Subexponential parameterized algorithms on graphs of bounded genus and H-minor-free graphs

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Abstract

We introduce a new framework for designing fixed-parameter algorithms with subexponential running time—$2^{O(\sqrt{k})}n^{O(1)}$. Our results apply to a broad family of graph problems, called *bidimensional problems*, which includes many domination and covering problems such as vertex cover, feedback vertex set, minimum maximal matching, dominating set, edge dominating set, clique-transversal set, and many others restricted to bounded genus graphs. Furthermore, it is fairly straightforward to prove that a problem is bidimensional. In particular, our framework includes as special cases all previously known problems to have such subexponential algorithms. Previously, these algorithms applied to planar graphs, single-crossing-minor-free graphs, and/or map graphs; we extend these results to apply to bounded-genus graphs as well. In a parallel development of combinatorial results, we establish an upper bound on the treewidth (or branchwidth) of a bounded-genus graph that excludes some planar graph H as a minor. This bound depends linearly on the size $|V(H)|$ of the excluded graph H and the genus $g(G)$ of the graph G, and applies and extends the graph-minors work of Robertson and Seymour.

Building on these results, we develop subexponential fixed-parameter algorithms for dominating set, vertex cover, and set cover in any class of graphs excluding a fixed graph H as a minor. In particular, this general category of graphs includes planar graphs, bounded-genus graphs, single-crossing-minor-free graphs, and any class of graphs that is closed under taking minors. Specifically, the running time is $2^{O(\sqrt{k})}n^h$, where h is a constant depending only on H, which is polynomial for $k = O(\log^2 n)$. We introduce a general approach for developing algorithms on H-minor-free graphs, based on structural results about H-minor-free graphs at the heart of Robertson and Seymour's graph-minors work. We believe this approach opens the way to further development on problems in H-minor-free graphs.

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1 Introduction

Dominating set is a classic NP-complete graph optimization problem which fits into the broader class of *domination* and *covering* problems on which hundreds of papers have been written; see e.g. the survey [24]. A sample application is the problem of finding sites for emergency service facilities such as fire stations. Here we suppose that we can afford to build k fire stations to cover a city, and we require that every building is covered by at least one fire station. This problem is k-dominating set (finding a dominating set of size k) in the graph where edges represent suitable pairings of fire stations with buildings. In this application, we can afford high running time (e.g., several weeks of real time) if the resulting solution builds fewer fire stations (which are extremely expensive). Thus, we prefer exact *fixed-parameter* algorithms (which run fast provided the parameter k is small) over approximation algorithms, even if the approximation were within an additive constant. The theory of fixed-parameter algorithms and parameterized complexity has been thoroughly developed over the past few years; see e.g. [12, 15, 17, 19, 21, 2].

In the last two years, several researchers have obtained exponential speedups in fixed-parameter algorithms for various problems on several classes of graphs. While most previous fixed-parameter algorithms have a running time of $O(2^k n^{O(1)})$ or worse, the exponential speedups results in subexponential algorithms with running times of $O(2^{\sqrt{k}} n^{O(1)})$. For example, the first fixed-parameter algorithm for k-dominating set in planar graphs [15] has running time $O(11^k |G|)$; subsequently, a sequence of subexponential algorithms and improvements have been obtained, starting with running time $O(4^{6\sqrt{34k}} n)$ [1], then $O(2^{27\sqrt{k}} n)$ [25], and finally $O(2^{15.13\sqrt{k}} k + n^3 + k^4)$ [19]. Other subexponential algorithms for other domination and covering problems on planar graphs also have been obtained [1, 2, 8, 28, 23].

However, all of these algorithms apply only to planar graphs. In another sequence of papers, these results have been generalized to other classes of graphs: map graphs [12], which include planar graphs; $K_{3,3}$-minor-free graphs and K_5-minor-free graphs [14], which include planar graphs; and single-crossing-minor-free graphs [13, 14], which include $K_{3,3}$ or K_5-minor-free graphs. These algorithms [12, 13, 14] apply to dominating set and several other problems related to domination, covering, and logic.

Algorithms for H-minor-free graphs for a fixed graph H have been studied extensively; see e.g. [9, 22, 10, 26, 30]. In particular, it is generally believed that several algorithms for planar graphs can be generalized to H-minor-free graphs for any fixed H [22, 26, 30]. H-minor-free graphs are very general. The deep Graph-Minor Theorem of Robertson and Seymour shows that any graph class that is closed under minors is characterized by excluding a finite set of minors. In particular, any graph class that is closed under minors excludes at least one minor H.

Our results. We introduce a framework for extending algorithms for planar graphs to apply to H-minor-free graphs for any fixed H. In particular, we design subexponential fixed-parameter algorithms for dominating set, vertex cover, and set cover (viewed as one-sided dominating in a bipartite graph) for H-minor-free graphs. Our framework consists of three components, as described below. We believe that many of these components can be applied to other problems and conjectures as well.

First we extend the algorithm for planar graphs to bounded-genus graphs. Roughly speaking, we study the structure of the solution to the problem in $k \times k$ grids, which form a representative substructure in both planar graphs and bounded-genus graphs, and capture the main difficulty of the problem for these graphs. Then using Robertson and Seymour's graph-minor theory, we repeatedly remove handles to reduce the bounded-genus graph down to a planar graph, which is essentially a grid.

Second we extend the algorithm to *almost-embeddable* graphs which can be drawn in a bounded-genus surface except for a bounded number of "local areas of non-planarity", called vortices, and for a bounded number of "apex" vertices, which can have any number of incident edges that are not properly embedded. Because the vortices have bounded pathwidth, their number is bounded, and the number of apexes is bounded, we are able to solve almost-embeddable graphs using our solution to bounded-genus graphs.

Third we apply a deep theorem of Robertson and Seymour which characterizes H-minor-free graphs as a tree structure of pieces, where each piece is an almost-embeddable graph. Using dynamic programming on such tree structures, analogous to algorithms for graphs of bounded treewidth, we are able to combine the pieces and solve the problem for H-minor-free graphs.

The first step of this procedure, for bounded-genus graphs, applies to a broad class of problems called "bidimensional problems". Roughly speaking, a parameterized problem is *bidimensional* if the parameter is large (linear) in a grid and closed under contractions. Examples of bidimensional problems include vertex cover, feedback vertex set, minimum maximal matching, dominating set, edge dominating set, clique-transversal set, and set cover. We obtain subexponential fixed-parameter algorithms for all of these problems in bounded-genus graphs. As an special case, this generalization settles an open problem about dominating set posed by Ellis, Fan, and Fellows [16]. Along the way, we establish an upper bound on the treewidth (or branchwidth) of a bounded-genus graph that excludes some planar graph H as a minor. This bound depends linearly on the size $|V(H)|$ of the excluded graph H and the genus $g(G)$ of the graph G, and applies and extends the graph-minors work of Robertson and Seymour.

This paper is organized as follows. First, we introduce the terminology used throughout the paper, and formally define tree decompositions, treewidth, and fixed-parameter tractability in Section 2. In Section 3 is devoted to graphs on surfaces. We construct a general framework for obtaining subexponential parameterized algorithms on graphs of bounded genus. First we introduce the concept of bidimensional problem, and then prove that every bidimensional problem has a subexponential parameterized algorithm on graphs of bounded genus. The proof techniques used in this section are very indirect and are based on deep Theorems from Robertson & Seymour's Graph Minors XI and XII. As a byproduct of our results we obtain a generalization of Quickly Excluding Planar Graph Theorem for graphs of bounded genus. In Section 4 we make a step further by developing subexponential algorithms for graphs containing no fixed graph H as a minor. The proof of this result is based on combinatorial bounds from the previous section, deep structural theorem from Graph Minors XIV, and complicated dynamic programming. Finally, in

Section 5, we present several extensions of our results and some open problems.

2 Background

2.1 Preliminaries

All the graphs in this paper are undirected without loops or multiple edges. The reader is referred to standard references for appropriate background [5].

Our graph terminology is as follows. A graph G is represented by $G = (V, E)$, where V (or $V(G)$) is the set of vertices and E (or $E(G)$) is the set of edges. We denote an edge e between u and v by $\{u, v\}$. We define n to be the number of vertices of a graph when this is clear from context.

