj$\mathbf{C}$, define the **bar complex** $\mathcal{B}(M, A, X)$ to be the simplicial object in $\mathbf{C}^B$ with nth degree $M \circ A^n(X)$, along with the obvious face and degeneracy maps. Then the diagonal $\operatorname{diag} \mathcal{B}(M, A, X)$ is an object in $\mathbf{C}^B$, and there is an augmentation map $\operatorname{diag} \mathcal{B}(M, A, X) \to M \circ_A (X)$ in $\mathbf{S}^B$.

Likewise, if A, B, and C are operads on some simplicial symmetric monoidal category $\mathbf{C}$, and given $M \in (B, A)\text{-biobj}\mathbf{C}$ and $N \in (A, C)\text{-biobj}\mathbf{C}$, define the **bar complex** $\mathcal{B}(M, A, N)$ to be the simplicial object in $(B, C)\text{-biobj}\mathbf{C}$ with nth degree $M \circ A^{\circ n} \circ N$, along with the obvious face and degeneracy maps. Then $\operatorname{diag} \mathcal{B}(M, A, N)$ is an object in $(B, C)\text{-biobj}\mathbf{C}$, and there is an augmentation map $\operatorname{diag} \mathcal{B}(M, A, N) \to M \circ_A N$ in $(B, C)\text{-biobj}\mathbf{C}$.

3.7.2 Realizations of bar complexes

For the remainder of this section, we shall let $\mathbf{C}$ denote *either* the category of simplicial sets or the category of simplicial R-modules.

Given a category $\mathbf{N}$ enriched over simplicial sets, let $s\mathbf{N}$ denote the category of simplicial objects in $\mathbf{N}$. Then there is a realization functor $\operatorname{diag}\colon s\mathbf{N} \to \mathbf{N}$, which is left adjoint to the functor $\mathbf{N} \to s\mathbf{N}$ which takes X to the simplicial object $[\mathbf{n}] \mapsto X^{\Delta[\mathbf{n}]}$.

3.7.3. Proposition. *If $A \in \operatorname{oper}\mathbf{C}$ is is Σ-cofibrant, then given a $\mathbf{C}$-cofibrant A-algebra $X \in \mathbf{C}^A$, and a Σ-cofibrant B, A-biobject $M \in (B, A)\text{-biobj}\mathbf{C}$, the B-algebra $\operatorname{diag} \mathcal{B}(M, A, X)$ is a cofibrant B-algebra.*

3.7.4. Corollary. *If $A \in \operatorname{oper}\mathbf{C}$ is Σ-cofibrant, and $X \in \mathbf{C}^A$ is $\mathbf{C}$-cofibrant, then $\operatorname{diag} \mathcal{B}(A, A, X) \to X$ is a cofibrant resolution of X as an A-algebra.*

Proof. This follows from Proposition 3.7.3 and the existence of a contracting homotopy in the underlying category $\mathbf{C}$ given by $\eta\colon A^n(X) \to A\big(A^n(X)\big) = A^{n+1}(X)$, which shows that the augmentation map is a weak equivalence of A-algebras. $\square$

3.7.5. Proposition. *If $A \in \operatorname{oper}\mathbf{C}$ is Σ-cofibrant, then given a Σ-cofibrant B, A-biobject $M \in (B, A)\text{-biobj}\mathbf{C}$ and a Σ-cofibrant A, C-biobject $N \in (A, C)\text{-biobj}\mathbf{C}$, the B, C-biobject $\operatorname{diag} \mathcal{B}(M, A, N)$ is a cofibrant B, C-biobject.*

3.7.6. Corollary. *If $A \in \operatorname{oper}\mathbf{C}$ is Σ-cofibrant, then $\operatorname{diag} \mathcal{B}(A, A, A)$ is a cofibrant resolution of A as an A-biobject.*

Proof. This follows from Proposition 3.7.5 and the existence of a contracting homotopy in the underlying category $\Sigma\mathbf{C}$ given by $\eta\colon \circ 1\colon A^{\circ(n+1)} \to A \circ A^{\circ(n+1)} = A^{\circ(n+2)}$, which shows that the augmentation map is a weak equivalence of A, A-biobjects. $\square$

To prove these, we need to describe the realization functor $\operatorname{diag}\colon s\mathbf{N} \to \mathbf{N}$ more explicitly. Let $\mathbf{N}$ be a simplicial category, and consider a simplicial object $Y_\bullet$ in $s\mathbf{N}$. The **latching object** $L_n Y_\bullet$ is defined to be the coequalizer in

$$\coprod_{0 \le i < j \le n} Y_{n-1} \rightrightarrows \coprod_{0 \le k \le n} Y_n \to L_n Y_\bullet,$$

where the top map takes the (i,j)th summand to the jth summand by s_i, and the bottom map takes the (i,j)th summand to the ith summand by s_{j-1}. Then it follows from the simplicial identities that there is a canonical map $L_nY_\bullet \to Y_{n+1}$, induced by the map which takes the ith summand in $\coprod Y_n$ to Y_{n+1} by s_i. We let $L_{-1} = \varnothing$.

It is well known that the realization $\operatorname{diag} Y_\bullet$ can be recovered from information about the latching spaces. In particular, if we let $\operatorname{diag}_0 Y_\bullet = Y_0$, and if we construct $\operatorname{diag}_n Y_\bullet$ inductively by a pushout square

$$\begin{tikzcd}
\Delta[n] \otimes L_{n-1}Y_\bullet \amalg_{\dot\Delta[n]\otimes L_{n-1}Y_\bullet} \dot\Delta[n] \otimes Y_n \arrow[r] \arrow[d] & \Delta[n] \otimes Y_n \arrow[d] \\
\operatorname{diag}_{n-1} Y_\bullet \arrow[r] & \operatorname{diag}_n Y_\bullet,
\end{tikzcd}$$

then $\operatorname{diag} Y_\bullet = \operatorname{colim}_n \operatorname{diag}_n Y_\bullet$. This leads to the following proposition.

3.7.7. Proposition. *If each map $L_{n-1}Y_\bullet \to Y_n$ for $n \geq 0$ is a cofibration, then $\operatorname{diag} Y_\bullet$ is cofibrant.*

Let A be an operad on $\mathbf{C}$. Write $s_i\colon A^{\circ n} \to A^{\circ(n+1)}$ for the maps $s_i = 1\circ\cdots\circ\eta\circ\cdots\circ 1$ induced by the unit map $\eta\colon 1 \to A$. Thus, $s_0 = \eta \circ 1 \circ \cdots \circ 1$ and $s_n = 1 \circ \cdots \circ 1 \circ \eta$. Define an object $K_n \in \Sigma\mathbf{C}$ to be the coequalizer in

$$\coprod_{0\leq i<j\leq n} A^{\circ(n-1)} \rightrightarrows \coprod_{0\leq k\leq n} A^{\circ n} \to K_n,$$

where the top map takes the (i,j)th summand to the jth summand by s_i, and the bottom map takes the (i,j)th summand to the ith summand by s_{j-1}. It follows from the simplicial identities that there is a canonical map $k\colon K_n \to A^{\circ(n+1)}$. We let $K_{-1} = \varnothing$.

3.7.8. Lemma. *For each $n \geq 0$, there is a pushout square in $\Sigma\mathbf{C}$ of the form*

$$\begin{tikzcd}
K_n \circ I \arrow[r, "1\circ\eta"] \arrow[d, "k\circ 1"'] & K_n \circ A \arrow[d] \\
A^{\circ(n+1)} \circ I \arrow[r] & K_{n+1}
\end{tikzcd}$$

such that the maps $1 \circ \eta\colon A^{\circ(n+1)} \circ I \to A^{\circ(n+2)}$ and $k \circ 1\colon K_n \circ A \to A^{\circ(n+2)}$ induce $k\colon K_{n+1} \to A^{\circ(n+2)}$.

Proof. The proof is by induction on $n \geq -1$. The case $n = -1$ is clear, since $K_0 = I$.

For $n \geq 0$ we consider the following diagram:

$$\begin{tikzcd}
\coprod\limits_{0\leq i<j\leq n} A^{\circ(n-1)} \circ A \arrow[r, shift left] \arrow[r, shift right] & \coprod\limits_{0\leq k\leq n} A^{\circ n} \circ A \arrow[r] & K_n \circ A \\
\coprod\limits_{0\leq i<j\leq n} A^{\circ(n-1)} \arrow[u, "1\circ\eta"] \arrow[r, shift left] \arrow[r, shift right] & \coprod\limits_{0\leq k\leq n} A^{\circ n} \arrow[u, "1\circ\eta"] \arrow[r] \arrow[d] & K_n \circ I \arrow[u, "1\circ\eta"] \arrow[d, "k\circ 1"] \\
& A^{\circ(n+1)} \arrow[r, equal] & A^{\circ(n+1)} \circ I.
\end{tikzcd}$$

The middle row is the coequalizer which defines K_n. The top row is $- \circ A$ applied to the middle row; since $- \circ A$ preserves colimits in $\mathbf{\Sigma C}$ the top row is a coequalizer. The colimit of the diagram consisting of all objects in the two left-most columns and all maps shown between them is clearly K_{n+1}. This colimit is also clearly the same as the colimit of the right-hand column, which is the desired push-out. □

3.7.9. Lemma. *If A is a Σ-cofibrant operad, then the map $k\colon K_n \to A^{\circ(n+1)}$ is a cofibration between cofibrant objects in $\mathbf{\Sigma C}$ for each $n \geq 0$.*

Proof. The proof proceeds by induction on n. For $n = 0$, it's clear since $K_0 = I \to A$ is a cofibration between cofibrant objects in $\mathbf{\Sigma C}$ since A is Σ-cofibrant. Suppose now we have that $K_{n-1} \to A^{\circ n}$ is a cofibration between cofibrant objects in $\mathbf{\Sigma C}$. Then since $I \to A$ is also cofibration between cofibrant objects, the result follows from Lemma 3.7.8 and Proposition 3.4.5. □

3.7.10. Lemma. *If A is a Σ-cofibrant operad, X a $\mathbf{C}$-cofibrant algebra, and M a Σ-cofibrant B, A-biobject, then the natural map $L_{n-1}\mathcal{B}(M, A, X) \to M \circ A^n(X)$ in $\mathbf{C}^B$ is a cofibration of B-algebras.*

