Information, Learning and Incentive Design for Urban Transportation Networks

by

Manxi Wu

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Abstract

Today's data-rich platforms are reshaping the operations of urban transportation networks by providing information and services to a large number of travelers. How can we model the travelers' strategic decisions in response to services provided by these platforms and develop tools to improve the aggregate outcomes in a socially desirable manner? In this thesis, we tackle this question from three aspects: 1) Game-theoretic analysis of the impact of *information platforms* (navigation apps) on the strategic behavior and learning processes of travelers in uncertain networks; 2) Market mechanism design for efficient carpooling and toll pricing in the presence of *autonomous driving technology*; 3) *Security* analysis and resource allocation for robustness under random or adversarial disruptions.

Firstly, we present game-theoretic analysis to evaluate the impact of multiple heterogeneous information platforms on travelers' selfish routing decisions, and the resulting network congestion. We compare the value of information provided by multiple platforms to their users, and capture the key trade-off between the gain from information about uncertain network state and the congestion externality resulting from other users. We also design an optimal information structure that induces socially preferred traffic flows. Next, we extend the static model to a dynamic setting that addresses the behavior of users who learn and strategically act in an uncertain environment, while adapting their decisions to the up-to-date information received from platforms. The resulting stochastic learning dynamics requires analyzing strategic and adaptive (hence, endogenous and non i.i.d.) data. We present new results for convergence and stability of such learning dynamics and develop conditions for convergence to complete information equilibrium.

Secondly, we design a market mechanism that enables efficient carpooling and optimal toll pricing in an autonomous transportation market. In this market, the transportation authority sets toll prices on edges, and riders organize carpooled trips using driverless cars and split payments. Riders have heterogeneous preferences, with the value of each trip depending on the travel time of chosen route and rider-specific parameters that capture their individual value of time and carpool disutilities. We identify sufficient conditions on the network topology and travelers' preferences under which a market equilibrium exists, and carpooling trips can be organized in a socially optimal manner. We also present an algorithm that computes a set of equilibrium trips, toll prices and payments that maximize rider utilities.

Finally, we analyze stylized game-theoretic models of attacker-defender interactions for the purpose of evaluating security risks in transportation networks. Our equilibrium analysis suggests an optimal resource allocation strategy to defend multiple infrastructure facilities against an adversarial attacker. To evaluate robustness against random disturbances, we also develop a class of machine learning models that predict the change of travelers' usage demand in congestion-prone multi-modal networks. These results have the potential to help mitigate the impact of transportation network disruptions and limit security risks.

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Chapter 1

Introduction

Transportation systems are rapidly embracing the era of digitalization. At the center of digitalization are various platforms powered by rich sources of data and ever-growing computing capabilities. These platforms inform decision-making of system operators, and create new services (e.g. navigation, ride-hailing, etc.) for a large number of users. In fact, the information and services provided by today's platforms are already re-shaping the system operations and travel patterns and thus have a significant impact on the performance and robustness of transportation networks.

Despite the advanced technological capabilities, platforms can cause inefficiencies and fragility to disruptions in transportation networks. For example, traffic information platforms (navigation apps) such as Google Maps, Apple Maps, and Waze can aggravate congestion in residential areas when many travelers follow the routing suggestions to take a shortcut during morning rush hours (Foderaro [2017]). Ride-hailing platforms such as Uber and Lyft contribute a significant proportion of total miles traveled in megacities as more and more travelers are shifting from taking public transit to ride-hailing services (Hawkins [2019]). Moreover, the increasingly deployed sensing and control units in infrastructures that support these platforms are making our transportation networks more vulnerable to cyber-security threats (Jacobs [2014]). These real-world issues bring forth the importance of understanding the influence of platforms on users' travel behavior, and their societal impacts.

This dissertation focus on two interrelated problems. First, we aim to understand the impact of information platforms and autonomous carpooling services on users' travel patterns, and the resulting performance of transportation networks. Second, we design information and incentive guidelines for platforms to induce socially desirable outcomes such as system efficiency and resiliency.

When studying the impact of platforms on travelers' behavior and system performance, we need to account for the following three key aspects (Fig 1-1):

1. Physical constraints and uncertainty of transportation networks. The usage and operations in transportation networks are subject to various physical constraints such as network topology, capacity, maximum safety speed, etc. For example, the driving time from an origin to a destination depends on the available routes, and the capacity of these routes; schedule of a subway line depends on design of lines and stations, maximum train capacity, and speed limits. These constraints are determined by the structure of the supporting infrastructure, and thus are difficult or very expensive to change. Moreover, transportation systems often suffer from uncertain disruptions including traffic accidents, random infrastructure failures, natural disasters, and malicious cyber-physical attacks. These disruptions can compromise the functionality of one or many critical facilities, and lead to efficiency and welfare loss.

2. Platforms and information technology. Information and services provided by platforms rely on ubiquitous information technologies including data collection, computation and communication in transportation networks. Information platforms use flow data collected from embedded road loop detectors and traffic cameras, as well as crowd-sourcing data from users' GPS-based phones. Powered by machine learning algorithms, these platforms provide their users accurate measurement of traffic flows and travel time prediction. Additionally, mobility platforms are supported by the intelligent matching and dynamic pricing algorithm that match trip requests and rider supply near real time. In the future, autonomous vehicles that are equipped with various sensors, computing, and control units can drive with little or no human interventions.

Moreover, system operators rely on information technologies to effectively manage traffic flows. For example, adaptive ramp metering automatically adjusts the discharge rate of on-ramp traffic into the main highway via traffic signals according to measured congestion level on highways. Traditional tolling infrastructure such as toll booths and microwavebased gantries have been deployed at specific locations to manage traffic demand. New satellite-based tolling technology does not require the installation of these expensive roadside infrastructure. Instead, satellite-based tolling uses on-board units (OBU) to record the vehicle location data, and charges distance- or time- based congestion pricing that precisely reflects the externality caused by the vehicle's travel pattern. In theory, such congestion pricing can induce socially optimal travel behavior.

3. Strategic nature of human agents. The decisions made by human agents, including travelers, service providers, and system operators, have substantial impacts on system performance of transportation networks. Human agents often have heterogeneous preferences, and their decisions are affected by the information environment, monetary incentives, as well as anticipated decisions made by other agents in the system.