The *(disjoint) union* of two disjoint graphs G_1 and G_2, $G_1 \cup G_2$, is the graph G with merged vertex and edge sets: $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

One way of describing classes of graphs is by using *minors*. Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is the subgraph of a some contraction of G. We use the notation $H \preceq G$ (resp. $H \preceq_c G$) for H is a minor (a contraction) of G. A graph class $\mathcal{C}$ is a *minor-closed* class if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is H*-minor-free* if $H \notin \mathcal{C}$.

For example, a planar graph is a graph excluding both $K_{3,3}$ and K_5 as minors.

2.2 Fixed-Parameter Algorithms (FPTs)

Developing fast algorithms for NP-hard problems is an important issue. Recently, Downey and Fellows [15] introduced a new approach to cope with this NP-hardness, called *fixed-parameter tractability.* For many NP-complete problems, the inherent combinatorial explosion can be attributed to a certain aspect of the problem, a *parameter.* The parameter is often an integer and small in practice. The running times of simple algorithms may be exponential in the parameter but polynomial in the rest of the problem size. For example, it has been shown that k-vertex cover (finding a vertex cover of size k) has an algorithm with running time $O(kn+1.271^k)$ [11] and hence this problem is fixed-parameter tractable. Alber et al. [1] demonstrated a solution to the planar k-dominating set in time $O(4^{6\sqrt{34k}}n)$. This result was the first nontrivial results for the parameterized version of an NP-hard problem where the exponent of the exponential term grows sublinearly in the parameter (see also [25] and [19] for further improvements of the time bound of [1]) and it initiated the extensive study of subexponential algorithms for various parameterized problems on planar graphs. Using this result, others could obtain exponential speedup of fixed parameter algorithms for many NP-complete problems on planar graphs (see e.g. [8, 27, 2, 6]). (See also Cai & Juedes [7] for discussions on lower bounds of subexponential algorithms on planar graphs.) Recently, Demaine et al. [14, 13, 12] extended these results to many NP-complete problems on map graphs and graphs excluding a single-crossing-graph such as K_5 or $K_{3,3}$ as a minor. As mentioned before, we extend these results for bounded genus graphs and more generally H-minor-free graphs for any fixed H.

2.3 Treewidth and branchwidth

The notion of treewidth was introduced by Robertson and Seymour [32] and plays an important role in their fundamental work on graph minors. To define this notion, first we consider the representation of a graph as a tree, which is the basis of our algorithms in this paper. A *tree decomposition* of a graph G, denoted by $TD(G)$, is a pair (χ, T) in which T is a tree and $\chi = \{\chi_i | i \in V(T)\}$ is a family of subsets of $V(G)$ such that: (1) $\bigcup_{i \in V(T)} \chi_i = V(G)$; (2) for each edge $e = \{u, v\} \in E(G)$ there exists an $i \in V(G)$ such that both u and v belong to χ_i; and (3) for all $v \in V(G)$, the set of nodes $\{i \in V(T) | v \in \chi_i\}$ forms a connected subtree of T. To distinguish between vertices of the original graph G and vertices of T in $TD(G)$, we call vertices of T *nodes* and their corresponding χ_i's *bags*. The maximum size of a bag in $TD(G)$ minus one is called the *width* of the tree decomposition. The *treewidth* of a graph G ($\mathbf{tw}(G)$) is the minimum width over all possible tree decompositions of G.

A *branch decomposition* of a graph (or a hyper-graph) G is a pair (T, τ), where T is a tree with vertices of degree 1 or 3 and τ is a bijection from the set of leaves of T to $E(G)$. The *order* of an edge e in T is the number of vertices $v \in V(G)$ such that there are leaves t_1, t_2 in T in different components of $T(V(T), E(T) - e)$ with $\tau(t_1)$ and $\tau(t_2)$ both containing v as an endpoint.

The *width* of (T, τ) is the maximum order over all edges of T, and the *branchwidth* of G, $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G. (In case where $|E(G)| \leq 1$, we define the branch-width to be 0; if $|E(G)| = 0$, then G has no branch decomposition; if $|E(G)| = 1$, then G has a branch decomposition consisting of a tree with one vertex – the width of this branch decomposition is considered to be 0).

It is easy to see that if H is a subgraph of G then $\mathbf{bw}(H) \leq \mathbf{bw}(G)$. The following result is due to Robertson & Seymour [(5.1) in [34]].

Theorem 2.1 ([34]). *For any connected graph G where $|E(G)| \geq 3$, $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \frac{3}{2}\mathbf{bw}(G)$.*

3 Graphs on surfaces

3.1 Preliminaries

In this subsection we remind a machinery developed in Graph Minor papers that is used in our proofs.

A *surface* Σ is a compact 2-manifold, without boundary. A *line* in Σ is subset homeomorphic to $[0, 1]$. An O-arc is a subset of Σ homeomorphic to a circle. Let G be a graph 2-cell embedded in Σ. To simplify notations we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider G as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H where $H \subseteq G$.

We call by *region* of G any connected component of $\Sigma - E(G) - V(G)$. (Every region is an open set.) We use the notation $V(G), E(G)$, and $R(G)$ for the set of the vertices, edges and regions of G.

If $\Delta \subseteq \Sigma$, then $\overline{\Delta}$ denotes the *closure* of Δ, and the boundary of Δ is $\mathbf{bd}(\Delta) = \overline{\Delta} \cap \overline{\Sigma - \Delta}$. An edge e (a vertex v) is incident with a region r if $e \subseteq \mathbf{bd}(r)$ ($v \subseteq \mathbf{bd}(r)$).

A subset of Σ meeting the drawing only in vertices of G is called G-*normal*. If an O-arc is G-normal then we call it *noose*. The length of a noose is the number of its vertices. $\Delta \subseteq \Sigma$ is an open disc if it is homeomorphic to $\{(x, y) : x^2 + y^2 < 1\}$. We say that a disc D is *bounded* by a noose N if $N = \mathbf{bd}(D)$. A graph G 2-cell embedded in a connected surface Σ is θ-*representative* if every noose of length $< \theta$ is contractable (null-homotopic in Σ).

A *separation* of a graph G is a pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and its order is $|V(A \cap B)|$. Tangles were introduced by Robertson & Seymour in [34]. A *tangle of order* $\theta \geq 1$ is a set $\mathcal{T}$ of separations of G, each of order θ, such that

(i) for every separation (A, B) of G of order $< \theta$, $\mathcal{T}$ contains one of (A, B), (B, A)

(ii) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$.

(iii) if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Let G be a graph embedded in a connected surface Σ. A tangle $\mathcal{T}$ of order θ is *respectful* if for every noose N in Σ with $|N \cap V(G)| < \theta$, there is a closed disc $\Delta \subseteq \Sigma$ with $\mathbf{bd}(\Delta) = N$ such that separation

$$(G \cap \Delta, G \cap \overline{\Sigma - \Delta}) \in \mathcal{T}.$$

Our proofs are based on the following results from Graph Minors papers by Robertson & Seymour.

Theorem 3.1 ((4.3) in [34]). *Let G be a graph with at least one edge. Then there is a tangle in G of order θ if and only if G has branch-width $\geq \theta$.*

Theorem 3.2 ((4.1) in [35]). *Let Σ be a connected surface, not a sphere, let $\theta \geq 1$, and let G be a θ-representative graph 2-cell embedded in Σ. Then there is a unique respectful tangle in G of order θ.*

Also in our proofs we use the notion of radial graph. Informally, the radial graph of a 2-cell embedded in Σ graph G is the bipartite graph R_G obtained by selecting a point in every region r of G and connecting it to every vertex of G incident to that region. However, a region maybe "incident more than once" with the same vertex, so one needs a more formal definition. A *radial drawing* R_G is a radial graph of a 2-cell embedded in Σ graph G if

1. $E(G) \cap E(R_G) = V(G) \subseteq V(R_G)$;

2. Each region $r \in R(G)$ contains a unique vertex $v_r \in V(R_G)$;

3. R_G is bipartite with a bipartition $(V(G), \{v_r\colon r \in R(G)\})$;

4. If e, f are edges of R_G with the same ends $v \in V(G)$, $v_r \in V(R_G)$, then $e \cup f$ does not bound a closed disc in $r \cup \{v\}$;

5. R_G is maximal subject to 1,2,3 and 4.

3.2 Bounding the representativity

Lemma 3.3. *Let G be a graph 2-cell embedded in a non-planar surface Σ of representativity at least θ. Then G contains as a contraction a partially triangulated $(\theta/4 \times \theta/4)$-grid.*

Proof sketch. By Theorem 3.2, G has a respectful tangle of order θ. Let $A(R_G)$ be the set of vertices, edges, and regions (collectively, *atoms*) in the radial graph R_G. According to Section 9 of [35] (see also [36]), the existence of a respectful tangle makes it possible to define a metric d on $A(R_G)$ as follows:

1. If $a = b$, then $d(a, b) = 0$.

2. If $a \neq b$, and a and b are interior to a contractible closed walk of radial graph of length $< 2\theta$, then $d(a, b)$ is half the minimum length of such a walk (here by *interior* we mean the direction in which the walk can be contracted).