Proof. First, note that $L_n\mathcal{B}(M, A, X) \simeq M \circ K_n(X)$. The map $K_n(X) \to A^n(X)$ is a cofibration in $\mathbf{C}$ by Lemma 3.7.9 and Proposition 3.4.4, and the lemma follows by applying $M \circ -$ and using Proposition 3.4.4. □

Proof of Proposition 3.7.3. From Lemma 3.7.10 and Proposition 3.7.7, we see that $\operatorname{diag}\mathcal{B}(M, A, X)$ is cofibrant in $\mathbf{C}^A$. □

3.7.11. Lemma. *The natural map $L_{n-1}\mathcal{B}(M, A, N) \to M \circ A^{\circ n} \circ N$ in (B, C)-biobj$\mathbf{C}$ is a cofibration of B, C-biobjects.*

Proof. First, note that $L_n\mathcal{B}(M, A, N) \simeq M \circ K_n \circ N$. The lemma then follows from Lemma 3.7.9. □

Proof of Proposition 3.7.5. From Lemma 3.7.11 and Proposition 3.7.7, we see that $\operatorname{diag}\mathcal{B}(M, A, N)$ is cofibrant in (B, C)-biobj$\mathbf{C}$. □

Chapter 4

Moduli spaces of algebra structures

4.1 Moduli spaces of algebra structures

In this section we define the endomorphism operad of a *diagram*, and hence the moduli space of algebra structures of a diagram. We then show that in certain cases maps between such moduli spaces are fibrations or equivalences.

As usual, $\mathbf{C}$ denotes *either* $\mathcal{S}$ or $\mathcal{M}_{\mathrm{R}}$.

4.1.1 Endomorphism operads for diagrams

In Section 2.2 we defined the endomorphism operad $\mathcal{E}_X$ of an object $X \in \mathbf{C}$. The endomorphism operad classifies algebra structures on X.

Let $\mathbf{D}$ be a small category, and let $X \colon \mathbf{D} \to \mathbf{C}$ denote a functor. An A**-algebra structure** on X consists of for each object $D \in \mathbf{D}$ an A-algebra structure on XD such that for each map $d \colon D \to D' \in \mathbf{D}$ the induced map $Xd \colon XD \to XD'$ is a map of A-algebras. Given a functor $F \colon \mathbf{B} \to \mathbf{D}$ we see that an A-algebra structure on X naturally restricts to an A-algebra structure on XF, where XF denotes the composite functor $XF \colon \mathbf{B} \to \mathbf{D}$.

Let $\mathbf{D}$ be a small category, and let $\mathbf{D}_0$ denote the set of objects in $\mathbf{D}$. Given a diagram $X \colon \mathbf{D} \to \mathbf{C}$, define the object $\mathcal{E}_X$ in $\Sigma\mathbf{C}$ to be the equalizer of the diagram

$$\mathcal{E}_X \to \prod_{\alpha \in \mathbf{D}_0} \mathfrak{Hom}(X_\alpha, X_\alpha) \rightrightarrows \prod_{\alpha \to \beta \in \mathbf{D}[\alpha,\beta]} \mathfrak{Hom}(X_\alpha, X_\beta).$$

4.1.2. Proposition. *Let $X \colon \mathbf{D} \to \mathbf{C}$ be a diagram. The object $\mathcal{E}_X$ in $\Sigma\mathbf{C}$ has the structure of an operad, induced as a subobject of the operad $\prod_{\alpha \in \mathbf{D}_0} \mathfrak{Hom}(X_\alpha, X_\alpha)$. Maps $A \to \mathcal{E}_X$ of operads are in one-to-one correspondence with A-algebra structures on the diagram X.*

The operad $\mathcal{E}_X$ is called the *endomorphism operad* of the diagram X. In particular, if $\mathbf{D}$ is the category with a single object α and a single morphism, then $\mathcal{E}_X$ is just the endomorphism operad of the space X_α, as defined in Section 2.2.

4.1.3. Proposition. *Given a diagram $X\colon \mathbf{D} \to \mathbf{C}$ and a functor $F\colon \mathbf{B} \to \mathbf{D}$ there is an induced map $\mathcal{E}_X \to \mathcal{E}_{XF}$.*

Given an operad A, let $A\{X\}$ denote the simplicial set defined by

$$A\{X\} = \mathrm{map}_{\mathrm{oper}\mathbf{C}}(A, \mathcal{E}_X).$$

We call $A\{X\}$ the *moduli space of A-algebra structures* on X.

Given a map $f\colon X \to Y \in \mathbf{C}$, we let $\mathcal{E}_f$ or $\mathcal{E}_{(X \xrightarrow{f} Y)}$ denote the endomorphism operad of the diagram $(\bullet \to \bullet) \to \mathbf{C}$ which takes the unique non-identity arrow of $(\bullet \to \bullet)$ to f, and let $A\{f\}$ or $A\{X \xrightarrow{f} Y\}$ denote the corresponding space of A-algebra structures. Likewise, given a string of maps $X_n \to \cdots \to X_0 \in \mathbf{C}$, we write $\mathcal{E}_{(X_n \to \cdots \to X_0)}$ for the endomorphism operad and $A\{X_n \to \cdots \to X_0\}$ for the space of A-algebra structures on this diagram.

4.1.4. Proposition. *Given a map $f\colon X \to Y$ in $\mathbf{C}$, we have that*

$$\mathcal{E}_f \simeq \mathfrak{Hom}(X, X) \times_{\mathfrak{Hom}(X,Y)} \mathfrak{Hom}(Y, Y).$$

More generally, we have the following.

4.1.5. Proposition. *Given a sequence of maps $X_n \to \cdots \to X_0$ in $\mathbf{C}$, we have that*

$$\mathcal{E}_{(X_n \to \cdots \to X_0)} \simeq \mathfrak{Hom}(X_n, X_n) \times_{\mathfrak{Hom}(X_n, X_{n-1})} \cdots \times_{\mathfrak{Hom}(X_1, X_0)} \mathfrak{Hom}(X_0, X_0).$$

4.1.6 Homotopical properties of endomorphism operads for diagrams

4.1.7. Proposition. *If $p\colon X \to Y$ is a fibration between cofibrant-fibrant objects in $\mathbf{C}$, then in*

$$\mathcal{E}_X \xleftarrow{f} \mathcal{E}_p \xrightarrow{g} \mathcal{E}_Y,$$

the map g is a fibration. If furthermore p is a trivial fibration between cofibrant-fibrant objects, then f and g are also weak equivalences.

Proof. Given Proposition 4.1.4, we see that to show that g is a fibration (resp. a trivial fibration), it suffices to show that the map $\mathfrak{Hom}(X, p)\colon \mathfrak{Hom}(X, X) \to \mathfrak{Hom}(X, Y)$ is. That $\mathfrak{Hom}(X, p)$ is a fibration (resp. a trivial fibration) follows from Corollary 3.4.4.

Given Proposition 4.1.4 and the fact that $\mathfrak{Hom}(X, p)$ is a fibration, we see that to show that f is a weak equivalence it suffices to show that the map

$$\mathfrak{Hom}(p, Y)\colon \mathfrak{Hom}(Y, Y) \to \mathfrak{Hom}(X, Y)$$

is. That $\mathfrak{Hom}(p, Y)$ is a weak equivalence follows from Corollary 3.4.4. □

4.1.8. Proposition. *If $i\colon X \to Y$ is a cofibration between cofibrant-fibrant objects in $\mathbf{C}$, then in*

$$\mathcal{E}_X \xleftarrow{f} \mathcal{E}_i \xrightarrow{g} \mathcal{E}_Y,$$

the map f is a fibration. If furthermore i is a trivial cofibration between cofibrant-fibrant objects, then f and g are also weak equivalences.

Proof. Given Proposition 4.1.4, we see that to show that f is a fibration (resp. a trivial fibration), it suffices to show that the map $\mathfrak{Hom}(i, Y)\colon \mathfrak{Hom}(Y, Y) \to \mathfrak{Hom}(X, Y)$ is. That $\mathfrak{Hom}(i, Y)$ is a fibration (resp. a trivial fibration) follows from Corollary 3.4.4.

Given Proposition 4.1.4 and the fact that $\mathfrak{Hom}(i, Y)$ is a fibration, we see that to show that g is a weak equivalence it suffices to show that the map

$$\mathfrak{Hom}(X, i)\colon \mathfrak{Hom}(X, X) \to \mathfrak{Hom}(X, Y)$$

is. That $\mathfrak{Hom}(X, i)$ is a weak equivalence follows from Corollary 3.4.4. □

We will actually need the following generalization of Proposition 4.1.7

4.1.9. Proposition. *If $X_n \to \cdots \to X_0$ are fibrations between cofibrant-fibrant objects in $\mathbf{C}$, then for any k such that $0 \leq k \leq n$ we have that in*

$$\mathcal{E}_{(X_n \to \cdots \to X_{k+1})} \xleftarrow{f} \mathcal{E}_{(X_n \to \cdots \to X_0)} \xrightarrow{g} \mathcal{E}_{(X_k \to \cdots \to X_0)},$$

the map g is a fibration. If furthermore all the maps in the sequence are trivial fibrations, then f and g are also weak equivalences.

Proof. Similar to the proof of Proposition 4.1.7. □

4.1.10 Homotopical properties of moduli spaces of algebra structures on diagrams

Propositions 4.1.7, 4.1.8, and 4.1.9 imply the following results about spaces of algebra structures.

4.1.11. Proposition. *If $i\colon X \to Y$ is a cofibration between cofibrant-fibrant objects in $\mathbf{C}$, and A is a cofibrant operad, then in*

$$A\{X\} \xleftarrow{f} A\{i\} \xrightarrow{g} A\{Y\},$$

the map f is a fibration. If furthermore i is a trivial cofibration between cofibrant-fibrant objects, then f and g are also weak equivalences.

4.1.12. Proposition. *If $p\colon X \to Y$ is a fibration between cofibrant-fibrant objects in $\mathbf{C}$, and A is a cofibrant operad, then in*

$$A\{X\} \xleftarrow{f} A\{p\} \xrightarrow{g} A\{Y\},$$

the map g is a fibration. If furthermore p is a trivial fibration between cofibrant-fibrant objects, then f and g are also weak equivalences.