Aggregate travel patterns and the resulting system performance are governed by the strategic interactions among human agents as well as the interaction between agents and platforms. On one hand, services provided by various platforms influence the strategies of users by re-shaping their information environment and incentives. For example, travelers account for incident information and route suggestions provided by the navigation services when planning their trips; commuters' travel patterns including departure time, route choices, and mode choices are influenced by toll prices and fees; riders make trips requests and drivers accept or reject requests based on the prices and information of predicted pick-up time provided by ride-hailing platforms. On the other hand, the technological capabilities of platforms also depend on the usage behavior of agents. Such situation is particularly relevant when the data acquired by platforms is crowd-sourced from users, or generated by users' decisions. In this case, platforms' knowledge of uncertain network environment and their operational decisions are in turn influenced by the decisions made by their users.

Classical methods in the field of economics and systems and control theory often capture one or two of these features, but not all three: standard economics theory and models such as games and mechanism design emphasize the strategic nature of human decision makers, but do not fully address the dynamics and other physical constraints of transportation networks. On the other hand, many well-known methods that are widely adopted to model and optimize the system dynamics in control of transportation networks focus much less on the role of

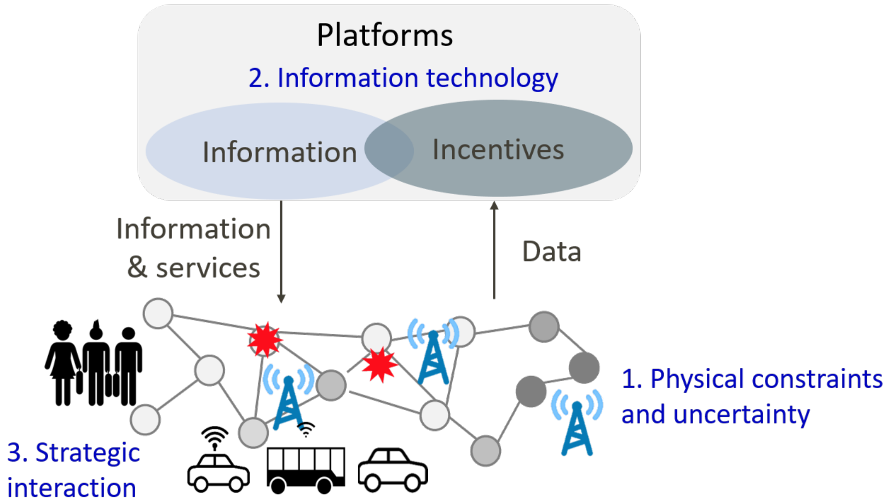


Figure 1-1: Platform-based services in transportation systems.

humans in the system and the impact of their strategic decisions.

This thesis contributes new models and design guidelines based on foundations in game theory, incentive design, and network optimization. We build models that capture the interaction between platforms and strategic users under the physical constraints and uncertainty in transportation networks. We also develop methods to analyze how the outcomes of strategic interactions are shaped by the information environment and incentive mechanisms in both static and dynamic settings.

This thesis consists of three parts: in the first part (Chapters 2 – 4), we focus on analyzing the role of information platforms on travelers' strategic decision making and learning processes; the second part (Chapter 5) designs an autonomous carpooling market; the third part (Chapters 6 – 7) focuses on evaluating system resiliency against disruptions. We conclude and discuss future research directions in Chapter 8. In the remainder of this chapter, we briefly discuss the main contributions in each of the parts.

1.1 Information Platforms

The first part of the thesis builds a new game-theoretic foundation to *(i)* evaluate the value of information for travelers to make route choices in uncertain environment; *(ii)* design optimal information structure; *(iii)* analyze the long-run outcomes of multi-agent learning dynamics.

Value of information in Bayesian routing games. Chapter 2 (Wu et al. [2017, 2020c]) develops a new Bayesian game approach to analyze the impact of information platforms (navigation apps) on travelers' routing decisions, and the resulting congestion costs in transportation networks with uncertain state. This work is motivated by the increasing prevalence of navigation services, which send information on traffic incidents, prediction of uncertain travel time, and route suggestions to their users. A heterogeneous information environment is created by multiple services with inherent technological differences in data collection and analysis. Moreover, this information environment also endogenously depends on the market shares of various platforms (i.e. the fraction of travelers with access to each information platform) that are governed by travelers' choices of platforms.

This Bayesian routing game model captures all three key aspects mentioned above: Firstly, the uncertain and congestible nature of the *physical traffic network* is modeled by the increasing edge cost functions that depend on the unknown network state. Secondly, the *technological capabilities* of information platforms are represented by the game common prior, which is induced from the heterogeneous and potentially correlated distributions of signals sent by platforms. Thirdly, travelers are *strategic* in choosing information platforms, and making routing decisions based on received information of the uncertain state. Their decisions govern the aggregate traffic loads and congestion.

Classical congestion games with complete information do not account for network uncertainty and heterogeneous information environment. We developed a new approach to characterize Bayesian Wardrop equilibrium for general network topology and information environment. This approach depends on *(i)* showing that the Bayesian routing game has a weighted potential function, where the weights are derived from the distribution of signals; *(ii)* applying techniques in sensitivity analysis to study the change of Bayesian Wardrop equilibrium with user sizes of information platforms.

Our approach leads to analysis of the relative value of information between any pair of information platforms evaluated by the difference of travel time costs of their respective users for any market share. Furthermore, we characterize the equilibrium market shares of different platforms in situations when travelers choose platforms by comparing the values of information.

Information design. Chapter 3 (Wu and Amin [2019a]) extends Chapter 2 to study the problem of information design for inducing socially preferred traffic flows. In our setup, a traffic authority send signals about the uncertain network state to travelers according to a designed information structure. Travelers can choose whether or not to receive the signal, and they make routing decisions to minimize their travel time costs based on the information of the state.

Our model extends classical Bayesian persuasion (Kamenica and Gentzkow [2011]) to incorporate agents' choice of information (i.e. the decision or receiving the signal or not). Given any information structure, travelers' routing strategy is a Bayesian Wardrop equilibrium in the heterogeneous information environment that is endogenously determined by the persuasion signal and travelers' choice of information. We provide an analytical characterization of the optimal information structure for any fraction of travelers receiving the signal. We find that under travelers' choice of information, the optimal information structure can induce the preferred traffic flow with only a fraction (less than 1) of travelers receiving the signal.