3. Otherwise, $d(a, b) = \theta$.

Assume for simplicity that θ is even. Let c be any vertex in G. For $0 \leq i < \theta/2$, define Z_{2i} to be the union of all atoms of distance at most $2i$ from c. (Notice that, in radial graphs, all closed walks have even length.) By Theorem 8.10 of [35], Z_{2i} is a nonempty simply connected set, for all i. (A patch of a surface is *simply connected* if it has no noncontractible closed curves.) Thus, the boundary ∂Z_{2i} of each Z_{2i} is a closed walk in the radial graph.

We claim that the closed walks ∂Z_{2i} and ∂Z_{2i+2} are vertex-disjoint. Consider any atom a on ∂Z_{2i} and an adjacent atom b outside Z_{2i}. The distance between a and b is 2 because there is a length-2 closed walk connecting them, doubling the edge (a, b). By Theorem 9.1 of [35], the metric satisfies the triangle inequality, and hence $d(s, b) \leq d(s, a) + 2 = 2i + 2$. In fact, this bound must hold with equality, because $b \notin Z_{2i}$. Therefore, every atom a on ∂Z_{2i} is surrounded on the exterior of Z_{2i} by atoms at distance exactly $2i + 2$ from c, so ∂Z_{2i} is strictly enclosed by ∂Z_{2i+2}.

Consider the "annulus" $\mathcal{A} = Z_{2\theta-2} - Z_\theta$. We claim that there are at least $\theta/2$ vertex-disjoint paths in the radial graph connecting vertices in ∂Z_θ to vertices in $\partial Z_{2\theta-2}$. By Menger's Theorem, the contrary implies the existence of a cut in $\mathcal{A}$ of size $< \theta/2$ separating the two sets, which implies the existence of a cycle of length $< \theta$, but such a cycle must be contained in Z_θ.

Now we form a $(\theta/2 \times \theta/2)$-grid in the radial graph. The row lines in the grid are formed by taking cycles enclosing c that are subsets of the closed walks ∂Z_{2i} for $i =$

$\theta, \theta+2, \theta+4, \ldots, 2\theta-2$. The column lines in the grid are formed by the $\theta/2$ vertex-disjoint paths found above. Therefore, we obtain a subdivision of the $(\theta/2 \times \theta/2)$-grid as a subgraph of the radial graph.

Finally, we transform this grid into a $(\theta/4 \times \theta/4)$-grid in the original graph G. Each grid edge in the radial graph corresponds in the original graph to a sequence of faces surrounding the edge. We replace this grid edge by the upper half of each face. In this way, each row line in the radial graph maps in the original graph to a curve above this row line. Two adjacent mapped row lines may touch but cannot properly cross, so row lines of distance 2 or more cannot overlap. Thus, by discarding the odd-numbered row lines, and similarly for the columns, we obtain a subdivision of the $(\theta/4 \times \theta/4)$-grid in the original graph. Because each Z_{2i} was simply connected, the grid is embedded in a planar patch on Σ, so if we apply contractions without deletions, we obtain a partially triangulated grid. $\square$

3.3 Bidimensional Parameters

In this section, we define a general framework of parameterized problems for which subexponential algorithms with small constants can be obtained. Our framework is sufficiently broad that an algorithmic designer only needs to check two simple properties of any desired parameter to determine the applicability and practicality of our approach.

A *partially triangulated* $(r \times r)$*-grid* is any graph obtained by adding edges between pairs of nonconsecutive vertices on a common face of a planar embedding of a $(r \times r)$-grid.

Definition 3.4. A *parameter* P is any function mapping graphs to nonnegative integers. The *parameterized problem* associated with P asks, for some fixed k, whether $P(G) \leq k$ for a given graph G.

Definition 3.5. Parameter P is called *minor bidimensional with density* δ if (i) contracting or deleting an edge in a graph G cannot increase $P(G)$, and (ii) there exists a function $f, f(x) = o(x)$ such that for the $(r \times r)$-grid R, $P(R) = (\delta r)^2 + f((\delta r)^2)$.

Parameter P is called *contraction bidimensional with density* δ if (i) contracting an edge in a graph G cannot increase $P(G)$, (ii) there exists a function $f, f(x) = o(x)$ such that for any partially triangulated $(r \times r)$-grid R, $P(R) \geq (\delta r)^2 + f((\delta r)^2)$, and δ is the smallest real number for which this inequality holds.

In either case, P is called *bidimensional*. The *density* δ of P is the minimum of the two possible densities (when both definitions are applicable). We call the sublinear function f *residual function* of P.

Many parameters are bidimensional, mention just a few. Examples of minor bidimensional parameters are

Vertex cover. A vertex cover of a graph G is a set C of vertices such that every edge of G has at least one endpoint in C. The *vertex cover problem* ($\mathcal{VC}$) is to find a minimum vertex cover. $\mathcal{VC}$ is minor bidimensional with problem with density $\delta = 1/\sqrt{2}$.

Feedback vertex set (FVS). A *feedback vertex set (FVS)* of a graph G is a set U of vertices such that every cycle of G passes through at least one vertex of U. The *feedback*

vertex set problem ($\mathcal{FVS}$) asks for a minimum a feedback vertex set of size $\leq k$. This is minor bidimensional problem with density $\delta \geq 1/2$.
Minimum maximal matching. A *matching* in a graph G is a set E' of edges without common endpoints. A matching in G is *maximal* if there is no other matching in G containing it. The *maximal matching problem* $\mathcal{MM}$ asks for a minimum maximal matching. $\mathcal{MM}$ is minor bidimensional with density $\delta \geq 1/\sqrt{8}$.

Examples of contraction bidimensional parameters are
Dominating set. A *dominating set* of a graph G is a set D of vertices of G such that each of the vertices of $V(G) - D$ is adjacent to at least one vertex of D. Minimum dominating set problem is contraction bidimensional with density $\delta = 1/3$.
Edge dominating set. The *edge dominating set problem* $\mathcal{EDS}$ that given a graph G asks for a minimum set $E' \subseteq E(G)$ of such that every edge in $E(G)$ shares at least one endpoint with some edge in E'. $\mathcal{EDS}$ is contraction bidimensional with density $\delta = 1/\sqrt{14}$.
Clique-transversal set. A *clique-transversal set* of a connected graph G is a subset of vertices intersecting all the maximal cliques of G. This is contraction bidimensional with density $\delta \geq 1/2\sqrt{2}$.

Almost all known techniques for obtaining subexponential parameterized algorithms on planar graphs are based on the following 'bounded treewidth approach' [1, 19, 25]:

(I1) Prove that $tw(G) \leq c\sqrt{P(G)}$ for some constant c;

(I2) Compute treewidth (or branchwidth) of G;

(I3) If treewidth is $> c\sqrt{P(G)}$ there is no solution to the problem. If treewidth is $\leq c\sqrt{P(G)}$ run standard dynamic programming on graph of bounded treewidth which takes $2^{O(\sqrt{P(G)})}n$ steps.

All previously known ways of obtaining the most important step (I1) are based on rather complicated techniques based on separators. Let us first give some hints why bidimensional parameters are important for the design of subexponential algorithms on planar graphs. We need the following result of Robertson, Seymour & Thomas. (Theorems (4.3) in [34] and (6.3) in [38].)

Theorem 3.6 ([38]). *Let $r \geq 1$ be an integer. Every planar graph with no (r, r)-grid as a minor has branch-width $\leq 4r - 3$.*

Since for every bidimensional parameter P and $(r \times r)$-grid R, $|V(R)| = O(P(R))$, by Theorem 3.6 we have the following proposition.