4.1.13. Proposition. *If the maps $X_n \to \cdots \to X_0$ are trivial fibrations between cofibrant-fibrant objects in $\mathbf{C}$, and A is a cofibrant operad, then for any k such that $0 \leq k \leq n$ we have that in*

$$A\{X_n \to \cdots \to X_{k+1}\} \xleftarrow{f} A\{X_n \to \cdots \to X_0\} \xrightarrow{g} A\{X_k \to \cdots \to X_0\},$$

the map f is a weak equivalence and g is a trivial fibration.

4.1.14 An application to C-cofibrancy

We make use of the space of A-algebra structures to prove the following handy result, which will be used in Section 4.2.

4.1.15. Proposition. *Let A be a cofibrant operad. If $f\colon X \to Y$ is a cofibration of A-algebras from a $\mathbf{C}$-cofibrant object X, then Y is also $\mathbf{C}$-cofibrant.*

4.1.16. Lemma. *Consider $X \xrightarrow{i} Z \xrightarrow{p} Y \in \mathbf{C}$ such that i is a cofibration, p is a fibration, and X is cofibrant in $\mathbf{C}$. Then the induced map of endomorphism operads*

$$g\colon \mathcal{E}_{(X\to Z\to Y)} \longrightarrow \mathcal{E}_{(X\to Y)}$$

is a fibration. Furthermore g is a weak equivalence if either i or p is.

Proof. By Corollary 3.4.4 we see that the induced map

$$h\colon \mathfrak{Hom}(Z,Z) \longrightarrow \mathfrak{Hom}(X,Z) \times_{\mathfrak{Hom}(X,Y)} \mathfrak{Hom}(Z,Y)$$

is a fibration, and is furthermore a weak equivalence if either i or p is. The result follows because g is a pull-back of h, namely

$$g = \mathfrak{Hom}(X,X) \times_{\mathfrak{Hom}(X,Z)} h \times_{\mathfrak{Hom}(Z,Y)} \mathfrak{Hom}(Y,Y).$$

$\square$

Proof of Proposition 4.1.15. Choose a factorization $X \xrightarrow{i} Z \xrightarrow{p} Y$ in $\mathbf{C}$ of the map $f\colon X \to Y \in \mathbf{C}^A$ into a cofibration i followed by a trivial fibration p. By Lemma 4.1.16 and the fact that A is a cofibrant operad we see that there is a trivial fibration

$$A\{X \xrightarrow{i} Z \xrightarrow{p} Y\} \longrightarrow A\{X \xrightarrow{f} Y\}.$$

In particular, this map is a surjective map of simplicial sets! Thus there exists an A-algebra structure on Z making $X \xrightarrow{i} Z \xrightarrow{p} Y$ into a diagram in $\mathbf{C}^A$. Since p is therefore a trivial fibration of A-algebras there is a lift in the square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dotted] & Y, \end{tikzcd}$$

showing that Y is a retract of Z in $\mathbf{C}^A$, hence a retract in $\mathbf{C}$. Thus Y is cofibrant in $\mathbf{C}$ since Z is. $\square$

4.2 Relation between moduli space and classification spaces

In this section we prove Theorems 1.2.10 and 1.2.15. We let $\mathbf{C}$ denote either the category of simplicial sets or the category of simplicial R-modules.

Given a closed model category $\mathbf{N}$, we will let $w\mathbf{N}$ denote the subcategory of weak equivalences of $\mathbf{N}$. We also let $w\mathbf{N}$ denote the nerve of this category. Note that any functor $\pi\colon \mathbf{N} \to \mathbf{N}'$ which takes weak equivalences to weak equivalences induces a corresponding map $\pi\colon w\mathbf{N} \to w\mathbf{N}'$.

The proof proceeds in several steps. The first step is to replace the categories $w\mathbf{C}$ and $w\mathbf{C}^A$ by more convenient subcategories. Let $w'\mathbf{C}$ denote the subcategory of $w\mathbf{C}$ having as objects the fibrant and cofibrant objects of $\mathbf{C}$, and having as morphisms all trivial fibrations between such objects. Let $w'\mathbf{C}^A \subset w\mathbf{C}^A$ denote the pre-image of $w'\mathbf{C}$ under π. Thus $w'\mathbf{C}^A$ is the subcategory of $w\mathbf{C}^A$ having as objects all fibrant A-algebras X such that $\pi X \in \mathbf{C}$ is cofibrant, and having as morphisms all trivial fibrations between such objects.

4.2.1. Lemma. *If A is a cofibrant operad, the horizontal maps in the diagram*

$$\begin{tikzcd} w'\mathbf{C}^A \arrow[r] \arrow[d] & w\mathbf{C}^A \arrow[d] \\ w'\mathbf{C} \arrow[r] & w\mathbf{C} \end{tikzcd}$$

are weak equivalences.

Lemma 4.2.1 is proved below.

Let $w\mathbf{C}^{\Delta[-]\otimes A}$ denote the bisimplicial set of 1.2.8. Since A is a cofibrant operad, then so are $\Delta[t] \otimes A$ for $t \geq 0$, and thus by Lemma 4.2.1 the inclusion $w'\mathbf{C}^{\Delta[t]\otimes A} \to w\mathbf{C}^{\Delta[t]\otimes A}$ is a weak equivalence.

Furthermore, each map $\Delta[t] \otimes A \to \Delta[t'] \otimes A$ induced by a map $\Delta[t] \to \Delta[t']$ of simplicial sets is a weak equivalence of cofibrant operads, and thus by Corollary 3.6.5 induces an equivalence of homotopy theories in the sense of Quillen, and thus by a result of Dwyer-Kan [3] induces an equivalence of simplicial sets $w\mathbf{C}^{\Delta[t']\otimes A} \to w\mathbf{C}^{\Delta[t]\otimes A}$. In particular, there are natural equivalences

$$\operatorname{diag} w'\mathbf{C}^{\Delta[-]\otimes A} \xrightarrow{\sim} \operatorname{diag} w\mathbf{C}^{\Delta[-]\otimes A} \simeq w\mathbf{C}^A.$$

The third step is to get a hold of the homotopy fiber of the map

$$\operatorname{diag} w'\mathbf{C}^{\Delta[-]\otimes A} \to w'\mathbf{C},$$

using the following variation on Quillen's Theorem B [15].

4.2.2. Lemma. *Let $\mathbf{M}^\bullet$ be a simplicial object in the category of categories, let $\mathbf{N}$ be a category, and let $\pi\colon \mathbf{M}^\bullet \to \mathbf{N}$ be a functor to $\mathbf{N}$ considered as a constant simplicial object. Then the diagram of simplicial sets*

$$\begin{tikzcd} \operatorname{diag}(\pi \downarrow X) \arrow[r] \arrow[d] & \operatorname{diag}\mathbf{M}^\bullet \arrow[d] \\ (\mathbf{N} \downarrow X) \arrow[r] & \mathbf{N} \end{tikzcd}$$

will be a homotopy pull-back if each map $\operatorname{diag}(\pi \downarrow X) \to \operatorname{diag}(\pi \downarrow X')$ *induced by* $f\colon X \to X' \in \mathbf{N}$ *is a weak equivalence.*

Lemma 4.2.2 is proved below.

Thus it will suffice to show that for each map $X \to X' \in w'\mathbf{C}$, the induced map $\operatorname{diag}(\pi \downarrow X) \to \operatorname{diag}(\pi \downarrow X')$ is an equivalence, and that $\operatorname{diag}(\pi \downarrow X)$ is weakly equivalent to $A\{X\}$.

Note that the bisimplicial set $(\pi \downarrow X)$ has (s,t)-bisimplices given by

$$(\pi \downarrow X)_{st} = \coprod_{Y_s \to \cdots \to Y_0 \to X \in w'\mathbf{C}} A\{Y_s \to \cdots \to Y_0\}_t.$$

We see from Proposition 4.1.13 that the vertical maps in

$$\begin{tikzcd} \displaystyle\coprod_{Y_s \to \cdots \to Y_0 \to X \in w'\mathbf{C}} A\{Y_s \to \cdots \to Y_0\} \\ \displaystyle\coprod_{Y_s \to \cdots \to Y_0 \to X \in w'\mathbf{C}} A\{Y_s \to \cdots \to Y_0 \to X\}\} \arrow[u] \arrow[d] \\ \displaystyle\coprod_{Y_s \to \cdots \to Y_0 \to X \in w'\mathbf{C}} A\{X\} \end{tikzcd}$$

are weak equivalences, and we see that the bottom space is just $(w'\mathbf{C} \downarrow X) \times A\{X\} \simeq A\{X\}$; thus, each fiber $(\pi \downarrow X)$ is weakly equivalent to $A\{X\}$.

To see what a map $f\colon X \to X' \in w'\mathbf{C}$ induces, we look at another diagram:

$$\begin{array}{ccc}
\coprod_{Y_s\to\cdots\to X} A\{Y_s \to \cdots \to Y_0\} & \xrightarrow{\quad b \quad} & \coprod_{Y_s\to\cdots\to X'} A\{Y_s \to \cdots \to Y_0\} \\
\sim\big\uparrow & & \sim\big\uparrow \\
\coprod_{Y_s\to\cdots\to X} A\{Y_s \to \cdots \to Y_0 \to X \xrightarrow{f} X'\} & \longrightarrow & \coprod_{Y_s\to\cdots\to X'} A\{Y_s \to \cdots \to Y_0 \to X'\} \\
\sim\big\downarrow & & \sim\big\downarrow \\
\coprod_{Y_s\to\cdots\to X} A\{X \xrightarrow{f} X'\} & \xrightarrow{\quad a \quad} & \coprod_{Y_s\to\cdots\to X'} A\{X'\}.
\end{array}$$

The maps marked by $\sim$ are weak equivalences by Proposition 4.1.13, and a is just the map

$$a\colon (w'\mathbf{C} \downarrow X) \times A\{X \xrightarrow{f} X'\} \to (w'\mathbf{C} \downarrow X') \times A\{X'\},$$

which is a weak equivalence by Proposition 4.1.11, and because $(w'\mathbf{C} \downarrow X)$ and $(w'\mathbf{C} \downarrow X')$ are contractible. Thus b is a weak equivalence as desired. The proof is complete.