Multi-agent strategic learning. In Chapter 4 (Wu and Amin [2019b], Wu et al. [2020a,b]), we extend Chapters 2 and 3 to study the role of information platforms in a learning dynamics induced by multiple strategic agents who repeatedly interact in games with unknown payoff-relevant parameters. An information platform repeatedly updates a Bayesian belief estimate of the unknown parameter based on the crowd-sourced history of players' realized strategies and payoffs, and provide the belief to all players for updating their strategies for future rounds of play. One application of such dynamics is learning in repeated routing games on transportation networks with uncertain state. In this application, the traffic information platform repeatedly updates the estimate of the latent network condition based on collected data on edge loads and travel time. Travelers adjust their routing decisions based on the updated estimates. The long-run outcomes of the dynamics are governed by the joint evolution of information update of the unknown environment on the platform and players' strategy updates in games.

Our work is motivated by the need to formulate a learning foundation that analyzes the long-run outcomes and efficiency for learning on information platforms based on crowd-sourcing data generated by players' repeated interactions. In this learning dynamics, agents are *strategic and myopic* in maximizing their stage-wise utilities based on the most recently updated information of the payoff distribution and their knowledge of opponents' play. Since learning of the unknown parameter is based on the observed strategies and payoffs, whether or not the information platform can identify the true parameter is subject to the endogenous data generating process of strategies and payoffs. In the application of repeated routing games, the platform can only observe the travel time of the set of edges that are utilized, and this set depends on the *network topology* and travelers' routing strategies. The travel time distribution of the remaining edges may not be consistently estimated. As a result, learning may not identify the true parameter even if the platform has the *computing capability* to process data sets with infinite sizes.

The main contribution of our work is to develop a new stochastic systems approach to study the long-run properties – convergence and stability (both local and global) – of Bayesian beliefs and strategies. Although there is rich literature on Bayesian learning (Acemoglu et al. [2011], Jadbabaie et al. [2013]) and learning in games (Fudenberg et al. [1998]), the coupled dynamics of beliefs and strategies have not been thoroughly studied in game-theoretic settings. We show that, with probability 1, the beliefs and strategies converge to a fixed point, which may or may not identify the true parameter. We find conditions, under which learning converges to a complete information equilibrium with probability 1. We also derive sufficient conditions that ensure robustness of fixed points under local perturbations of both beliefs and strategies. In the application of repeated routing games, we show that when learning does not lead to complete information equilibrium, the system suffers from long-run inefficiency in that the social cost of a fixed point is higher than that of a complete information equilibrium.

1.2 Autonomous Carpooling Market

We have studied the impact of information and information design on agents' strategic decisions in transportation networks with uncertainty. In Chapter 5 (Amin et al. [2021]), we study welfare-improving incentive design in autonomous transportation networks. Particularly, we focus on designing an autonomous carpooling market that incentivizes riders to take carpooled trips and share network capacity in a socially optimal manner. In this market, a transportation authority sets toll prices on edges, and riders organize carpooled trips using driverless cars. A market equilibrium corresponds to the situation where no rider has an incentive to deviate from the organized trips or opt-out, and riders' payments cover the toll prices plus the trip costs. A market equilibrium, when it exists, organizes trips that maximize social welfare.

This autonomous carpooling market model again addresses the above-mentioned three key features: First, the routes taken by carpooling trips are subject to the *topology and capacity constraints* of the physical road network. Second, the market design leverages the advanced ability of *autonomous driving technology* in enabling frictionless carpooling, and *micro-tolling technologies* (specifically satellite-based tolling) that are increasingly deployed (Yu [2020]). Third, riders make *strategic decisions* when participating the market, and taking carpooled trips.

The key challenge is that the carpooling market needs to satisfy incentive constraints of all riders, who have heterogeneous preferences (in terms of value of time and disutilities of sharing carpooled trips), while accounting for the topology and capacity constraints of the transportation network. In classical settings of market design, the price of a commodity is an invisible hand that balances the market demand and supply. In our setting, the toll set on each edge (which can be viewed as the price of a unit capacity on that edge) influences the trip organization on any route going through that edge. Therefore, the toll price on each edge not only affects the traffic demand on that edge, but also governs how capacities are shared in all neighboring edges.

We develop a new approach to analyze market equilibrium that accounts for the network constraints by combining ideas from market design (Kelso Jr and Crawford [1982], Gul and Stacchetti [1999]) and network flow optimization (Dantzig and Fulkerson [2003]). Our work makes two contributions: *(i)* We characterize sufficient conditions on network topology and riders' carpool preferences that guarantee the existence of market equilibrium. These conditions provide important guidelines for the system designers and policy makers on how to deploy infrastructure facilities that support an efficient and stable autonomous carpooling service. *(ii)* We provide an algorithm that enables a carpooling platform to implement a market equilibrium efficiently. In this equilibrium, riders are incentives to truthfully report their private preferences to the platform, and only pay for the minimum tolls compared to all other equilibria.

1.3 System Resilience

The final part of this thesis focuses on improving the resiliency of transportation networks. We provide game-theoretic analysis to design an optimal security strategy to protect critical infrastructure facilities against malicious disruption. We also develop a machine learning approach to predict the aggregate demand shift between modes – driving and public transit – in response to recurrent congestion in transportation networks. These results are useful for preparing the operators and travelers in face of non-recurrent (random or malicious) and recurrent disturbances.

Optimal security resource allocation. Chapter 6 (Wu and Amin [2018]) formulates a two-player game to model the *strategic* interaction between a system operator (defender) and a malicious adversary (attacker). In this game, the defender allocates security resources on a set of infrastructure to prevent it from being compromised by the attacker. A facility will be compromised if it is attacked but not secured, and the consequence of a compromised facility is evaluated by the loss of system performance. In this model, we represent the system performance as a generic value associated to each scenario of a compromised facility and the scenario where no facility is compromised. We emphasize that, practically, system performance depends on the *network structure* of of infrastructure facilities, and the change of *aggregate usage behavior* of humans in the system.