Proposition 3.7. *Let P be a bidimensional parameter. Then for any planar graph G, $\mathbf{tw}(G) = O(\sqrt{P(G)})$.*

The class of bidimensional parameterized problems contains all known from the literature planar graph parameters with subexponetial parameterized algorithms. Recently, Cai et al. [6] defined a class *Planar $TMIN_1$* and proved that for every graph G and parameter $P \in$ *Planar $TMIN_1$*, $\mathbf{tw}(G) = O(\sqrt{P(G)})$. Every problem in *Planar $TMIN_1$*

can be expressed as a special type of dominating set problem on bipartite graphs (we refer to [6] for definitions and further properties of $Planar\ TMIN_1$) and Proposition 3.7 yields immediately the result of Cai et al.

Notice that density assigns a real number in $(0, 1]$ to any bidimensional parameter. This assignment defines a *total* order on all such parameters. It is tempting to wonder whether *every* parameter admitting a $2^{O(\sqrt{k})}n^{O(1)}$-time algorithm is bidimensional.

To extend Proposition˜refthm:btwappr on graphs of bounded genus more work need to be done.

If P is a bidimentional parameter with density δ and residual function f then we define the *normalization factor* of P as

$$\min\{\beta \mid (\frac{\delta}{\beta}r)^2 \leq (\delta r)^2 + f(\delta r), \text{ for any } r \geq 1\ \}.$$

Lemma 3.8. *Let P be a contraction (minor) bidimentional parameter with density δ. Then $P(G) < (\frac{\delta}{\beta}r)^2$ implies that G excludes the $(r \times r)$-grid as a minor (and all partial triangulations of the $(r \times r)$-grid as contractions).*

Proof. If P is minor bidimentional and H is the $(r\times r)$-grid and $H \preceq G$, then $P(H) \leq P(G)$ and as $P(H) = (\delta r)^2 + f(\delta r)$, we have that $(\frac{\delta}{\beta}r)^2 > P(G) \geq (\delta r)^2 + f(\delta r)$ which contradicts to the definition of β.

If P is contraction bidimentional, H is any partial triangulation of the $(r \times r)$-grid, and $H \preceq G$, then $P(H) \leq P(G)$ and as $P(H) = (\delta r)^2 + f(\delta r)$, we have that $(\frac{\delta}{\beta}r)^2 > P(G) \geq (\delta r)^2 + f(\delta r)$ which contradicts to the definition of β. □

Let G be a graph and let $v \in V(G)$. Also suppose we have a partition $\mathcal{P}_v = (N_1, N_2)$ of the set of the neighbors of v. Define the *splitting* of G with respect to v and P_v to be the graph obtained from G by

(i) removing v and its incident edges

(ii) introducing two new vertices v^1, v^2 and

(iii) connecting v^i with the vertices in $N_i, i = 1, 2$.

If H is the result of the consecutive application of the above operation on some graph G then we say that H is a *splitting* of G. If additionally in such a splitting process we do not split vertices that are results of previous splittings then we say that H is a *fair splitting* of G.

We say a parameter P is α*-splittable*, if for every graph G and for each vertex $v \in V(G)$ the result of splitting G' with respect to v has $P(G') \leq P(G) + \alpha$.

Many natural graph problems are α-splittable for small α. Examples of 1-splittable problems are dominating set, vertex cover, edge dominating set, independent set, clique-transversal set and feedback vertex set among many others.

For the proof of our main result on properties of bidimentional parameters we need two technical lemmas used in induction on the genus.

It is convenient to work with *Euler genus*. The Euler genus $\mathbf{eg}(\Sigma)$ of a nonorientable surface Σ is equal to the nonorientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The Euler genus $\mathbf{eg}(\Sigma)$ of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ.

The following lemma is very useful in proofs by induction on the genus. The first part of the lemma follows from Lemma 4.2.4 (corresponding to nonseparating cycle) and the second part follows from Proposition 4.2.1 (corresponding to surface separating cycle) in [29].

Lemma 3.9. *Let G be a connected graph 2-cell embedded in a non-planar surface Σ, and let N be a noncontractible noose on G. Then there is a fair splitting G' of G affecting the set $S = (v_1, \dots, v_\rho)$ of the vertices of G met by N such that one of the following holds*

1. *G' can be 2-cell embedded in a surface with Euler genus strictly smaller than $\mathbf{eg}(\Sigma)$.*
2. *each connected component G_i of G' can be 2-cell embedded in a surface with Euler genus strictly smaller than $\mathbf{eg}(\Sigma)$ and is a contraction of some graph G_i^* obtained from G after $\leq \rho$ splittings.*

The following lemma is a direct consequence of the definition of branchwidth.

Lemma 3.10. *Let G be a graph and let G' be the splitting of a vertex in G. Then $\mathbf{bw}(G') \leq \mathbf{bw}(G) + 1$.*

Theorem 3.11. *Suppose that P is an α-splittable bidimensional parameter (for $\alpha \geq 0$) with density δ and normalization factor β ($\delta \leq 1$ and $\beta \geq 1$). Then for any graph G 2-cell embedded in a surface Σ of Euler genus $\mathbf{eg}(\Sigma)$, $\mathbf{bw}(G) \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1)\sqrt{P(G)+1} + 8\alpha(\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1))^2$.*

Proof. We use induction on the Euler genus of Σ.

In case $\mathbf{eg}(\Sigma) = 0$, Lemma 3.8 implies that if $P(G) < (\frac{\delta}{\beta}r)^2$, then G excludes the $(r \times r)$-grid as a minor. Indeed, this is obvious in case P is minor bidimentional. If P is contraction bidimentional, then it is enough to observe that if the planar graph G can be transformed to H via a sequence of edge contractions or removals, then by applying only the contractions in this sequence we get a partial triangulation of H. Using now Theorem exclude-grid we get that if $P(G) < (\frac{\delta}{\beta}r)^2$, then $\mathbf{bw}(G) \leq 4r - 6$. If we set $r = \lfloor \frac{\beta}{\delta}\sqrt{P(G)} \rfloor + 1$, we have that $\mathbf{bw}(G) \leq 4\lfloor \frac{\beta}{\delta}\sqrt{P(G)} \rfloor - 2$. As $\alpha, \beta, \delta \geq 0$, the induction base is done.

Suppose now that $\mathbf{eg}(\Sigma) \geq 1$ and that induction hypothesis holds for any graph 2-cell embedded in a sphere with Euler genus less than $\mathbf{eg}(\Sigma)$. Let G be a graph embedded in Σ. We set $k = P(G)$ and we claim that the representativity of G is $\leq 4\lfloor \frac{\beta}{\delta}\sqrt{k+1} \rfloor$. Lemma 3.8 implies that if $k < (\frac{\delta}{\beta}r)^2$, then G excludes any triangulation of the $(r \times r)$-grid as a contraction. By the contrapositive of Lemma 3.3, this implies that the representativity of G is $< 4r$. If we set $r = \lfloor \frac{\delta}{\beta}\sqrt{k+1} \rfloor + 1$, we have that the representativity of G is $\leq 4\lfloor \frac{\beta}{\delta}\sqrt{k+1} \rfloor$. Let N be a minimum size non-contractible noose N on Σ meeting ρ vertices of G where $\rho \leq 4\lfloor \frac{\beta}{\delta}\sqrt{k+1} \rfloor$. By Lemma 3.9, there is a fair splitting along the vertices met by N such that one of the conditions (1) or (2) holds (see Figure 1). Let G'

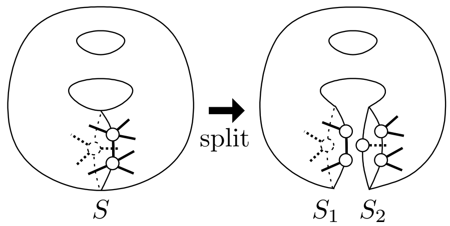

Figure 1: Splitting a noose.

be the resulting graph and let Σ' be a sphere such that $\mathbf{eg}(\Sigma') \leq \mathbf{eg}(\Sigma) - 1$ and every component of G' is 2-cell embedable in Σ'. We claim that in each of the cases (1), (2), $\mathbf{bw}(G') \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+\alpha\rho+1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2$.