4.2.3 Proofs of technical lemmas

We still need to prove several lemmas. In the following, let $\mathbf{N}$ be a closed model category. Let $\mathbf{N}_f$ denote the full subcategory of $\mathbf{N}$ which contains all fibrant objects. Let $\mathbf{N}_u$ denote a full subcategory of $\mathbf{N}$ such that

1. $\mathbf{N}_u$ contains all cofibrant objects of $\mathbf{N}$,

2. $\mathbf{N}_u$ is **stable under cofibrations**. That is, if $X \in \mathbf{N}_u$ and $i\colon X \to Y \in \mathbf{N}$ is a cofibration, then $Y \in \mathbf{N}_u$.

Let $\mathbf{N}_{uf} = \mathbf{N}_u \cap \mathbf{N}_f$.

If $\mathbf{N}'$ is a full subcategory of a model category $\mathbf{N}$, let $w\mathbf{N}' \subseteq \mathbf{N}'$ denote the subcategory consisting of all the objects and all weak equivalences between them. Let $fw\mathbf{N}' \subseteq \mathbf{N}'$ denote the subcategory consisting of all the objects and all trivial fibrations between them.

4.2.4. Lemma. *The map $w\mathbf{N}_{uf} \to w\mathbf{N}$ is a weak equivalence.*

4.2.5. Lemma. *The map $\pi\colon fw\mathbf{N}_{uf} \to w\mathbf{N}_{uf}$ is a weak equivalence.*

Proof of Lemma 4.2.1. To show that $w'\mathbf{C} \to w\mathbf{C}$ is a weak equivalence, let $\mathbf{N} = \mathbf{C}$ and let $\mathbf{N}_u$ denote the subcategory of cofibrant objects. Then $w'\mathbf{C} = fw\mathbf{N}_{uf}$ and Lemmas 4.2.4 and 4.2.5 apply.

To show that $w'\mathbf{C}^A \to w\mathbf{C}^A$ is a weak equivalence, let $\mathbf{N} = \mathbf{C}^A$ and let $\mathbf{N}_u$ denote the subcategory of $\mathbf{C}$-cofibrant objects. Then since A is assumed to be a cofibrant operad, Proposition 4.1.15 shows that $\mathbf{N}_u$ satisfies the desired properties. Thus $w'\mathbf{C}^A = fw\mathbf{N}_{uf}$ and Lemmas 4.2.4 and 4.2.5 apply. $\square$

Proof of Lemma 4.2.4. We first prove that the inclusion $i\colon w\mathbf{N}_f \to w\mathbf{N}$ is a weak equivalence. Using the functorial factorization of Proposition 3.1.5 we can define a functor $j\colon w\mathbf{N} \to w\mathbf{N}_f$ and natural transformations $1 \to ij$ and $1 \to ji$, which show that i is a simplicial homotopy equivalence.

To show that the inclusion $i'\colon w\mathbf{N}_{uf} \to w\mathbf{N}_f$ is a weak equivalence, we again use the functorial factorization to define a functor $j'\colon w\mathbf{N}_f \to w\mathbf{N}_{uf}$ and natural transformations $i'j' \to 1$ and $j'i' \to 1$, which show that i' is a simplicial homotopy equivalence. This part of the proof makes use of the fact that $\mathbf{N}_u$ contains all cofibrant objects. $\square$

Proof of Lemma 4.2.5. By Quillen's Theorem A it suffices to show that for each $X \in \mathbf{N}_{uf}$ the nerve of $(X \downarrow \pi)$ is contractible.

Each category $(X \downarrow \mathbf{N})$ is a closed model category, with fibrant initial object X. It is easy to see that $(X \downarrow \pi)$ is equivalent to the full subcategory of $fw(X \downarrow \mathbf{N})_{uf}$ consisting of objects which are weakly equivalent to X. Here $(X \downarrow \mathbf{N})_u$ denotes the full subcategory of objects $X \to Y$ such that $Y \in \mathbf{N}_u$. Thus $(X \downarrow \mathbf{N})_u$ is stable under cofibrations. The result now follows from the following lemma, taking $(X \downarrow \mathbf{N})$ for $\mathbf{N}$ and $(X \downarrow \mathbf{N})_u$ for $\mathbf{N}_u$. $\square$

4.2.6. Lemma. *Let $\mathbf{N}$ be a closed model category with fibrant initial object X, and with a subcategory $\mathbf{N}_u$ stable under cofibrations which contains X. Then the full subcategory $\mathbf{N}_0$ of $fw\mathbf{N}_{uf}$ containing the initial object has contractible nerve.*

Proof. It suffices to show that any functor $F\colon \mathbf{D} \to \mathbf{N}_0$ from the subdivision category of a simplicial set induces a null homotopic map on nerves. The *subdivision* $\mathbf{D}$ of a simplicial set K is the category whose objects are the non-degenerate simplices $k \in K_n$ for some n, and whose morphisms $k \to l$ correspond to pairs of non-degenerate simplices such that l is a face of k. Note that $\mathbf{D}$ is actually a partial order on the set of non-degenerate simplices of K, with the 0-simplices of K as the maximal elements

The category $\mathbf{N}^{\mathbf{D}}$ of $\mathbf{D}$-diagrams in $\mathbf{N}$ admits a model category structure with the following properties.

1. A map of diagrams $F \to G$ is a weak equivalence if and only if each object $d \in \mathbf{D}$ is taken to a weak equivalence $f(d) \to g(d)$ in $\mathbf{N}$.
2. A map of diagrams $F \to G$ is a cofibration if and only if each object $d \in \mathbf{D}$ is taken to a cofibration $f(d) \to g(d)$ in $\mathbf{N}$.

Furthermore, I claim that if $i\colon \mathbf{D}' \to \mathbf{D}$ is the inclusion of a subcategory of $\mathbf{D}$ with the property that every non-empty subset of the set of objects of $\mathbf{D}'$ has a unique maximal element, then the restriction of a fibration $F \to G$ of $\mathbf{D}$-diagrams to $\mathbf{D}'$ is a fibration of $\mathbf{D}'$-diagrams.

Let $\bar{X}\colon \mathbf{D} \to \mathbf{N}$ denote the initial diagram, defined by $\bar{X}(d) = X$. Given a diagram $F\colon \mathbf{D} \to \mathbf{N}_0 \subseteq \mathbf{N}$, let $F \times \bar{X}\colon \mathbf{D} \to \mathbf{N}$ be the functor defined on objects by

$$(F \times \bar{X})(d) = F(d) \times X.$$

This functor actually lands in $f\mathbf{N}_f$, since X is fibrant. Factor the unique map $\bar{X} \to F \times \bar{X}$ into maps of diagrams

$$\bar{X} \xrightarrow{i} G \xrightarrow{p=p_1\times p_2} F \times \bar{X}$$

where i is a trivial cofibration and p is a fibration of $\mathbf{D}$-diagrams. We claim that the diagram G and the maps of diagrams $\bar{X} \xleftarrow{p_2} G \xrightarrow{p_1} F$ land in $\mathbf{N}_0$, and thus define a functor

$$\mathbf{D} \times (\bullet \leftarrow \bullet \rightarrow \bullet) \to \mathbf{N}_0$$

which homotops F to the constant functor.

First, we show that for each object d in $\mathbf{D}$, the maps $p_1(d)\colon G(d) \to F(d)$ and $p_2(d)\colon G(d) \to X$ are trivial fibrations. Since $F(d)$ and X are fibrant, it is enough to show that $p(d)\colon G(d) \to F(d) \times X$ is a fibration. Let $\mathbf{D}'$ be the subcategory of $\mathbf{D}$ consisting of the single object d and the identity map. Then the restriction of p to $\mathbf{D}'$ is a fibration of diagrams, hence $p(d)$ is a fibration as desired.

Next, we show that for each map $f\colon d_0 \to d_1$ in $\mathbf{D}$, the map $G(f)\colon G(d_0) \to G(d_1)$ is a trivial fibration. Since $G(f)$ is by construction a weak equivalence, it is enough to show that it is a fibration. Let $\mathbf{D}' = (d_0 \xrightarrow{f} d_1)$ be a subcategory of $\mathbf{D}$. Then the restriction of $p\colon G \to F'$ to $\mathbf{D}'$ is a fibration of $\mathbf{D}'$ diagrams, hence the map

$$G(d_0) \to G(d_1) \times_{F(d_1)\times X} \big(F(d_0) \times X\big)$$

is a fibration. Since by hypothesis the map $F(f)$ is a fibration and X is fibrant, this implies that $G(f)$ is a fibration as desired.

Finally, we note that G is fibrant since there is a fibration p to a fibrant object, and that G lies in the subcategory $\mathbf{N}_u$, since there is a cofibration i from an object in $\mathbf{N}_u$. $\square$

Proof of Lemma 4.2.2. The key element of Quillen's proof of Theorem B [15]is the fact that for any functor $X\colon \mathbf{D} \to \mathcal{S}$ such that for each $d\colon D \to D' \in \mathbf{D}$ the induced map Xd is a weak equivalence, the square

$$\begin{tikzcd} XD \arrow[r] \arrow[d] & Y \arrow[d] \\ \{D\} \arrow[r] & \mathbf{D} \end{tikzcd}$$

is a homotopy pull-back, where Y denotes the diagonal of the simplicial object in $\mathcal{S}$ defined by

$$t \mapsto \coprod_{D_0 \to \ldots \to D_t \in \mathbf{D}} XD_0.$$

Using this one may easily show that, given a functor $X_\bullet \colon \mathbf{D} \to \mathcal{S}^{\Delta^{\mathrm{op}}}$ to bisimplicial sets, the corresponding square

$$\begin{tikzcd} \operatorname{diag} X_\bullet D \arrow[r] \arrow[d] & \operatorname{diag} Y_\bullet \arrow[d] \\ \{D\} \arrow[r] & \mathbf{D} \end{tikzcd}$$

is also a weak equivalence. The lemma now follows by an argument similar to Quillen's [15, §1]. $\square$

Chapter 5

Cohomology of operads

5.1 Abelian group objects in simplicial R-module operads

In this section we describe abelian group objects in the category of biobjects over an operad, and show the existence of an abelianization functor.