We capture the defender and attacker's *technological capabilities* by their cost of defending and attacking a single facility. We parametrically characterize the optimal security strategy for the defender with respect to the defense and attack costs. We also provide precise conditions on cost parameters under which the defender can completely deter the attacker by proactively allocating security resources. These results enable the defender to assess the relative risks of multiple facilities, and deploy security resources in a manner that minimizes the total loss of system efficiency in face of an attack.

Aggregate demand prediction. Chapter 7 (Brenner* et al. [2021]) develops a class of machine learning methods to predict the aggregate demand ratio between driving and taking public transit in a transportation network. This work is motivated by the need to evaluate system efficiency when experiencing recurrent disturbances such as seasonal weather, demand fluctuations, nominal freeway operations, etc. (Skabardonis et al. [2003]) Since the system performance largely depends on travelers' decisions, it is crucial to predict how the aggregate behavior changes with those disturbances.

We predict the aggregate demand ratio based on the input data of travel time costs on all (road and transit) segments in the entire network. Due to the high dimensionality of input data, our prediction model combines a logistic regression with a class of dimension reduction techniques including naive subset variable selection, LASSO, Ridge, principal component analysis, and random forest (Friedman et al. [2001]). Our prediction method achieves less than 2% Root Mean Squared Error in predicting the share of driving demand (relative to the subway ridership) in San Francisco Bay area during morning rush hours.

Chapter 2

Value of Information in Bayesian Routing Games

2.1 Introduction

Travelers are increasingly relying on traffic navigation services (information platforms) to make their route choice decisions. In the past decade, numerous services have come to the forefront, including Waze/Google maps, Apple maps, INRIX, etc. These platforms provide their subscribers with costless information about the uncertain network condition (state), which affects the travel time costs. The network state is typically influenced by exogenous factors such as weather, incidents, and road conditions. The information provided by the platforms can be especially useful in making travel decisions. Experiential evidence suggests that the accuracy levels of information platforms are less than perfect, and exhibit heterogeneities due to the inherent technological differences in data collection and analysis procedures. Moreover, travelers may use different information platforms or choose not to use them at all, depending on factors such as marketing, usability, and availability. Therefore, we can reasonably expect that travelers face an environment of asymmetric and incomplete information about the network state.

Importantly, information heterogeneity can directly influence the travelers' route choice decisions, and the resulting congestion externalities. Consider an example where some travelers are informed by their platforms that a certain route has an incident. Taking a detour

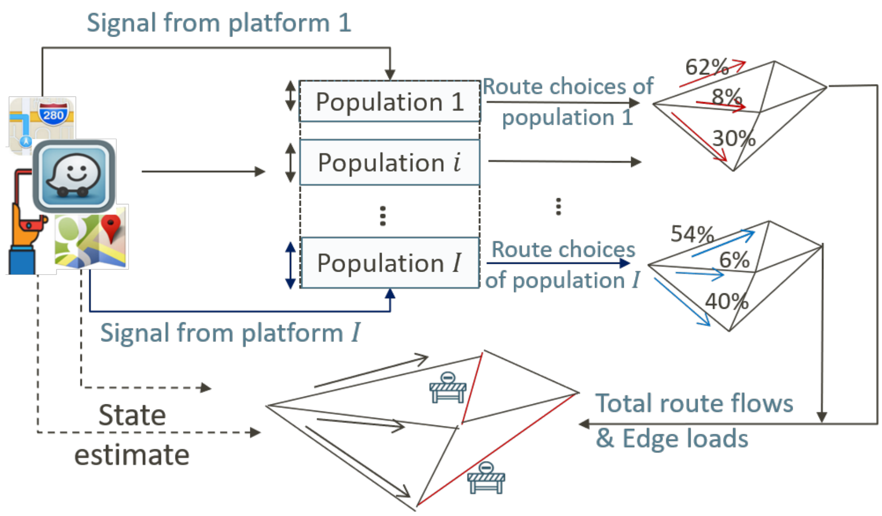


Figure 2-1: Impact of traffic information platforms on strategic routing decisions and network congestion.

based on this information may not only reduce their own travel time, but also benefit the uninformed travelers by shifting traffic away from the affected route. However, if too many travelers take the detour, then this alternate route will also start getting congested, limiting the benefits of information. Thus, the question arises as to *how the heterogeneous information environment impacts the travelers' strategic route choices and the resulting costs?*

This chapter formulates a Bayesian routing game to study this question. In this game (demonstrated in Fig. 2-1), multiple information platforms send private signals of the unknown network state to their respective traveler (user) population. These signals create a heterogeneous information environment that is captured by the common prior of the Bayesian game. In equilibrium, travelers of each population make selfish route choices in network that minimize their costs based on the receive private signals. The aggregate route choices determine the edge loads and congestion costs.

Our model builds a game-theoretic foundation for analyzing the impact of heterogeneous information environment on travel behavior and network congestion. Prior work that studied the effects of information on travelers' departure time or route choices have mainly considered simple information environments (i.e. one population being completely informed and the other being uninformed, Arnott et al. [1991], Ben-Akiva et al. [1991]) or focused on specific network structures (Acemoglu et al. [2016]). Our Bayesian routing game model is general in that we make no assumptions on the topology of the traffic network or the cost functions, and the signals can be realized from any distribution with or without correlation. In Chapters 3 – 4, we further extend this Bayesian routing game model. Particularly, Chapter 3 studies the design of information structure that governs the signal distribution with the goal of inducing socially desirable Bayesian equilibrium route flows. Chapter 4 extends this static games to a stochastic dynamics to analyze the role of information platforms in strategic learning processes.

In this chapter, we develop a new approach to fully characterize equilibrium routing strategies for general networks and information environment. This approach leads to a precise evaluation of how the relative value of information provided by different apps change with their market shares – sizes of travelers who use each app. Moreover, our results can be applied to predict the market shares of both existing and new apps, based on how the relative accuracies of information provided by them shape the overall network congestion.

Our results extend the rich literature on congestion games with complete information to the setting with heterogeneous information environment. The well-known results include the equivalence between complete information congestion games and potential games (Rosenthal [1973], Monderer and Shapley [1996b], and Sandholm [2001]), analysis of equilibrium and inefficiency (Roughgarden and Tardos [2004], Koutsoupias and Papadimitriou [1999], Correa et al. [2007], Acemoglu and Ozdaglar [2007], and Nikolova and Stier-Moses [2014]). In our setting, populations hold different estimate of expected costs of routes due to their heterogeneous private beliefs, which are derived from the common prior according to Bayes' rule. This feature makes our Bayesian routing game equivalent to a weighted potential game, where the weights are derived from the common prior that describes the information environment. We analyze the change of equilibrium structure and the value of information with respect to population sizes through sensitivity analysis of the convex optimization problem that minimizes the weighted potential function.