Case (1): We apply the induction hypothesis on G' and get that $\mathbf{bw}(G') \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma')+1)\sqrt{P(G')+1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma')+1)^2$. As G' is obtained from G after $\leq \rho$ splittings and P is an α-splittable parameter, we have $P(G') \leq k + \alpha\rho$. Taking in mind that $eg(\Sigma') \leq \mathbf{eg}(\Sigma) - 1$, we obtain $\mathbf{bw}(G') \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+\alpha\rho+1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2$.

Case (2): We apply the induction hypothesis on each of the connected components of G. Let G_i be such a component. We get that $\mathbf{bw}(G_i) \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma')+1)\sqrt{P(G_i)+1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma')+1)^2$. As G_i is a contraction of some graph G_i^* obtained from G after $\leq \rho$ splittings and P is an α-splittable parameter, we get that $P(G_i) \leq P(G_i^*) \leq k+\alpha\rho$. Again since $\mathbf{eg}(\Sigma') \leq \mathbf{eg}(\Sigma)-1$, we have $\mathbf{bw}(G_i) \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+\alpha\rho+1}+8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2$. Notice that $\mathbf{bw}(G') = \max_i(\mathbf{bw}(G_i))$ which in turn implies that $\mathbf{bw}(G') \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+\alpha\rho+1}+ 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2$.

As G' is the result of ρ consecutive vertex splittings on G, Lemma 3.10 yields that

$\mathbf{bw}(G) \leq \mathbf{bw}(G') + \rho$. Therefore,

$$\begin{aligned}
\mathbf{bw}(G) &\leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+\alpha\rho+1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2 + \rho \\
&\leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+\alpha(4\frac{\beta}{\delta}\sqrt{k+1})+1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2 + 4\frac{\beta}{\delta}\sqrt{k+1} \\
&= 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{(\sqrt{k+1})(\sqrt{k+1}+4\alpha\frac{\beta}{\delta})} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2 + 4\frac{\beta}{\delta}\sqrt{k+1} \\
&\leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{(\sqrt{k+1}+4\alpha\frac{\beta}{\delta})(\sqrt{k+1}+4\alpha\frac{\beta}{\delta})} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2 + 4\frac{\beta}{\delta}\sqrt{k+1}, \\
&\qquad \text{since } \alpha, \beta, \delta \geq 0 \\
&= 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)(\sqrt{k+1}+4\alpha\frac{\beta}{\delta}) + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2 + 4\frac{\beta}{\delta}\sqrt{k+1} \\
&= 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+1} + 16\alpha(\frac{\beta}{\delta})^2\mathbf{eg}(\Sigma) + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2 + 4\frac{\beta}{\delta}\sqrt{k+1} \\
&= 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1)\sqrt{k+1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma)^2 + 2\mathbf{eg}(\Sigma)) \\
&= 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1)\sqrt{k+1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma)^2 + 2\mathbf{eg}(\Sigma) + 1) \text{ since } \alpha, \beta, \delta \geq 0 \\
&= 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1)\sqrt{k+1} + 8\alpha(\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1))^2
\end{aligned} \tag{1}$$

□

Theorem 3.11 is a general theorem which applies for any α-splittable bidimensional parameter. For minor bidimensional parameters the bound for branchwidth can be further improved.

Theorem 3.12. *Suppose that P is a minor bidimensional parameter with density δ and normalization factor β ($\delta \leq 1$ and $\beta \geq 1$). Then for any graph G 2-cell embedded in a surface Σ of Euler genus $\mathbf{eg}(\Sigma)$, $\mathbf{bw}(G) \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1)\sqrt{P(G)+1}$.*

Proof. The proof is very similar to the proof of Theorem 3.11. The only difference is that instead of a fair splitting along the vertices of a minimum size non-contractible noose, we just remove vertices of the noose from the graph. Since the parameter is minor bidimensional, the parameter can not increase by this operation. The rest of the proof goes the same. Let G be a graph 2-cell embedded in a surface Σ of Euler genus $\mathbf{eg}(\Sigma)$ and let $k = P(G)$. We have the following inequality which is simpler than inequality (1).

$$\mathbf{bw}(G) \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+1} + \rho \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k+1} + 4\frac{\beta}{\delta}\sqrt{k+1} = 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1)\sqrt{k+1}$$

□

3.4 Combinatorial Results and Further Improvements

As a consequence of Theorem 3.12, we establish an upper bound on the treewidth (or branchwidth) of a bounded-genus graph that excludes some planar graph H as a minor.

As part of their seminal Graph Minors series, Robertson and Seymour proved the following:

Theorem 3.13 ([33]). *If G excludes a planar graph H as a minor, then the branchwidth of G is at most b_H and the treewidth of G is at most t_H, where b_H and t_H are constants depending only on H.*

The current best estimate of these constants is the exponential upper bound $t_H \leq 20^{2(2|V(H)|+4|E(H)|)^5}$ [38]. However, it is known that planar graphs can be excluded "quickly" from planar graphs. More precisely, the following result says that, for planar graphs, the constants depend only linearly on the size of H:

Theorem 3.14 ([38]). *If G is planar and excludes a planar graph H as a minor, then the branchwidth of G is $4(2|V(H)| + 4|E(H)|) - 3$.*

Note that the parameter $P(G) = |V(G)|$ is minor bidimensional with δ and β equal to 1. Thus Theorems 3.11 and 3.12 immediately implies the following generalization of Theorem 3.6 for graphs of bounded genus.

Theorem 3.15. *If G is a graph of genus $g(G)$ with branchwidth more than $4r(g(G)+1)$, then G has a $(r \times r)$-grid as a minor.*

In the same way, we are able to quickly exclude any planar graph from bounded-genus graphs. In other words, we generalize Theorem 3.14 as follows:

Theorem 3.16. *If G is a graph of genus $g(G)$ that excludes a planar graph H as a minor, then its branchwidth is at most $b^{\text{genus}}_{H,g(G)} = 4(2|V(H)| + 4|E(H)|)(g(G) + 1)$.*

3.5 Algorithmic Consequences

As we already discussed, the combinatorial upper bounds for branchwidth/treewidth in are used for constructing subexponential parameterized algorithms as follows. Let G be a graph P be a parameterized problem we need to solve on G. First one constructs a branch/tree decomposition of G that is optimal or 'almost' optimal. A $(\theta, \gamma, \lambda)$-*approximation scheme* for branchwidth/treewidth consists of, for every w, an $O(2^{\gamma w} n^{\lambda})$-time algorithm that, given a graph G, either reports that G has branchwidth/treewidth at least w or produces a branch/tree decomposition of G with width at most θw. For example, the current best schemes are a $(3 + 2/3, 3.698, 3 + \epsilon)$-approximation scheme for treewidth [3] and a $(3, \lg 27, 2)$-approximation scheme for branchwidth [37].

If the branchwidth/treewidth of a graph is 'large', then combinatorial upper bounds come into play and we conclude that P has no solution on G. Otherwise we run dynamic programming on graphs of bounded branchwidth/treewidth and compute $P(G)$.

Thus conclude with the following theorem which is the main algorithmic result of this section:

Theorem 3.17. *Let P be a bidimensional parameter with density $> \delta$. Suppose there is an algorithm for the associated parameterized problem that runs in $O(2^{aw}n^b)$ time given a tree/branch decomposition of the graph G with width w. Suppose also that we have a $(\theta, \gamma, \lambda)$-approximation scheme for treewidth/branchwidth. Set $\tau = 1$ in the case of branchwidth and $\tau = 1.5$ in the case of treewidth. Then the parameterized problem asking whether $P(G) \leq k$ can be solved in $O(2^{\max\{a\theta,\gamma\}\tau 4\frac{\beta}{\delta}(g(G)+1)(\sqrt{k+1}+\mu\alpha\frac{\beta}{\delta}(g(G)+1))}n^{\max\{b,\lambda\}})$ time for minor bidimensional parameter $P(G)$ with density δ and normalization factor β, where μ is 0 if P is minor bidimensional and is 2 if P is α-splittable contraction bidimensional.*

The first condition of the theorem holds with small values of a and b for many examples of bidimensional parameters; see [1, 2, 8, 14, 19, 28]. Observe that the correctness of our algorithms is simply based on Theorems 3.11 and 3.12, despite their nonalgorithmic natures, and $(\theta, \gamma, \lambda)$-approximation scheme for branch/tree decomposition. We note that the time bounds we provide do not contain any hidden constants, and the constants are reasonably low for a broad collection of problems.