5.1.1 Abelian group objects and pointed objects

Let $\mathbf{D}$ be a complete category, and let X be an object in $\mathbf{D}$. An **abelian group object** in $\mathbf{D}$ over X is a map $\pi\colon K \to X \in \mathbf{D}$ together with addition, unit, and inverse maps $+\colon K \times_X K \to K$, $(-1)\colon K \to K$, $0\colon X \to K$ in $(\mathbf{D} \downarrow X)$, which satisfy the usual abelian group axioms. We let $\mathrm{ab}(\mathbf{D} \downarrow X)$ denote the category of abelian group objects in $\mathbf{D}$ over X.

Again, suppose X is an object in some category $\mathbf{D}$. A **pointed object** in $\mathbf{D}$ over X is a map $\pi\colon K \to X$ together with a map $s\colon X \to K$ such that $\pi s = 1_X$. We let $\mathrm{pt}(\mathbf{D} \downarrow X)$ denote the category of pointed objects over X. There is an obvious forgetful functor $\mathrm{ab}(\mathbf{D} \downarrow X) \to \mathrm{pt}(\mathbf{D} \downarrow X)$.

Let $X \in \Sigma\mathcal{M}_{\mathrm{R}}$ be a Σ-object on simplicial R-modules. An abelian group object in $\Sigma\mathcal{M}_{\mathrm{R}}$ over X is a Σ-object K equipped with a direct sum splitting $K = X \oplus \bar{K}$; the X-summand maps in via the 0-section, and the $\bar{K}$ summand is the kernel of the projection $\pi\colon K \to X$. Conversely, every Σ-object K over X equipped with such a direct sum splitting in an abelian group object in a unique way. Thus the functor

$$\mathrm{ab}(\Sigma\mathcal{M}_{\mathrm{R}} \downarrow X) \to \Sigma\mathcal{M}_{\mathrm{R}}$$

sending $K \mapsto \bar{K}$ is an equivalence of categories.

5.1.2 Abelian group objects in categories of biobjects

Let A and B be operads over simplicial R-modules. Let M be an A, B-biobject. A map $\pi\colon K \to M \in (A, B)\text{-biobj}\mathcal{M}_{\mathrm{R}}$ equipped with a section $0_K\colon M \to K \in$

(A,B)-biobj$\mathcal{M}_\mathrm{R}$ can be an abelian group object over M in *at most* one way. Thus the forgetful functor

$$\mathrm{ab}((A,B)\text{-biobj}\mathcal{M}_\mathrm{R} \downarrow M) \to \mathrm{pt}((A,B)\text{-biobj}\mathcal{M}_\mathrm{R} \downarrow M)$$

is inclusion of a full subcategory. We give a characterization of the objects in this full subcategory.

An A,B-biobject is equivalent to a Σ-object K equipped with maps

$$\lambda\colon A[n] \otimes K[i_1] \otimes \cdots \otimes K[i_n] \to K[m]$$

and

$$\rho\colon K[n] \otimes B[i_1] \otimes \cdots \otimes B[i_n] \to K[m]$$

for $n \geq 0$, $i_j \geq 0$, $m = \sum i_j$, which satisfy certain unit, equivariance, and associativity relations. An **ideal** $I \subseteq K$ is a collection of submodules $\{I[n] \subseteq K[n]\}$ such that

1. $I[n]$ is preserved by the action of Σ_n on $K[n]$,
2. if $a \in A[n]$ and $k_1 \in K[i_1], \ldots, k_n \in K[i_n]$ such that $k_j \in I[i_j]$ for at least one value of j where $1 \leq j \leq n$, then $\lambda(a \otimes k_1 \otimes \cdots \otimes k_n) \in I[m]$,
3. if $k \in I[n]$ and $b_1 \in B[i_1], \ldots, b_n \in B[i_n]$, then $\rho(k \otimes b_1 \otimes \cdots \otimes b_n) \in I[m]$.

It is not hard to see that if $I \subseteq K$ is an ideal, then the Σ-object K/I defined by

$$(K/I)[n] = K[n]/I[n]$$

has a canonical A,B-biobject structure making the quotient map $K \to K/I$ into a map of A,B-biobjects. Conversely, the kernel of a map of A,B-biobjects is an ideal.

Suppose $\pi\colon K \to M$ is a map of A,B-biobjects; let I be the kernel of this map. Let $J_K \subseteq K$ denote the set of elements of K which are of the form $\lambda(a \otimes k_1 \otimes \cdots \otimes k_n)$, where $k_j \in I[i_j]$ for at least *two* distinct indices j in the set $\{1, \ldots, n\}$. Let $(J_K) \subseteq K$ denote the smallest ideal containing J_K.

5.1.3. Proposition. *An object $K \in \mathrm{pt}((A,B)\text{-biobj}\mathcal{M}_\mathrm{R} \downarrow M)$ is an abelian group object if and only if $J_K = 0$.*

5.1.4. Corollary. *There is an* ***abelianization*** *functor*

$$\mathcal{A}\colon ((A,B)\text{-biobj}\mathcal{M}_\mathrm{R} \downarrow M) \to \mathrm{ab}((A,B)\text{-biobj}\mathcal{M}_\mathrm{R} \downarrow M)$$

which is left adjoint to the forgetful functor.

Proof. We define $\mathcal{A}$ as follows. Given $K \to M \in (A,B)$-biobj$\mathcal{M}_\mathrm{R}$, let $M * K \in (A,B)$-biobj$\mathcal{M}_\mathrm{R}$ denote the coproduct in the category of A,B-biobjects of M and K. Clearly $M * K$ is equipped with a projection map $M * K \to M$ and a section $M \to M * K$. Define

$$\mathcal{A}(K) = M * K/(J_{M*K}).$$

$\square$

5.1.5. Proposition. *Limits and colimits exist in* $\mathrm{ab}((A,B)\text{-biobj}\mathcal{M}_{\mathrm{R}} \downarrow M)$, *and are computed in* $\mathrm{pt}(\Sigma\mathcal{M}_{\mathrm{R}} \downarrow M)$.

Proof. The statement about limits is clear. For formal reasons, in any category of abelian group objects finite coproducts are isomorphic to finite products; thus finite coproducts exist and are computed in the underlying category $(\Sigma\mathcal{M}_{\mathrm{R}} \downarrow M)$.

Filtered colimits and reflexive coequalizers may also be computed in the underlying category: we already know that filtered colimits and reflexive coequalizers in (A,B)-biobj$\mathcal{M}_{\mathrm{R}}$ are computed in Σ-objects, and since these colimits commute with finite limits we see that the filtered colimit (resp. reflexive coequalizer) in (A,B)-biobj$\mathcal{M}_{\mathrm{R}}$ of a diagram of abelian group objects is itself an abelian group object.

The proposition follows, since any colimit can be built out of finite coproducts, filtered colimits, and reflexive coequalizers. □

5.1.6. Corollary. *The forgetful functor*

$$\mathrm{ab}((A,B)\text{-biobj}\mathcal{M}_{\mathrm{R}} \downarrow M) \to \mathrm{ab}(\Sigma\mathcal{M}_{\mathrm{R}} \downarrow M)$$

is exact.

5.1.7 Abelian group objects in the category of operads

In Section 5.2 we will show that for an operad A over simplicial R-modules, the category $\mathrm{ab}(\mathrm{oper}\mathcal{M}_{\mathrm{R}} \downarrow A)$ is naturally equivalent to the category $\mathrm{ab}(A\text{-biobj}\mathcal{M}_{\mathrm{R}} \downarrow A)$.

5.1.8. Proposition. *There is an abelianization functor*

$$\mathcal{D}\colon (\mathrm{oper}\mathcal{M}_{\mathrm{R}} \downarrow A) \to \mathrm{ab}(\mathrm{oper}\mathcal{M}_{\mathrm{R}} \downarrow A)$$

which is left adjoint to the forgetful functor.

5.2 The fundamental exact sequence

Let $\mathbf{M}$ denote a complete right-closed monoidal category.

5.2.1 An equivalence of categories of abelian group objects

Let K be an abelian group object in $(\mathrm{mon}\mathbf{M} \downarrow A)$. Altogether, K has the following structure maps:

$$\eta_A : I \longrightarrow A,$$

$$\mu_A : A \circ A \longrightarrow A,$$

which make A into a monoid,

$$\eta_K : I \longrightarrow K,$$

$$\mu_K : K \circ K \longrightarrow K,$$

which make K into a monoid,

$$\pi : K \longrightarrow A,$$

which is a map of monoids, and

$$0 : A \longrightarrow K,$$

$$(-1) : K \longrightarrow K,$$

$$+ : K \times_A K \longrightarrow K,$$

which are maps of monoids which make K into an abelian group object as a monoid over A.

Recall that if K is an abelian group object over A, the functor $(\mathbf{M} \downarrow A)[-, K]$ lands in abelian groups. In particular, given an object $X \in (\mathbf{M} \downarrow A)$ and maps $f, g \in (\mathbf{M} \downarrow A)[X, K]$, we can add them, obtaining $f + g : X \to K$; the map $0\pi_X$ is the zero element of this abelian group, where $\pi_X : X \to A$. Furthermore, given maps $f_1, f_2 : X \to K$ and $g_1, g_2 : Y \to K$ in $(\mathbf{M} \downarrow A)$ we have an equality

$$\mu_K(f_1 \circ g_1) + \mu_K(f_2 \circ g_2) = \mu_K\big((f_1 + f_2) \circ (g_1 + g_2)\big),$$

where both sides are elements in $(\mathbf{M} \downarrow A)[X \circ Y, K]$; this equality follows from the fact that $+$ is a map of monoids over A.

Let $\ell : A \circ K \to K$ and $r : K \circ A \to K$ be defined by $\ell = \mu_K(0 \circ 1)$ and $r = \mu_K(1 \circ 0)$ respectively. The maps ℓ and r make K into a A-biobject over A, and since $+$ commutes with ℓ and r we see that K is an abelian group object in $({}_A\mathbf{M}_A \downarrow A)$.

5.2.2. Lemma. *Given K in* $\mathrm{ab}(\mathrm{mon}\mathbf{M} \downarrow A)$, *the identities*

$$\mu_K = \ell(\pi \circ 1) + r(1 \circ \pi) \tag{5.2.3}$$

and

$$\eta_K = 0\eta_A \tag{5.2.4}$$

hold.