Finally, our work also contributes to the study of value of information in the theory of economics. In a classical paper, Blackwell [1953] showed that for a single decision maker, more informative signal always results in higher expected utility. In game-theoretic settings, it is generally difficult to determine whether the value of information in equilibrium for individual players and/or society is positive, zero or negative (see Hirshleifer [1971], and Haenfler [2002]). However, the value of information is guaranteed to be positive when certain conditions are satisfied; see for example Neyman [1991], and Lehrer and Rosenberg [2006]. Since travelers are non-atomic players in our game, the relative value of information between any two information platforms is equivalent to the value of information for an individual traveler when her usage of platforms changes unilaterally. We give precise conditions on the population sizes under which the value of information in our Bayesian routing game is positive, zero, or negative.

This chapter is organized as follows: Sec. 2.2 presents the equilibrium analysis in an illustrative example, where two populations (one population is completely informed, and the other population is uninformed) make routing decisions on a two route network. Sec. 2.3 introduces the general Bayesian routing game and the notion of Bayesian Wardrop equilibrium. We provide equilibrium characterization in Sec. 2.4, and the analysis of relative value of information in Sec. 2.5. We characterize the market share of information platforms in Sec. 2.6. The proofs of all results are in Appendix A.

2.2 An Illustrative Example

In this section, we motivate our analysis using a simple game of two traveler populations routing over a network of two parallel routes $R = \{r_1, r_2\}$ (Fig. 2-2a). The network state s is uncertain, and it belongs to the set $S = \{\mathbf{a}, \mathbf{n}\}$, where state $\mathbf{a}$ ("accident") corresponds to an incident on r_1; and state $\mathbf{n}$ ("nominal") indicates no incident on r_1. Route r_2 is not prone to incidents. The state is drawn from a prior probability distribution θ.

The cost (travel time) of each route $r \in R$ is an affine increasing function of the route flow f_r. The cost function of f_1 depends on the network state s, and the cost function of r_2 is state-independent (Fig. 2-2b). The cost functions are given by:

$$c_1^s(f_1) = \begin{cases} \alpha_1^{\mathbf{a}} f_1 + b_1, & \text{if } s = \mathbf{a}, \\ \alpha_1^{\mathbf{n}} f_2 + b_2, & \text{if } s = \mathbf{n}, \end{cases} \tag{2.1a}$$

$$c_2(f_2) = \alpha_2 f_2 + b_2. \tag{2.1b}$$

For ease of presentation, we assume that r_1 is "shorter" than r_2 in that its free-flow travel time is smaller, i.e. $b_1 < b_2$. In addition, the rate of congestion in r_1 is smaller than r_2 in state $\mathbf{n}$, but larger in state $\mathbf{a}$, i.e. $\alpha_1^{\mathbf{a}} > \alpha_2 > \alpha_1^{\mathbf{n}}$.

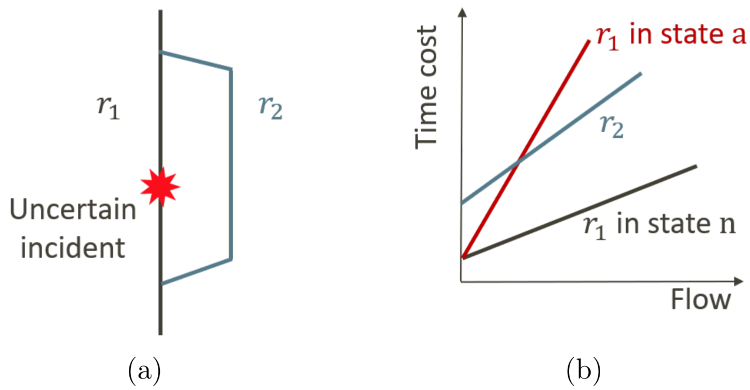


Figure 2-2: (a) Two route network; (b) Cost functions.

The network faces travelers with total demand of D. To avoid triviality, we assume that the demand is sufficiently high so that players do not take one route exclusively:

$$D > \frac{b_2 - b_1}{\alpha_1^{\mathbf{n}}}. \tag{2.2}$$

Travelers are separated into two populations, and they are heterogeneously informed: Population 1 receives a signal $t^1 \in T^1 = \{\mathbf{a}, \mathbf{n}\}$ from their information platform, which has complete information of the uncertain state, i.e. $p(t^1 = s|s) = 1$ for both $s = \mathbf{a}$ and $s = \mathbf{n}$. Population 2 has no information of the state. The size of population 1 is $\lambda^1 D$ and the size of population 2 is $\lambda^2 D$ with $\lambda^2 = 1 - \lambda^1$.

In this example, we represent travelers' routing strategy profile as $q = (q_r^1(\mathbf{a}), q_r^1(\mathbf{n}), q_r^2)_{r\in R}$, where $q_r^1(\mathbf{a})$ (resp. $q_r^1(\mathbf{n})$) is the traffic demand of population 1 that uses route r given the signal $\mathbf{a}$ (resp. $\mathbf{n}$), and q_r^2 is the demand of population 2 on route r. The aggregate flow on each route is $f_r(t^1) = q_r^1(t^1) + q_r^2$ for $t^1 \in T^1$.