4 H-minor free graphs

In this section we show how the results on graphs of bounded genus can be generalized on graphs with excluded minors.

4.1 Clique Sums

Suppose G_1 and G_2 are graphs with disjoint vertex-sets and $k \geq 0$ is an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k and let G'_i $(i = 1, 2)$ be obtained from G_i by deleting some (possibly no) edges from $G_i[W_i]$ with both endpoints in W_i. Consider a bijection $h : W_1 \to W_2$. We define a *k-sum* G of G_1 and G_2, denoted by $G = G_1 \oplus_k G_2$ or simply by $G = G_1 \oplus G_2$, to be the graph obtained from the union of G'_1 and G'_2 by identifying w with $h(w)$ for all $w \in W_1$. The images of the vertices of W_1 and W_2 in $G_1 \oplus_k G_2$ form the *join set*.

In the rest of this section, when we refer to a vertex v of G in G_1 or G_2, we mean the corresponding vertex of v in G_1 or G_2 (or both). It is worth mentioning that $\oplus$ is not a well-defined operator and it can have a set of possible results. See Figure 2 for an example of a 5-sum operation.

The following lemma shows how the treewidth changes when we apply a clique-sum operation, whose intuition will play an important role in our FPT results.

Lemma 4.1 (Folklore). *For any two graphs G and H, $\mathbf{tw}(G \oplus H) \leq \max\{\mathbf{tw}(G), \mathbf{tw}(H)\}$.*

4.2 Characterizations of H-minor-free graphs

In this section, we describe the deep theorem of Robertson and Seymour on graphs excluding a fixed graph H as a minor. Intuitively, Robertson-Seymour's theorem says for

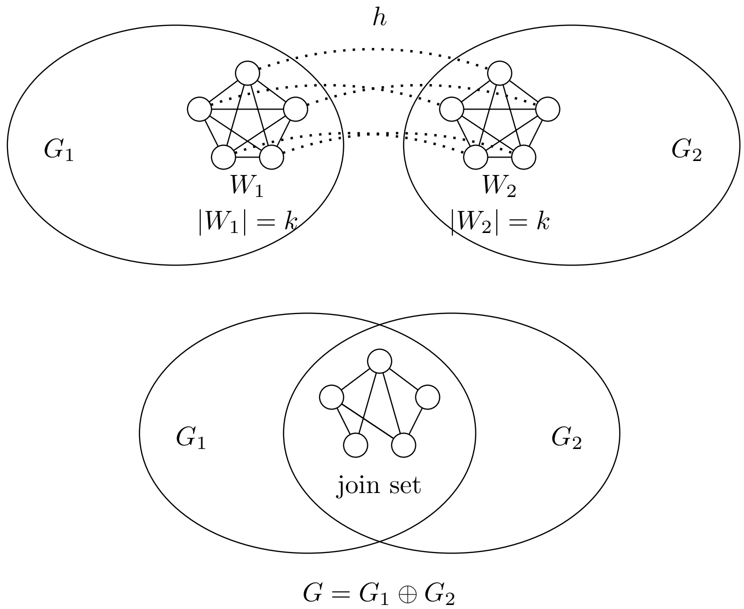

Figure 2: Example of 5-sum of two graphs.

every graph H, every H-minor-free graph can be expressed as a tree-structure of "pieces", where each piece is a graph which can be drawn in a surface in which H cannot be drawn, except for a bounded number of "apex" vertices and a bounded number of "local areas of non-planarity" called *vortices*. Here the bounds only depend on H.

Roughly speaking we say a graph G is *h-almost embeddable* in a surface S if there exists a set X of size at most h of vertices, called *apex vertices* or *apices*, such that $G - X$ can be obtained from a graph G_0 embedded in S by attaching at most h graphs of pathwidth at most h to G_0 along the boundary cycles $C_1, \cdots, C_h$ in an orderly way. More precisely:

Definition 4.2. A graph G is *h-almost embeddable* in S if there exists a vertex set X of size at most h called *apices* such that $G - X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_h$, where

- G_0 has an embedding in S;
- the graphs G_i, called *vortices*, are pairwise disjoint;
- there are (not necessarily distinct) faces $F_1, \ldots, F_h$ of G_0 in S, and there are pairwise disjoint disks $D_1, \ldots, D_h$ in S, such that for $i = 1, \ldots, h$, $D_i \subset F_i$ and $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap D_i$; and
- the graph G_i has a path decomposition $(\mathcal{B}_u)_{u \in U_i}$ of width less than h, such that $u \in \mathcal{B}_u$ for all $u \in U_i$. We note that the sets $\mathcal{B}_u$ are ordered by the ordering of their indices u as points in C_i, where C_i is the boundary cycle of F_i in G_0.

An h-almost embeddable graph is called *apex-free* if the set X of apices is empty.

Now, the deep result of Robertson and Seymour is as follows.

Theorem 4.3 ([31]). *For every graph H there exists an integer $h \geq 0$ only depending on $|V(H)|$ such that every H-minor-free graph can be obtained by at most h-sums of graphs of size at most h and h-almost-embeddable graphs in some surfaces in which H cannot be embedded.*

In particular, if H is fixed, any surface in which H cannot be embedded has bounded genus. Thus, the summands in the theorem are h-almost-embeddable graphs in bounded-genus surfaces.

This structural theorem plays an important role in obtaining the rest of the results of this paper. From the algorithmic point of view, because Robertson and Seymour [31] have shown that every minor-closed class of graphs has a polynomial-time membership test, one can observe the following theorem used by Grohe [20, Lemma 15]. Also, it is claimed that we can construct the clique-sum decompostion algorithmically using the proof of the theorem [39].

Theorem 4.4. *For any graph H, there is an algorithm with running time $O(n^{h+5})$ that either computes a clique-sum decomposition as in Theorem 4.3 for any given H-minor-free graph G, or outputs that G is not H-minor-free.*

Theorem 4.3 is very general and has not appeared in print so far. However already several nice applications (see e.g. [4, 20]) are known. In this paper we show an algorithmic consequence of this theorem and how this approach can be viewed as a guideline for solving other problems on H-minor-free graphs.

4.3 Almost embeddable graphs and k-dominating set

Definition 4.5. A vertex w is called *r-dominated* by a set S, if the distance from w to a vertex $v \in S$ is at most r.

We need the following result proved in [12].

Lemma 4.6 ([12]). *Let $\rho, k, r \geq 1$ be integers and G be a planar graph having a r-dominating set of size k and with a $(\rho \times \rho)$-grid as a minor. Then $k \geq (\frac{\rho - 2r}{2r+1})^2$.*

Lemma 4.7. *For any constant r, if a graph G of genus g has an r-dominating set of size at most k, then the treewidth of G is at most $O(g\sqrt{k} + g^2)$.*

Proof. By Lemma 4.6, r-dominating set is a 1-splittable bidimensional parameter. Now the lemma follows directly from Theorem 3.11. □ □

Now, we extend this result for apex-free h-almost embeddable graphs. Before expressing this result, we mention this simple lemma.

Lemma 4.8. *Consider an apex-free h-almost-embeddable graph $G = G_0 \cup G_1 \cup \cdots \cup G_h$. Suppose further that, for each $1 \leq i \leq h$, $U_i = \{u_i^1, u_i^2, \ldots, u_i^{m_i}\}$ forms a path in G_0. Then* $\mathbf{tw}(G) \leq (h^2 + 1)(\mathbf{tw}(G_0) + 1) - 1$.