Proof. To show 5.2.3, note that we have that

$$\ell(\pi \circ 1) = \mu_K(0 \circ 1)(\pi \circ 1) = \mu_K(0\pi \circ 1)$$

and

$$r(1 \circ \pi) = \mu_K(1 \circ 0)(1 \circ \pi) = \mu_K(1 \circ 0\pi),$$

whence

$$\begin{aligned}
\ell(\pi \circ 1) + r(1 \circ \pi) &= \mu_K(0\pi \circ 1) + \mu_K(1 \circ 0\pi) \\
&= \mu_K\big((0\pi + 1) \circ (1 + 0\pi)\big) \\
&= \mu_K(1 \circ 1) \\
&= \mu_K.
\end{aligned}$$

Equation 5.2.4 from the fact that 0 is a map of monoids over A. □

5.2.5. Theorem. *The functor*

$$\mathrm{ab}(\mathrm{mon}\mathbf{M} \downarrow A) \longrightarrow \mathrm{ab}({}_A\mathbf{M}_A \downarrow A)$$

which forgets structure is an equivalence of categories.

Proof. It is easy to see that given K in $\mathrm{ab}({}_A\mathbf{M}_A \downarrow A)$ we can give K the structure of a monoid using $\mu_K = \ell(\pi \circ 1) + r(1 \circ \pi)$ and $\eta_K = 0\eta_A$; this produces a functor $\mathrm{ab}({}_A\mathbf{M}_A \downarrow A) \to \mathrm{ab}(\mathrm{mon}\mathbf{M} \downarrow A)$. Now by Lemma 5.2.2 we see that we get an equivalence of categories. □

5.2.6 Derivations

Given a monoid A in $\mathbf{M}$ and an abelian group object $K \in \mathrm{ab}(\mathrm{mon}\mathbf{M} \downarrow A)$, a **derivation** from A to K is a map $f \in (\mathbf{M} \downarrow A)[A, K]$ such that

$$f\mu_A = \ell(1 \circ f) + r(f \circ 1).$$

We write $\mathrm{Der}(A, K)$ for the set of derivations from A to K.

5.2.7. Lemma. *Let K be an object in* $\mathrm{ab}(\mathrm{mon}\mathbf{M} \downarrow A)$. *Then*

$$\mathrm{Der}(A, K) \simeq (\mathrm{mon}\mathbf{M} \downarrow A)[A, K].$$

Proof. By Lemma 5.2.2 we see that

$$\begin{aligned}
\mu_K(f \circ f) &= \big(\ell(\pi \circ 1) + r(1 \circ \pi)\big)(f \circ f) \\
&= \ell(\pi f \circ f) + r(f \circ \pi f) \\
&= \ell(1 \circ f) + r(f \circ 1).
\end{aligned}$$

It follows that f is a derivation if and only if $f\mu_A = \mu_K(f \circ f)$. In particular, a map of monoids is a derivation.

Conversely, if f is a derivation, we have

$$\begin{aligned}
f\eta_A &= f\mu_A(\eta_A \circ \eta_A) \\
&= \big(\ell(1 \circ f) + r(f \circ 1)\big)(\eta_A \circ \eta_A) \\
&= \ell(\eta_A \circ f\eta_A) + r(f\eta_A \circ \eta_A) \\
&= f\eta_A + f\eta_A,
\end{aligned}$$

whence $f\eta_A = 0\eta_A = \eta_K$, and thus f is a map of monoids. $\square$

5.2.8 Fundamental exact sequence

We suppose there are adjoint pairs

$$\mathcal{A} : (\mathbf{M} \downarrow A) \rightleftarrows \mathrm{ab}(\mathbf{M} \downarrow A) : \mathcal{U},$$

$$\mathcal{A}_\ell : ({}_A\mathbf{M} \downarrow A) \rightleftarrows \mathrm{ab}({}_A\mathbf{M} \downarrow A) : \mathcal{U}_\ell,$$

$$\mathcal{A}_r : (\mathbf{M}_A \downarrow A) \rightleftarrows \mathrm{ab}(\mathbf{M}_A \downarrow A) : \mathcal{U}_r,$$

and

$$\mathcal{A}_b : ({}_A\mathbf{M}_A \downarrow A) \rightleftarrows \mathrm{ab}({}_A\mathbf{M}_A \downarrow A) : \mathcal{U}_b,$$

as well as

$$\mathcal{D} : (\mathrm{mon}\mathbf{M} \downarrow A) \rightleftarrows \mathrm{ab}(\mathrm{mon}\mathbf{M} \downarrow A) : \mathcal{U}_{\mathrm{mon}},$$

where $\mathcal{U}_.$ denote the obvious forgetful functors.

Let us also assume that the forgetful functors

$$\mathrm{ab}({}_A\mathbf{M}_A \downarrow A) \to \mathrm{ab}({}_A\mathbf{M} \downarrow A) \to \mathrm{ab}(\mathbf{M} \downarrow A)$$

and

$$\mathrm{ab}({}_A\mathbf{M}_A \downarrow A) \to \mathrm{ab}(\mathbf{M}_A \downarrow A) \to \mathrm{ab}(\mathbf{M} \downarrow A)$$

are exact.

The above conditions are satisfied in the case when $\mathbf{M}$ is $\Sigma\mathcal{M}_{\mathrm{R}}$, as was shown in Section 5.1.

5.2.9. Proposition. *In* $\mathrm{ab}({}_A\mathbf{M}_A \downarrow A)$ *there is an exact sequence*

$$\mathcal{A}_b(A \circ A \circ A \circ A) \xrightarrow{d} \mathcal{A}_b(A \circ A \circ A) \longrightarrow \mathcal{D}(A) \longrightarrow 0.$$

Proof. Define d by $d = \mathcal{A}_b(d_0) - \mathcal{A}_b(d_1) + \mathcal{A}_b(d_2)$, where $d_0 = \mu \circ 1 \circ 1$, $d_1 = 1 \circ \mu \circ 1$ and $d_2 = 1 \circ 1 \circ \mu$. Let K be an abelian group object in $({}_A\mathbf{M}_A \downarrow A)$. If we let $d^* = \mathrm{ab}({}_A\mathbf{M}_A \downarrow A)[d, K]$, then by standard adjunctions we get an exact sequence

$$(\mathbf{M} \downarrow A)[A \circ A, K] \xleftarrow{d^*} (\mathbf{M} \downarrow A)[A, K] \longleftarrow \mathrm{ab}({}_A\mathbf{M}_A \downarrow A)[\mathrm{Cok}\, d, K] \longleftarrow 0,$$

where $d^*(f) = \ell(1 \circ f) - f\mu_A + r(f \circ 1)$. Now using Lemma 5.2.7 we see that $\mathrm{Cok}\, d$ has the same universal property as $\mathcal{D}(A)$. $\square$

Recall from Section 2.1 that since $\mathbf{M}$ is right-closed, so is $(\mathbf{M} \downarrow A)$, where the function object functor is denoted $\mathcal{F}_A(-,-)$.

5.2.10. Proposition. *The functor* $\mathcal{F}_A(-,-) : (\mathbf{M} \downarrow A)^{\mathrm{op}} \times (\mathbf{M} \downarrow A) \to (\mathbf{M} \downarrow A)$ *underlies a functor*

$$\mathcal{F}_A(-,-) : ({}_A\mathbf{M} \downarrow A)^{\mathrm{op}} \times ({}_A\mathbf{M} \downarrow A) \longrightarrow ({}_A\mathbf{M}_A \downarrow A)$$

which has the property that for X in $({}_A\mathbf{M}_A \downarrow A)$ and Y in $({}_A\mathbf{M} \downarrow A)$, then

$$({}_A\mathbf{M} \downarrow A)[X,Y] \simeq ({}_A\mathbf{M}_A \downarrow A)[X, \mathcal{F}_A(A,Y)].$$

Proof. This is immediate from Proposition 2.1.23. $\square$

5.2.11. Proposition. *Taking $\mathcal{F}_A(-,-)$ as in 5.2.10, if X and Y are abelian group objects in $({}_A\mathbf{M}_A \downarrow A)$ and $({}_A\mathbf{M} \downarrow A)$ respectively, then $\mathcal{F}_A(A,Y)$ is naturally an abelian group object, and we have a bijection*

$$\mathrm{ab}({}_A\mathbf{M} \downarrow A)[X,Y] \simeq \mathrm{ab}({}_A\mathbf{M}_A \downarrow A)[X, \mathcal{F}_A(A,Y)].$$

Proof. This follows from Proposition 5.2.10 and the fact that

$$\mathcal{F}_A(A,Y) \times_A \mathcal{F}_A(A,Y) \simeq \mathcal{F}_A(A, Y \times_A Y).$$

$\square$

5.2.12. Proposition. *Given X in $\mathrm{ab}({}_A\mathbf{M}_A \downarrow A)$, the natural map $f : \mathcal{A}_\ell(X) \to \mathcal{A}_b(X)$ is an isomorphism.*

Proof. Using Propositions 5.2.10 and 5.2.11 we see that, given an abelian group object Y in $({}_A\mathbf{M} \downarrow A)$, there are natural bijections

$$\begin{aligned} \mathrm{ab}({}_A\mathbf{M} \downarrow A)[\mathcal{A}_b(X), Y] &\simeq \mathrm{ab}({}_A\mathbf{M}_A \downarrow A)[\mathcal{A}_b(X), \mathcal{F}_A(A,Y)] \\ &\simeq ({}_A\mathbf{M}_A \downarrow A)[X, \mathcal{F}_A(A,Y)] \\ &\simeq ({}_A\mathbf{M} \downarrow A)[X,Y]. \end{aligned}$$

Thus $\mathcal{A}_b$ has the universal property which characterizes $\mathcal{A}_\ell$. $\square$

Let $X_\bullet$ be the simplicial object in $\mathrm{ab}({}_A\mathbf{M}_A \downarrow A)$ defined by

$$X = \mathcal{A}_b\mathcal{B}(A,A,A).$$

5.2.13. Proposition. *The augmentation map $\mathcal{A}_b\mathcal{B}(A,A,A) \to \mathcal{A}_b(A)$ has a contracting homotopy in the underlying category $\mathrm{ab}({}_A\mathbf{M} \downarrow A)$, and hence the chain complex*

$$\ldots \longrightarrow \mathcal{A}_b(A \circ A \circ A \circ A) \to \mathcal{A}_b(A \circ A \circ A) \to \mathcal{A}_b(A \circ A) \to \mathcal{A}_b(A) \to 0$$

in $\mathrm{ab}({}_A\mathbf{M}_A \downarrow A)$ is acyclic.