This routing game with heterogeneously informed traveler populations admits a Bayesian Wardrop equilibrium, which will be defined formally in Section 2.3. We denote $q^* = (q_r^{1*}(\mathbf{a}), q_r^{1*}(\mathbf{n}), q_r^{2*})_{r\in R}$ as an equilibrium strategy profile, and $f^* = (f_r^*(t^1))_{r\in R, t^1\in T^1}$ as an induced equilibrium flow vector. In equilibrium, the self-interested travelers take routes that minimize their travel time costs. Since population 1 has complete information, their travelers know the exact travel time costs in both states. However, population 2 travelers are uninformed, and they make their route choices based on the expected cost of each route, evaluated according to the prior distribution of states. Travelers in each population, given the signal it receives, can either exclusively take one of the two routes that has a lower cost, or split on both routes when the two routes have identical costs. Thus, there are $3^3 = 27$ possible cases as follows:

$$c_1^{\mathbf{a}}(f_1^*(\mathbf{a}))\begin{cases} < c_2(f_2^*(\mathbf{a})), & \Rightarrow \quad q_1^{1*}(\mathbf{a}) = \lambda^1 D, \quad q_2^{1*}(\mathbf{a}) = 0, \\ = c_2(f_2^*(\mathbf{a})), & \Rightarrow \quad q_1^{1*}(\mathbf{a}) \in [0, \lambda^1 D], \quad q_2^{1*}(\mathbf{a}) = \lambda^1 D - q_1^{1*}(\mathbf{a}), \\ > c_2(f_2^*(\mathbf{a})), & \Rightarrow \quad q_1^{1*}(\mathbf{a}) = 0, \quad q_2^{1*}(\mathbf{a}) = \lambda^1 D, \end{cases}$$

$$c_1^{\mathbf{n}}(f_1^*(\mathbf{n}))\begin{cases} < c_2(f_2^*(\mathbf{n})), & \Rightarrow \quad q_1^{1*}(\mathbf{n}) = \lambda^1 D, \quad q_2^{1*}(\mathbf{n}) = 0, \\ = c_2(f_2^*(\mathbf{n})), & \Rightarrow \quad q_1^{1*}(\mathbf{n}) \in [0, \lambda^1 D], \quad q_2^{1*}(\mathbf{n}) = \lambda^1 D - q_1^{1*}(\mathbf{n}), \\ > c_2(f_2^*(\mathbf{n})), & \Rightarrow \quad q_1^{1*}(\mathbf{n}) = 0, \quad q_2^{1*}(\mathbf{n}) = \lambda^1 D, \end{cases}$$

$$\mathbb{E}_\theta[c_1^s(f_1^*(t^1))]\begin{cases} < \mathbb{E}_\theta[c_2(f_2^*(t^1))], & \Rightarrow \quad q_1^{2*} = \lambda^2 D, \quad q_2^{2*} = 0, \\ = \mathbb{E}_\theta[c_2(f_2^*(t^1))], & \Rightarrow \quad q_1^{2*} \in [0, \lambda^2 D], \quad q_2^{2*} = \lambda^2 D - q_1^{2*}, \\ > \mathbb{E}_\theta[c_2(f_2^*(t^1))], & \Rightarrow \quad q_1^{2*} = 0, \quad q_2^{2*} = \lambda^2 D. \end{cases}$$

where $\mathbb{E}_\theta[c_1^s(f_1^*(t^1))] = \sum_{s\in S}\theta(s)c_1^s(f_1^*(s))$ and $\mathbb{E}_\theta[c_2^s(f_2^*(t^1))] = \sum_{s\in S}\theta(s)c_2(f_2^*(s))$.

The rest of this chapter develops an approach to characterize how the equilibrium strategies and route flows change with population sizes (i.e. λ^1 varies from 0 to 1). In this example, we find that there exists a threshold size of population 1, $0 < \underline{\lambda}^1 = \frac{(\alpha_2 D + b_2 - b_1)}{D}\left(\frac{1}{\alpha_1^{\mathbf{n}}+\alpha_2} - \frac{1}{\alpha_1^{\mathbf{a}}+\alpha_2}\right) < 1$, such that the qualitative structure of equilibrium routing strategies is different based on whether $\lambda^1 \in [0, \underline{\lambda}^1)$ or $\lambda^1 \in [\underline{\lambda}^1, 1]$.[1]

When $\lambda^1 \in [0, \underline{\lambda}^1)$, the game admits a unique equilibrium:

$$q_1^{1*}(\mathbf{a}) = 0, \quad q_1^{1*}(\mathbf{n}) = \lambda^1 D, \text{ and } q_1^{2*} = \frac{\alpha_2 D + b_2 - b_1}{\bar{\alpha}_1 + \alpha_2} - \lambda^1 D\frac{\theta(\mathbf{n})(\alpha_1^{\mathbf{n}} + \alpha_2)}{\bar{\alpha}_1 + \alpha_2},$$

where $\bar{\alpha}_1 = \theta(\mathbf{a})\alpha_1^{\mathbf{a}} + \theta(\mathbf{n})\alpha_1^{\mathbf{n}}$. This equilibrium corresponds to the following outcome: in

[1]Detailed analysis for this simple routing game and some interesting variants are available in Wu et al. [2017].

state $\mathbf{n}$ (resp. state $\mathbf{a}$), all travelers in population 1 exclusively take route r_1 (resp. route r_2), and travelers in population 2 take both routes. The induced equilibrium flow on r_1 is given by $f_1^* = q_1^{2*}$ in state $\mathbf{a}$, and $f_1^* = \lambda^1 D + q_1^{2*}$ in state $\mathbf{n}$. The remaining demand is the flow on route r_2.

On the other hand, when $\lambda^1 \in [\underline{\lambda}^1, 1]$, the equilibrium set may not be singleton, and can be represented as follows:

$$q_1^{1*}(\mathbf{a}) = \chi, \quad q_1^{1*}(\mathbf{n}) = \underline{\lambda}^1 D + \chi, \text{ and } q_1^{2*} = \frac{\alpha_2 D + b_2 - b_1}{\alpha_1^{\mathbf{a}} + \alpha_2} - \chi,$$

where

$$\max\left\{0, \lambda^1 D - \frac{\alpha_1^{\mathbf{a}} D + b_1 - b_2}{\alpha_1^{\mathbf{a}} + \alpha_2}\right\} \leq \chi \leq \min\left\{\frac{\alpha_2 D + b_2 - b_1}{\alpha_1^{\mathbf{a}} + \alpha_2}, \lambda^1 D - \underline{\lambda}^1 D\right\}.$$

In this case, both populations split their demand on the two routes. Moreover, the equilibrium route flow on each route is unique and it does not change with λ^1: $f_1^* = \frac{\alpha_2 D + b_2 - b_1}{\alpha_1^{\mathbf{a}} + \alpha_2}$ if $t^1 = \mathbf{a}$, and $f_1^* = \frac{\alpha_2 D + b_2 - b_1}{\alpha_1^{\mathbf{n}} + \alpha_2}$ if $t^1 = \mathbf{n}$.