Proof. Consider a bag $\mathcal{B}$ of the tree decomposition of G_0. For each vertex in $\mathcal{B} \cap U_i$, say u_i^j, we add to $\mathcal{B}$ the corresponding bag $\mathcal{B}_{u_i^j}$ of the path decomposition of G_i. Because $u_i^j \in \mathcal{B} \cap U_i$, this addition increases the size of $\mathcal{B}$ by at most $|U_i| - 1 \leq h$. We perform this addition for all $1 \leq i \leq h$, for a total increase in treewidth of at most h^2 per vertex in $\mathcal{B}$. It can be easily seen that the resulting set of bags $\mathcal{B}$ form a tree decomposition of G, because each U_i forms a path in G_0. $\square$

Lemma 4.9. *For any constant r, an apex-free h-almost-embeddable graph G embedded on a surface of genus g with a set $S \subset V(G)$ of size at most k which r-dominates every vertex of G which is not in a vortex has treewidth at most $O(h^2 g\sqrt{k+h} + g^2) = O(g\sqrt{k})$ (g and h are constants).*

Proof. Consider an apex-free h-almost embeddable graph $G = G_0 \cup G_1 \cup \cdots \cup G_h$ in a surface Σ of genus g. Suppose $U_i = \{u_i^1, u_i^2, \ldots, u_i^{m_i}\}$. Let G_0' be the graph obtained from G_0 by adding new vertices $c_1, c_2, \cdots, c_h$ and edges (c_i, u_i^j) and (u_i^j, u_i^{j+1}) (where $j+1$ is treated modulo m_i) for all $1 \leq i \leq h$ and $1 \leq j \leq m_i$. Notice that by adding these edges, vertices U_i, $1 \leq i \leq h$, form a path in G_0. If G has the aforementioned r-dominating set of size k, then G_0' has an r-dominating set of size at most $k+h$: just delete all vertices in the r-dominating set that are in $G_i - G_0$, $1 \leq i \leq h$, and add instead all new vertices $c_1, c_2, \cdots, c_h$ to the r-dominating set. Notice that G_0' is embeddable on Σ, since G_0 is embeddable. Thus, according to Lemma 4.7 it has treewidth at most $O(g\sqrt{k+h} + g^2)$. By Lemma 4.8, the treewidth of $G' = G_0' \cup G_1 \cup \cdots \cup G_h$ is $O((h^2+1)(g\sqrt{k+h}+1)+g^2-1)$. U_i forming a path in G_0. Because G is a subgraph of G', the lemma follows. $\square$

4.4 H-minor-free graphs and dominating set

The main result of this section is as follows.

Theorem 4.10. *One can test whether an H-minor-free graph G^* has a dominating set of size at most k in time $2^{O(\sqrt{k})} n^{2h+3}$, where h depends only on $|V(H)|$.*

Before mentioning the proof of the above Theorem, we need some definitions and lemmas.

Definition 4.11. Let G be an h-almost embeddable on a surface of genus g in a clique-sum decomposition of a graph G^*. Suppose the set of apices in G is X. Assume G has clique-sums with graphs $G_1, \cdots, G_p$ via joinsets $W_1, \cdots, W_p$, where $|W_i| \leq h$, $1 \leq i \leq p$. A clique W_i is called *fully dominated* by a set $S \subseteq V(G)$ if $V(G_i) - X \subseteq N_{G^*}(S)$, otherwise clique W_i is called *partially dominated* by S. A vertex v of G is *fully dominated* by a set S if $N_{G^*[V(G)-X]}(v) \subseteq N_{G^*}(S)$.

We note that in the above definition, the only edges that appear in G, but may not appear in G^* are the edges among vertices of $|W_i|$, $1 \leq i \leq p$.

Theorem 4.12. *Let G be an h-almost embeddable on a surface of genus g in a clique-sum decomposition of a graph G^*. Assume G has clique-sums with graphs $G_1, \cdots, G_p$ via join* *sets $W_1, \cdots, W_p$, where $|W_i| \leq h$, $1 \leq i \leq p$. Suppose G^* has a dominating set of size at most k. Then there is a subset $S \subseteq V(G)$ of size at most h such that if we remove all fully dominated vertices which are not included in any partially dominated clique W_i from G and obtain graph $\hat{G}$, $\mathbf{tw}(\hat{G}) = O(h^2 g\sqrt{k+h} + g^2) = O(g\sqrt{k})$.*

Proof. Suppose X is the set of apices in G, so that $G - X$ is an apex-free h-almost embeddable graph. Let D be a dominating set of size k of G^* and let $S = X \cap D$. We claim that S is our desired set. The rest of the proof is as follows: we construct a set $\hat{D}$ of size at most k for $\hat{G} - X$ which 2-dominates every vertex v of $\hat{G} - X$ which is not included in any vortex. Then since $\hat{G} - X$ is an apex-free h-almost-embeddable on a surface of genus g with a 2-dominating-type set of size at most k desired by Lemma 4.9, it has treewidth at most $O(h^2 g\sqrt{k+h} + g^2)$. Then we can add vertices of X to all bags and still have a tree decomposition of width $O(h^2 g\sqrt{k+h} + g^2)$, as desired. We construct $\hat{D}$ from D as follows. First, we set $\hat{D} = D \cap V(G)$. For each $1 \leq i \leq p$, if $D \cap (V(G_i) - W_i) \neq \emptyset$ and $W_i \not\subseteq X$, we add an arbitrary vertex $w \in W_i - X$ to $\hat{D}$. Here we say a vertex v of D is *mapped* to a vertex w of $\hat{D}$ if $v = w$ or if $v \in D \cap (V(G_i) - W_i)$ and vertex $w \in W_i - X$ is the one that we have added to $\hat{D}$. One can easily observe that since each new vertex in $\hat{D}$ is in fact accounted by a unique vertex in D, $|\hat{D}| \leq k$. It only remains to show that D is a 2-dominating set for $\hat{G} - X$. If a vertex $v \in V(\hat{G}) - X$ is not fully-dominated, then there exists a vertex $w \in N_G(v)$ which is not dominated by S and thus not dominated by X (since $S = D \cap X$). It means v is 2-dominated by a vertex u of $\hat{G} - X$ which dominates w (we note that u can be originally a vertex u' in $(V(G_i) - W_i) \cap D$ which is mapped to u in $\hat{D}$). Also, we note that for each clique W_i in which there is a mapped vertex of D, this vertex dominates all vertices of $W_i - X$ in $\hat{G} - X$ and thus we keep the whole clique $W_i - X$ in G. It only remains to show that every vertex of a partially dominated clique W_i is 2-dominated by a vertex of $\hat{G} - X$. We consider two cases: if $W_i \cap S = \emptyset$, since $V(G_i) - W_i \neq \emptyset$, there must exists a (mapped) vertex of $\hat{D}$ in $W_i - X$ and we are done. Now assume $W_i \cap S \neq \emptyset$. If $W_i \subset X$ then $W_i \cap (V(\hat{G}) - X) = \emptyset$ and we are done (since there is no clique in $\hat{G} - X$ at all.) Otherwise, there exists a vertex $W_i - X$. If $(V(G_i) - W_i) \subseteq N_{G^*}(S) \neq \emptyset$, then $V(G_i) \cap D \neq \emptyset$. Thus there exists a mapped vertex $w \in W_i - X$ and we have 1-dominated vertices of $W_i - X$. As mentioned before if $D \cap (W_i - X) \neq \emptyset$, vertices $W_i - X$ are 1-dominated and we are done. The only remaining case is the case in which there exists a vertex $w \in W_i - X$ which is dominated by a vertex $x \in V(G)$ and by assumption $w \notin N_{G^*}(S)$ (we note that in this case, there is no dominating vertex in $V(G_i) - W_i$ for any i for which $w \in W_i$.) It means vertex x is not fully dominated and thus it remains in $\hat{G}$. In addition, vertex x 2-dominates all vertices of $W_i - X$, since W_i is a clique in G and thus all vertices of $W_i - X$ are 2-dominated. This completes the proof of the theorem. □

Now, we are ready to prove Theorem 4.10.

Proof of Theorem 4.10. First, using the $O(n^{h+5})$-time algorithm of Theorem 4.4, we obtain the clique-sum decomposition of graph G^*. In fact, this clique-sum decomposition can be considered as a generalized tree decomposition of G^*.

More precisely, we consider the clique-sum decomposition as a rooted tree. We try to find a k-dominating set in this graph using a two level dynamic programming. Suppose a graph G is an h-almost embeddable on a surface of genus g in a clique-sum decomposition of a graph G^*. Assume G has clique-sums with graphs $G_0, \cdots, G_p$ via join sets $W_0, \cdots, W_p$, where $|W_i| \leq h$, $0 \leq i \leq p$. Also assume that G_0 is considered as the parent of G and $G_1, \cdots, G_p$ are considered as children of G. Now, we define a coloring very similar to Alber et al. [1] as follows.