Proof. Recalling Proposition 5.2.12, define $h : X_n \to X_{n+1}$ by $h = \mathcal{A}_\ell(1 \circ \cdots \circ 1 \circ \eta)$, where $\eta : I \to A$ is the unit of the monoid A. Then it is easy to check that h is the desired contracting homotopy. $\square$

5.2.14. Theorem. *There is a short exact sequence*

$$0 \to \mathcal{D}(A) \longrightarrow \mathcal{A}_b(A \circ A) \longrightarrow \mathcal{A}_b(A) \to 0.$$

Proof. This is an immediate consequence of Propositions 5.2.13 and 5.2.9. $\square$

5.2.15. Example. Let us illustrate Theorem 5.2.14 in the "classical" case of associative algebras. Thus, let $\mathbf{M}$ denote the category of R-modules, which is monoidal under tensor product over R. A monoid A in $\mathbf{M}$ is thus an associative R-algebra. The category ${}_A\mathbf{M}_A$ is then just the category of A-bimodules, the category of abelian group objects $\mathrm{ab}({}_A\mathbf{M}_A \downarrow A)$ is equivalent to ${}_A\mathbf{M}_A$, and the abelianization functor $\mathcal{A}_b\colon ({}_A\mathbf{M}_A \downarrow A) \to {}_A\mathbf{M}_A$ takes a bimodule $M \to A$ over A to M. Then Theorem 5.2.14 reduces to a short exact sequence

$$0 \to \mathcal{D}(A) \to A \otimes A \to A \to 0.$$

5.3 Model category structures for abelian group objects

Let A be an operad on simplicial R-modules.

5.3.1. Proposition. *There is a simplicial model category structure on the category* $\mathrm{ab}(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)$ *(which by Proposition 5.2.5 is equivalent to* $\mathrm{ab}(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)$*) with the following structure.*

1. *A map $f\colon K \to L$ is a* ***weak equivalence*** *(resp. a* ***fibration****) if for every subgroup $H \subseteq \Sigma_n$ the induced map on invariants $f[m]^H\colon K[m]^H \to L[m]^H$ is a weak equivalence (resp. a fibration) of the underlying simplicial sets.*

2. *The* ***cofibrations*** *(resp.* ***trivial cofibrations****) are generated by the abelianizations of the generators for* $(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)$*, which are the same as the abelianizations of the generators for* $(A\text{-}\mathrm{biobj}\mathcal{M}_\mathrm{R} \downarrow A)$*.*

Proof. Every object in $\mathrm{ab}(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)$ is a map $K \to A$ of Σ-objects; because of the existence of a zero-section $A \to K$, each map $K[m]^H \to A[m]^H$ of fixed point sets is surjective, and hence $K \to A$ is a fibration of Σ-objects. Thus we may apply Proposition 3.1.5 to the adjunction

$$\mathcal{D} : (\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A) \rightleftarrows \mathrm{ab}(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A) : \mathcal{U}$$

where we take E to be the identity functor on $\mathrm{ab}(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)$.

The fact that the abelianizations of the generators of $(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)$ are the same as the abelianizations of the generators of $(A\text{-}\mathrm{biobj}\mathcal{M}_\mathrm{R} \downarrow A)$ follows from a simple adjunction argument. If $K \in \mathrm{ab}(A\text{-}\mathrm{biobj}\mathcal{M}_\mathrm{R} \downarrow A) = \mathrm{ab}(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)$, then

$$\begin{aligned}
\mathrm{ab}(A\text{-}\mathrm{biobj}\mathcal{M}_\mathrm{R} \downarrow A)[\mathcal{A}_b(A \circ X \circ A), K] &\simeq (A\text{-}\mathrm{biobj}\mathcal{M}_\mathrm{R} \downarrow A)[A \circ X \circ A, K] \\
&\simeq (\Sigma\mathcal{M}_\mathrm{R} \downarrow A)[X, K] \\
&\simeq (\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)[FX, K] \\
&\simeq \mathrm{ab}(\mathrm{oper}\mathcal{M}_\mathrm{R} \downarrow A)[\mathcal{D}(FX), K].
\end{aligned}$$

$\square$

5.3.2. Proposition. *A map $i\colon K \to L$ in $\mathrm{ab}(\mathrm{oper}\mathcal{M}_{\mathrm{R}} \downarrow A)$ is a cofibration if and only if i is injective and the cokernel of i is cofibrant.*

5.3.3. Proposition. *In the adjoint pairs of functors*

$$\mathcal{A}_b : (A\text{-biobj}\mathcal{M}_{\mathrm{R}} \downarrow M) \rightleftarrows \mathrm{ab}(A\text{-biobj}\mathcal{M}_{\mathrm{R}} \downarrow M) : \mathcal{U}_b$$

and

$$\mathcal{D} : (\mathrm{oper}\mathcal{M}_{\mathrm{R}} \downarrow A) \rightleftarrows \mathrm{ab}(\mathrm{oper}\mathcal{M}_{\mathrm{R}} \downarrow A) : \mathcal{U}$$

the right adjoints preserve fibrations and weak equivalences and the left adjoints preserve cofibrations and trivial cofibrations.

Proof. This is clear, because the right adjoints preserve fibrations and weak equivalences. □

5.4 Equivalence of Quillen and Hochschild cohomologies

In this section we prove Theorem 1.3.8 which states that Quillen cohomology and Hochschild cohomology of an operad are essentially the same up to a dimension shift.

5.4.1 Computing Quillen cohomology of operads

Let $A \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ be an operad, and let U be a coefficient system over A. We compute the Quillen cohomology group $H^t_{\mathrm{oper}}(A, U)$ by choosing a weak equivalence $p\colon B \to A \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ from a cofibrant operad B. Since all abelian group objects are fibrant, we have that

$$\begin{aligned} H^t_{\mathrm{oper}}(A,U) &= \mathrm{Ho}\,(\mathrm{oper}\mathcal{M}_{\mathrm{R}} \downarrow A)\big(B, K(U,t)\big) \\ &\simeq \pi_0\,\mathrm{map}_{(\mathrm{oper}\mathcal{M}_{\mathrm{R}}\downarrow A)}\big(B, K(U,t)\big). \end{aligned}$$

5.4.2 Quillen cohomology of biobjects

Let $A \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ be an operad, let M be an A-biobject, and let U be a coefficient system over M. We compute the Quillen cohomology group $H^t_{A\text{-biobj}}(M, U)$ by choosing a weak equivalence $p\colon N \to M \in \mathrm{oper}\mathcal{M}_{\mathrm{R}}$ from a cofibrant A-biobject N. Since all abelian group objects are fibrant, we have that

$$\begin{aligned} H^t_{A\text{-biobj}}(M,U) &= \mathrm{Ho}\,(A\text{-biobj}\mathcal{M}_{\mathrm{R}} \downarrow M)\big(N, K(U,t)\big) \\ &\simeq \pi_0\,\mathrm{map}_{(A\text{-biobj}\mathcal{M}_{\mathrm{R}}\downarrow M)}\big(N, K(U,t)\big). \end{aligned}$$

5.4.3. Proposition. *If A is a Σ-cofibrant operad, then $\mathrm{diag}\,\mathcal{B}(A,A,A) \to A$ is a weak equivalence of A-biobjects in which the domain is a cofibrant biobject. More* *generally, if $f\colon A \to B$ is a weak equivalence of Σ-cofibrant operads, then*

$$g\colon \operatorname{diag}\mathcal{B}(B, A, B) \to B$$

is a weak equivalence of B-biobjects in which the domain is a cofibrant biobject.

Proof. The first part is just Corollary 3.7.6. The second part follows from Proposition 3.7.5 and from factoring g into $\operatorname{diag}\mathcal{B}(B, A, B) \to \operatorname{diag}\mathcal{B}(B, B, B) \to B$; the first map is an equivalence by Proposition 3.4.5 and the fact that A and B are Σ-cofibrant, and the second map is an equivalence by Corollary 3.7.6. □

5.4.4. Corollary. *If $f\colon A \to B$ is a weak equivalence of Σ-cofibrant operads, then there is a natural isomorphism* $\operatorname{Hoch}^*(A, f^*U) \simeq \operatorname{Hoch}^*(B, U)$.

5.4.5. Proposition. *The Hochschild cohomology of the trivial operad I is given by*

$$\operatorname{Hoch}^n(I, U) = \begin{cases} U[1] & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

5.4.6 Proof of Theorem 1.3.8

Choose a cofibrant operad B and weak equivalence $p\colon B \to A$. Since cofibrant operads are Σ-cofibrant by Proposition 3.5.3, it follows from Corollary 5.4.4 that $\operatorname{Hoch}^*(B, p^*U) \simeq \operatorname{Hoch}^*(A, U)$. Thus without loss of generality we may assume that A is a cofibrant operad.

The result now follows from Theorem 5.2.14 and the following Lemma.

5.4.7. Lemma. *If A is a cofibrant operad, then there is a diagram in the category* $\mathrm{ab}((A\text{-biobj}\mathcal{M}_{\mathrm{R}} \downarrow A))$ *of the form*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D}(A) \arrow[r, "i"] \arrow[d, equal] & P \arrow[r] \arrow[d, "f"] & \mathcal{A}_b(\operatorname{diag}\mathcal{B}(A, A, A)) \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathcal{D}(A) \arrow[r] & \mathcal{A}_b(A \circ A) \arrow[r] & \mathcal{A}_b(A) \arrow[r] & 0,
\end{tikzcd}$$

where $P = \mathcal{A}_b(\operatorname{diag}\mathcal{B}(A, A, A)) \times_{\mathcal{A}_b(A)} \mathcal{A}_b(A \circ A)$ is the pullback of the right-hand square, the rows are exact, the objects in top row are cofibrant, i is a cofibration, and f and g are weak equivalences.