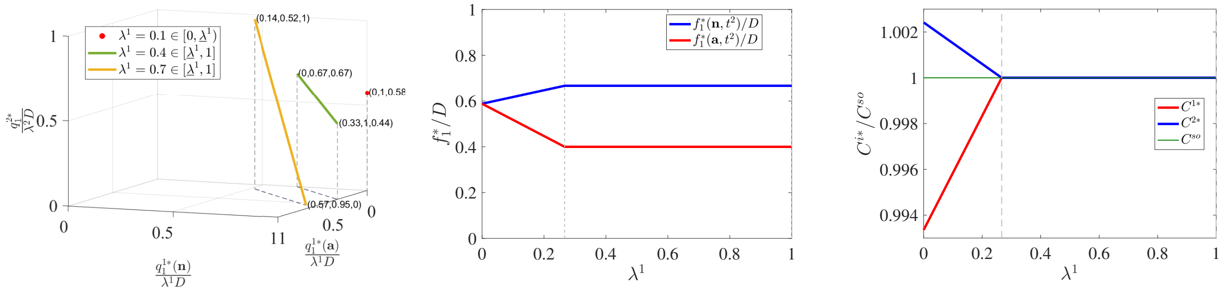
From the characterized equilibrium, we obtain the following two observations: Firstly, when $\lambda^1 \in [0, \underline{\lambda}^1)$, we have $q_1^{1*}(\mathbf{n}) - q_1^{1*}(\mathbf{a}) = \lambda^1 D$, i.e. population 1 shifts all its demand to r_2 when receiving the signal about the incident on r_1. However, if $\lambda^1 \in [\underline{\lambda}^1, 1]$, we have $q_1^{1*}(\mathbf{n}) - q_1^{1*}(\mathbf{a}) = \underline{\lambda}^1 D < \lambda^1 D$, i.e. the change in the received signal only influences a part of travelers in population 1. One can say that the information impacts the entire demand of population 1 in the first regime, but not in the second regime.

Secondly, for any feasible $\lambda = (\lambda^1, \lambda^2)$, we can calculate the equilibrium population costs, denoted $C^{i*}(\lambda)$. If $\lambda^1 \in [0, \underline{\lambda}^1)$, it is easy to check that $C^{2*}(\lambda) - C^{1*}(\lambda) > 0$, i.e. when the state information is only available to a small fraction of travelers, the informed travelers have an advantage over the uninformed ones (i.e. information has positive value). In this case, travelers prefer to receive information of the unknown state. On the other hand, if the size of informed population exceeds the threshold $\underline{\lambda}^1$, then $C^{1*}(\lambda) = C^{2*}(\lambda)$. Thus, all travelers face identical cost in equilibrium. The advantage of being informed is zero (i.e. information has zero value).

The two observations in this example show that the equilibrium structure and the advantage of receiving information depends on the size of the informed population. When information is only received by a small fraction of travelers, these travelers can exclusively take the shorter route in nominal state, and shift to the other route to avoid incident when it happens. This gives them the advantage over the remaining uninformed travelers who must take both routes in both states and thus experience higher cost. However, when the state information is shared with a sufficiently high fraction of travelers, the cost of r_2 becomes high when too many informed travelers take it in incident state. Therefore, informed travelers can no longer exclusively take r_2 to avoid the incident due to the congestion imposed by others who have the same state information. Consequently, travelers in both populations experience the same cost in equilibrium.

The rest of the chapter extends the observations in this example to general settings with multiple (heterogeneous and possibly correlated) information platforms and general network topology. Our analysis reveals the fundamental trade-off faced by travelers between the gain of receiving information of the unknown state and the congestion externality imposed by others with the same information. This trade-off governs how equilibrium structure and the value of information change with population sizes.

We illustrate the equilibrium strategy profile, equilibrium route flows and equilibrium costs for any population size $\lambda^1 \in [0, 1]$ in Fig. 2-3 using the following parameters: $\alpha_1^{\mathbf{n}} = 1$, $\alpha_1^{\mathbf{a}} = 3$, $\alpha_2 = 2$, $b_1 = b_2 = 20$, $D = 1$, and $\theta(\mathbf{a}) = 0.2$. The costs are normalized by the socially optimal cost, denoted C^{so}, which is the minimum cost achievable by a social planner with complete information of the state. The population size threshold $\underline{\lambda}^1 = \frac{4}{15}$. We can see that for $\lambda^1 \in [0, \underline{\lambda}^1)$, the equilibrium strategy profile is unique, the equilibrium route flows change with population sizes and population 1 experiences lower cost than population 2. For $\lambda^1 \in [\underline{\lambda}^1, 1]$, the set of equilibria is a one-dimensional subset of strategy profiles, the equilibrium route flows do not depend on population sizes, and the two populations experience the same costs.



(a) (b) (c)

Figure 2-3: Effects of varying population 1 size on equilibrium structure and costs: (a) Population strategies, (b) Flow on route r_1, (c) Population costs.

2.3 Model

2.3.1 Environment

To generalize the simple routing game in Section 2.2, we consider a transportation network modeled as a directed graph. For ease of exposition, we assume that the network has a single origin-destination pair. All our results apply to networks with multiple origin-destination pairs. Let E denote the set of edges and R denote the set of routes. The finite set of network states, denoted S, represents the set of possible network conditions, such as incidents, weather, etc. The network state, denoted s, is randomly drawn by a fictitious player "Nature" from S according to a distribution $\theta \in \Delta(S)$, which determines the prior probability of each state. For any edge $e \in E$ and state $s \in S$, the state-dependent edge cost function $c_e^s(\cdot)$ is a positive, increasing, and differentiable function of the load through the edge e. The state can impact the edge costs in various ways.