Colorings. The subproblems in our first level dynamic program are defined by a coloring of the vertices in W_i. Each vertex will be assigned one of 3 colors, labelled 0, $\uparrow$1, and $\downarrow$1. The meaning of the coloring of a vertex v is as follows. Color 0 represents that vertex v is a chosen in the dominating set. Colors $\downarrow$1 and $\uparrow$1 represent that the vertex v is not a dominating set, but has distance exactly 1 to a chosen dominating c. Such a vertex v must have a neighbor n in the dominating set; we say that vertex n *resolves* vertex v. Color $\downarrow$1 for vertex v represents that the dominating vertex n is in the subtree of the clique-sum decomposition rooted at the current graph G, whereas $\uparrow$1 represents that dominating vertex n is elsewhere in the clique-sum decomposition. Intuitively, the vertices colored $\downarrow$1 have already been resolved, whereas the vertices colored $\uparrow i$ still need to be assigned to a dominating.

Locally valid colorings. A coloring of the vertices of W_i in respect with sets $S_1, S_2 \subseteq V(G)$ is called *locally valid* if the following properties hold:

- for any two adjacent vertices v and w in the bag, if v is colored 0, w is colored $\downarrow$1; and
- if $v \in S_1$ then v is colored 0; and
- if $v \in S_2$ then v is not colored 0.

Dynamic program subproblems. Our first-level dynamic program has one subproblem for each graph G in the clique-sum decomposition and for each coloring c of the vertices in W_0. Thus, the number of subproblems is $n \cdot 3^w$. We define $S(G, c)$ to be the size of the minimum dominating set of the vertices in bags in the subtree rooted at G subject to the following restrictions:

1. Vertices assigned color 0 are chosen as dominating vertices, while vertices assigned any other color are not chosen as dominating vertices.
2. Vertices colored $\uparrow$1 are resolved for free by virtual dominating vertices. In other words, vertices colored $\uparrow$1 can be ignored and do not have to be resolved.

If we solve every such subproblem, then in particular, we solve the subproblems involving the root node of the clique-sum decomposition and in which every vertex is colored 0 or $\downarrow$1 . The final dominating set of size k is given by the best solution to these subproblems.

Induction Step Suppose for each coloring c of $|W_i| \leq h$, $0 \leq i \leq p$, we know $S(G_i, c)$. First, we note that obtaining all colorings of a graph of size at most h, takes at most $O(h^h)$ time which is constant. Thus, we mainly focus on almost embeddable graphs. First, we guess a subset X of size at most h. Then for each subset S of X, we set vertices of S in the dominating set and forbid vertices of $X - S$ to be in the dominating set. Now we remove all fully dominated vertices of $G - X$ which are not included in any partially dominated clique W_i from G to obtain $\hat{G}$. By Theorem 4.12, $\mathbf{tw}(\hat{G})$ is in $O(\sqrt{k})$. We can obtain such a tree-decomposition of width $3+2/3$ times optimum, in time $O(2^{3.698}n^3)$ by a result of Amir [3]. We note that all vertices which are absent in this tree-decomposition are those which are fully dominated and thus in any minimum dominating set which includes S, they will not appear except the following case; still it is possible that at most $|X| - |S| = O(h)$ vertices which are fully dominated or they belong to $V(G_i) - W_i$ where W_i is fully dominated appear in the dominating set to dominate vertices of X. Call such a set of vertices S'. W.l.o.g. we can also guess such a set S' of size at most h among discarded vertices which have at least one neighbor in $X - S$ to be in the dominating set. In the other hand, for any partially dominated clique W_i, we know that all of its vertices are present in the tree-decomposition; since they form a clique, there exists a bag α_i in any tree-decomposition, which contains the whole vertices of W_i. We find α_i in our tree-decomposition and map W_i and G_i to this bag. Now, for which coloring c of W_0 (we assume W_0 is contained in all bags, since its size is at most h), we run the dynamic programming of Alber et al. [1] on the tree-decomposition, provided that each coloring of the bags are locally valid with respect to $S \cup S'$ and $X - S$ and coloring c of W_0. In addition, for each bag α_i to which we mapped G_i, we also take into account the $S(G_i, c')$ for the current coloring c' of W_i. Using this dynamic programming, we can obtain $S(G, c)$ for each coloring c' of W_0. The running time for each coloring c of W_0 and each choice of S is $O(4^{O(\sqrt{k})}n)$ according to Alber et al. We have 3^h choices for c, $O(n^{h+1})$ choices for X, $O(2^h)$ choices for S and finally $O(n^{h+1})$ choices for S'. Thus the running time of this inductive step is in $O(6^h 4^{O(\sqrt{k})} n^{2h+2})$. There are at most $O(n)$ graphs in the clique-sum decomposition of G. It means the total running time of the algorithm is in $O(6^h 4^{O(\sqrt{k})} n^{2h+3}) + O(n^{h+5})$ (for creating the clique-sum decomposition) $= O(4^{O(\sqrt{k})} n^{2h+3})$ as desired. □

5 Conclusions and Future Work

Theorem 4.10 can be used to obtain subexponential algorithms not only for dominating set problems.

For example, for vertex cover one can use the following reduction. For a graph G let G' be the graph obtained from G by adding a path of length two between any pair of adjacent vertices. The following lemma is obvious.

Lemma 5.1. *For any K_h-minor free graph G, $h \geq 4$, and integer $k \geq 1$*

- *G' is K_h-minor free,*
- *G has vertex cover of size $\leq k$ if and only if G has a dominating set of size $\leq k$.*

Combining Lemma 5.1 with Theorem 4.10 we conclude that parameterized vertex cover can be solved in subexponential time on graphs with an excluded minor.

Another example is set cover problem. Given a collection $C = (C_1, C_2, \dots, C_m)$ of subsets of a finite set $S = (s_1, s_2, \dots, s_n)$, a set cover is a subcollection $C' \subseteq C$ such that $\cup_{C_i \in C'} = S$. Minimum set cover (SC) problem is to find a cover of minimum size. For a SC problem (C, S) its graph G_S is a bipartite graph with bipartition (C, S). Vertices s_i and C_j are adjacent in G_S if and only if $s_i \in C_j$. Theorem 4.10 can be used to prove that CS with G_S H-minor free for some fixed graph H, can be solved in subexponential time. In fact, for a given graph G_S we construct an auxiliary graph A_S by adding new vertices v, u, w and making adjacent v to $\{u, w, C_1, C_2, \dots, C_m\}$. Then

- (C, S) has a set cover of size $\leq k$ if and only if A_S has a dominating set of size $\leq k + 1$.
- If G_S is K_h-minor free then A_S is K_{h+1}-minor free.

We believe that we can generalize Theorem 4.10 in order to obtain a fixed-parameter algorithm with exponential speed-up for the (k, r)-*center problem* on H-minor-free graphs. The (k, r)-center problem is a generalization of the dominating set problem in which one asks whether an input graph G has $\leq k$ vertices (called centers) such that every vertex of G is within distance $\leq r$ from some center. Demaine et al. [12] consider this problem for planar graph and map graphs and present a generalization of dynamic programming mentioned in the proof of Theorem 4.10 to solve the (k, r)-center problem for graphs of bounded treewidth/branchwidth. Using this dynamic programming and a generalization of Lemma 4.12, one can obtain the desired result for H-minor-free graphs. As Demaine et al. [12] mentioned, in fact, using the solution for the (k, r)-center problem on H-minor-free graphs, we can solve the dominating set problem in constant powers of H-minor-free graphs, the most general class of graphs so far for which one can obtain the exponential speed-up.

However it is an open and tempting question if our technique can be generalized to solve in subexponential time on graphs with excluded minors every problem solved in subexponential time on bounded genus graphs.

We also suspect that there is a strong connection between bidimensional parameters and the existence of linear-size kernels for the corresponding parameterized problems in bounded-genus graphs.

The final question is if the upper bounds Theorems 3.11 and 3.12 can be extended to larger graph classes. The first step in this direction was obtained in [18] for minor-closed graph families: A graph family $\mathcal{F}$ has domination-treewidth property if there is some function $f(d)$ such for that every graph $G \in \mathcal{F}$ with dominating set of size $\leq k$, $\mathbf{tw}(G) \leq f(k)$. It was shown that a minor-closed graph family has domination-treewidth property if and only if this is bounded local treewidth family. We conjecture that for any bidimensional parameter P and minor-closed graph family $\mathcal{F}$, $\mathbf{tw}(G) = O(\sqrt{P(G)})$ for every $G \in \mathcal{F}$ if and only if $\mathcal{F}$ is of bounded local treewidth.

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