Proof. The object $\operatorname{diag}\mathcal{B}(A, A, A)$ is a cofibrant A-biobject by Proposition 3.7.6, since A is Σ-cofibrant by Proposition 3.5.3. Thus $\mathcal{A}_b(\operatorname{diag}\mathcal{B}(A, A, A))$ is cofibrant by Proposition 5.3.3. Hence it follows by Proposition 5.3.2 that i is a cofibration. Likewise $\mathcal{D}(A)$ is cofibrant by Proposition 5.3.3, and hence P is cofibrant.

By Proposition 5.2.13 the map g has a contracting homotopy in the underlying category $\mathrm{ab}(\Sigma\mathcal{M}_{\mathrm{R}} \downarrow A)$, and hence the pullback f of this map has one also. Thus f and g are weak equivalences as desired. □

Appendix A

Free monoids

In this appendix we describe a construction of the free operad. We present this construction as a special case of the construction of a free monoid in a monoidal category $\mathbf{M}$ which satisfies the **left distributivity law**

$$(A \amalg B) \circ C \simeq (A \circ C) \amalg (B \circ C),$$

and such that the monoidal product $\circ$ commutes with countable directed colimits in each variable.

Our construction is a generalization of the Barr-Wells construction of a free triple [1], which corresponds to the case of a free monoid in the monoidal category $\mathbf{M} = \mathbf{C}^{\mathbf{C}}$ of endofunctors on $\mathbf{C}$. Our proof is in fact an improvement on theirs; they construct the free triple on a functor A as at best a *retract* of the object we call FA.

There is another construction of the free operad as a "sum over trees", which is described in [8], [7].

For the sake of readability, in what follows we will write "+" for the coproduct in the category $\mathbf{M}$, and "$\circ$" will take precedence over "+"; e.g., $A \circ B + C \circ D$ should be read as $(A \circ B) + (C \circ D)$.

Construction of the Free Monoid. We first construct functors $F_n \colon \mathbf{M} \to \mathbf{M}$ inductively, by

$$F_0 A = I,$$

$$F_1 A = I + A,$$

and in general

$$F_{n+1} A = I + A \circ F_n A.$$

We let $\eta_n \colon I \to F_n A$ and $e_n \colon A \circ F_{n-1} A \to F_n A$ denote the inclusions of these two summands.

The functors F_n come with maps $i_n \colon F_n A \to F_{n+1} A$ defined inductively by

$$i_0 \colon F_0 A = I \longrightarrow I + A = F_1 A,$$

and

$$i_n = I + A \circ i_{n-1} \colon I + A \circ F_{n-1} A \longrightarrow I + A \circ F_n A.$$

Then we can define a functor $F\colon \mathbf{M} \to \mathbf{M}$ as the colimit

$$FA = \operatorname{colim}_n(F_0A \xrightarrow{i_0} F_1A \xrightarrow{i_1} F_2A \xrightarrow{i_2} \cdots).$$

A.0.1. Remark. Note that if the monoidal product in $\mathbf{M}$ distributes over coproducts on both sides, then this construction simplifies to

$$FA = I + A + A^{\circ 2} + A^{\circ 3} + \cdots .$$

A.0.2. Proposition. *Let $\mathbf{M}$ be a cocomplete monoidal category satisfying the left distributivity law, with the property that the functor $- \circ -\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ commutes with countable directed colimits in each variable. Then the above functor $F\colon \mathbf{M} \to \mathbf{M}$ is the free monoid triple. Furthermore, the free monoid FA on A satisfies a "recursive formula"*

$$FA \simeq I + A \circ FA.$$

Proof. Let $A \in \mathbf{M}$. We have already constructed FA. We shall show below that FA has the structure of a monoid, and is in fact the free monoid on A.

It is easy to show that the diagrams

$$\begin{tikzcd} I \arrow[r, equal] \arrow[d, "\eta_n"'] & I \arrow[d, "\eta_{n+1}"] \\ F_nA \arrow[r, "i_n"] & F_{n+1}A \end{tikzcd} \qquad \begin{tikzcd} A \circ F_{n-1}A \arrow[r, "A\circ i_{n-1}"] \arrow[d, "e_{n-1}"'] & A \circ F_nA \arrow[d, "e_n"] \\ F_nA \arrow[r, "i_n"] & F_{n+1}A \end{tikzcd}$$

commute. Thus the η's and e's induce maps $\eta\colon I \to FA$ and $e\colon A \circ FA \to FA$, and there is an isomorphism

$$\eta + e\colon I + A \circ FA \xrightarrow{\sim} FA;$$

this is the "recursion formula".

To define a product map, we inductively define maps

$$\mu_{mn}\colon F_mA \circ F_nA \to F_{m+n}A$$

by setting

$$\mu_{0n} = \mathrm{id}\colon F_nA \to F_nA$$

and

$$\mu_{m+1,n}\colon F_{m+1}A \circ F_nA = F_nA + A \circ F_mA \circ F_nA \xrightarrow{F_nA + A\circ\mu_{mn}} F_nA + A \circ F_{m+n}A \xrightarrow{(i,e)} F_{m+n+1}A.$$

It is straightforward to show that the diagrams

$$\begin{tikzcd} F_mA \circ F_nA \arrow[r, "i_m \circ F_nA"] \arrow[d, "\mu_{mn}"'] & F_{m+1}A \circ F_nA \arrow[d, "\mu_{m+1,n}"] \\ F_{m+n}A \arrow[r, "i_{m+n}"] & F_{m+n+1}A \end{tikzcd} \qquad \begin{tikzcd} F_mA \circ F_nA \arrow[r, "F_mA \circ i_n"] \arrow[d, "\mu_{mn}"'] & F_m \circ F_{n+1}A \arrow[d, "\mu_{m,n+1}"] \\ F_{m+n}A \arrow[r, "i_{m+n}"] & F_{m+n+1}A \end{tikzcd}$$

commute. Since $FA \circ FA \simeq \operatorname{colim}_{m,n} F_mA \circ F_nA$ the μ's fit together to give a map $\mu \colon FA \circ FA \to FA$.

We claim that (FA, η, μ) is a monoid. The unit axioms follow from the commutative diagrams above. To prove the associativity axiom, it suffices to show that for all $m, n, p \geq 0$ the two ways of going around the square

$$\begin{tikzcd} F_mA \circ F_nA \circ F_pA \arrow[r, "\mu_{mn} \circ F_pA"] \arrow[d, "F_mA \circ \mu_{np}"'] & F_{m+n}A \circ F_pA \arrow[d, "\mu_{m+n,p}"] \\ F_mA \circ F_{n+p}A \arrow[r, "\mu_{m,n+p}"] & F_{m+n+p}A \end{tikzcd}$$

are the same. The proof is by induction on m. The case when $m = 0$ is straightforward. For general m we note that the above square is the coproduct of the two squares:

$$\begin{tikzcd} I \circ F_nA \circ F_pA \arrow[r, "\sim"] \arrow[d, "1 \circ \mu_{n,p}"'] & F_nA \circ F_pA \arrow[d, "\mu_{n,p}"] \\ I \circ F_{n+p}A \arrow[r, "\sim"] & F_{n+p}, \end{tikzcd}$$

and

$$\begin{tikzcd} A \circ F_{m-1}A \circ F_nA \circ F_pA \arrow[r, "A \circ \mu_{m-1,n} \circ F_pA"] \arrow[d, "A \circ F_{m-1}A \circ \mu_{n,p}"'] & A \circ F_{m+n-1}A \circ F_pA \arrow[d, "A \circ \mu_{m+n-1,p}"] \\ A \circ F_{m-1}A \circ F_{n+p}A \arrow[r, "A \circ \mu_{m-1,n+p}"] & A \circ F_{m+n+p-1}A. \end{tikzcd}$$

The commutativity of each of these squares follows from the inductive hypothesis.

Finally, we show that FA is the free monoid on A. Let $\alpha \colon A \to FA$ be the map defined by

$$\alpha \colon A \xrightarrow{e_1} I + A = F_1A \xrightarrow{i} FA.$$

To show that FA is a free monoid on A it suffices to show that for any monoid B the map

$$\phi \colon \mathrm{mon}\mathbf{M}[FA, B] \to \mathbf{M}[A, B]$$

which sends $g \mapsto g\alpha$ is a bijection.

We show that ϕ is an injection. Suppose $g \colon FA \to B$ is a map of monoids, and let $f = g\alpha$. Let g_n denote the composite $F_nA \to FA \xrightarrow{g} B$; we will show by induction on n that the maps g_n are determined by the value of f, and hence so is g. Clearly the map $g_1 = (\eta, f) \colon F_1A = I + A \to B$ is determined by f. Now suppose g_n is given.

Then from the commutativity of the diagram

$$\begin{tikzcd}
 & F_1A \circ F_nA \arrow[r, "g_1 \circ g_n"] \arrow[d, "\mu_{1n}"] & B \circ B \arrow[d, "\mu"] \\
I + A \circ F_nA \arrow[ur, "(\eta_1 \circ \eta_n, e_1 \circ F_nA)"] \arrow[r, "\sim"] & F_{n+1}A \arrow[r, "g_{n+1}"] & B
\end{tikzcd}$$

it follows that g_{n+1} is determined.

We show that ϕ is a surjection. Thus, given $f\colon A \to B$, we want to construct a map of monoids $g\colon FA \to B$ such that $g\alpha = f$. We build maps $g_n\colon F_nA \to B$ inductively by setting

$$g_1\colon F_1A = I + A \xrightarrow{(\eta, f)} B$$

and

$$g_{n+1}\colon F_{n+1}A = I + A \circ F_nA \xrightarrow{I + f \circ g_n} I + B \circ B \xrightarrow{(\eta, \mu)} B.$$

By induction on n one shows that $g_{n+1}i_n = g_n$. Let $g\colon FA \to B$ be the map induced by the g_n's. To show that g is indeed a map of monoids, we must check that the square

$$\begin{tikzcd}
F_mA \circ F_nA \arrow[r, "g_m \circ g_n"] \arrow[d, "\mu_{mn}"'] & B \circ B \arrow[d, "\mu"] \\
F_{m+n}A \arrow[r, "g_{m+n}"] & B
\end{tikzcd}$$

commutes. The proof is by induction on m. $\square$

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