The network serves a set of non-atomic travelers with a fixed total demand D. We assume that each traveler is subscribed exclusively to one of the traffic information platforms in the set $I = \{1, \cdots, I\}$. We refer to the set of travelers subscribed to the platform $i \in I$ as population i. All travelers within a population receive an identical signal from their information platform. Let λ^i denote the ratio of population i's size and the total demand D. We also consider degenerate situations when the sizes of one or more populations approach 0. Thus, a vector of population sizes $\lambda = \left(\lambda^1, \ldots, \lambda^I\right)$ satisfies $\sum_{i\in I} \lambda^i = 1$ and $\lambda^i \geq 0$ for any $i \in I$. The size vector λ is considered as given in our analysis of equilibrium structure and costs (Sections 2.4 and 2.5). In Section 2.6 we consider a more general situation where λ results from the travelers' choices of information platforms.

Each information platform $i \in I$ sends a noisy signal t^i of the state to population i. The signal received by each population determines its type (private information). We assume that the type space of population i is a finite set, denoted as T^i. Note that the type spaces T^i and the state space S need not be of the same size. Let $t \triangleq \left(t^1, t^2, \ldots, t^I\right)$ denote a type profile, i.e. vector of signals received by the traveler populations; thus, $t \in T \triangleq \prod_{i \in I} T^i$. The joint probability distribution of the state s and the vector of signals t is denoted $\pi \in \Delta(S \times T)$, and it is the common prior of the game. The marginal distribution of π on states is consistent with the common prior, i.e. $\sum_{t \in T} \pi(s,t) = \theta(s)$ for all $s \in S$. The conditional probability of type profiles t on the state s is given by $p(t|s) = \frac{\pi(s,t)}{\theta(s)}$, i.e. the joint distribution of signals received by the populations when the network state is s. In our modeling environment, the signals of different information platforms can be correlated, conditional on the state. Each population i generates a belief about the state s and the other populations' types t^{-i} based on the signal received from the information system $i \in I$. We denote the population i's belief as $\beta^i(s, t^{-i}|t^i) \in \Delta(S \times T^{-i})$.

The routing strategy of each population $i \in I$ is a function of its type, denoted as $q^i(t^i) = (q^i_r(t^i))_{r \in R}$. One way to describe the generation of routing strategies is that each information platform $i \in I$ sends a noisy signal t^i of the state to its subscribed population, and the individual route choices of non-atomic travelers results in an aggregate routing strategy $q^i(t^i)$. An alternative viewpoint is that $q^i(t^i)$ is a direct result of strategy route recommendations sent by each information platform to its subscribed population. That is, each information platform $i \in I$ routes travelers in population i according to the function $q^i(t^i)$. For our purpose, these two viewpoints are equivalent in that given any population $i \in I$, and any type $t^i \in T^i$, the demand of travelers on route $r \in R$ is $q^i_r(t^i)$.

We say that a routing strategy profile $q \triangleq (q^i)_{i \in I}$ is *feasible* if it satisfies the following constraints:

$$\sum_{r \in R} q^i_r(t^i) = \lambda^i D, \quad \forall t^i \in T^i, \quad \forall i \in I, \tag{2.4a}$$

$$q_r^i(t^i) \geq 0, \quad \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I. \tag{2.4b}$$

For a given size vector λ, let $Q^i(\lambda)$ denote the set of all feasible strategies of population i. From (2.4a)-(2.4b), we know that the set of feasible strategy profiles $Q(\lambda) \triangleq \prod_{i \in I} Q^i(\lambda)$ is a convex polytope.

2.3.2 Bayesian Routing Games

The Bayesian routing game for a fixed size vector λ can be defined as $\Gamma(\lambda) \triangleq (I, S, T, \pi, Q(\lambda), C)$:

- I: Set of populations, $I = \{1, 2, \ldots, I\}$
- S: Set of states with prior distribution $\theta \in \Delta(S)$
- $T = \prod_{i \in I} T^i$: Set of population type profiles with element $t = (t^i)_{i \in I} \in T$
- $\pi = (\pi(s,t))_{s \in S, t \in T}$: Joint probability distribution of the state s and the type profile t
- $Q(\lambda) = \prod_{i \in I} Q^i(\lambda)$: Set of feasible strategy profiles for a given size vector λ, with element $q = (q^i)_{i \in I} \in Q(\lambda)$
- $C = \{c_e^s(\cdot)\}_{e \in E, s \in S}$: Set of state-dependent edge cost functions

All parameters including the common prior π are common knowledge, except that populations privately receive signals about the network state from their respective information platform. The game is played as shown in Fig. 2-4.

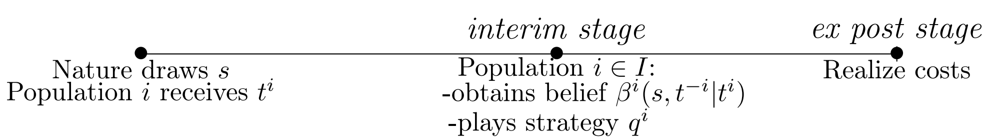


Figure 2-4: Timing of the game $\Gamma(\lambda)$.

For any $i \in I$ and $t^i \in T^i$, the interim belief of population i is derived from the common prior:

$$\beta^i(s, t^{-i}|t^i) = \frac{\pi(s, t^i, t^{-i})}{\Pr(t^i)}, \quad \forall s \in S, \quad \forall t^{-i} \in T^{-i}, \tag{2.5}$$

where $\Pr(t^i) = \sum_{s \in S} \sum_{t^{-i} \in T^{-i}} \pi(s, t^i, t^{-i})$. For a strategy profile $q \in Q(\lambda)$, the induced route flow is denoted $f \triangleq (f_r(t))_{r \in R, t \in T}$, where $f_r(t)$ is the aggregate flow assigned to the route $r \in R$ by populations with type profile t, i.e.

$$f_r(t) = \sum_{i \in I} q_r^i(t^i), \quad \forall r \in R, \quad \forall t \in T. \tag{2.6}$$

Note that the dependence of f on q is implicit and is dropped for notational convenience.

Again, for the strategy profile $q \in Q(\lambda)$, we denote the induced edge load as $w \triangleq (w_e(t))_{e \in E, t \in T}$, where $w_e(t)$ is the aggregate load on the edge e assigned by populations with type profile t:

$$w_e(t) = \sum_{r \ni e} \sum_{i \in I} q_r^i(t^i) \overset{(2.6)}{=} \sum_{r \ni e} f_r(t), \quad \forall e \in E, \quad \forall t \in T. \tag{2.7}$$

The corresponding cost of edge $e \in E$ in state $s \in S$ is $c_e^s(w_e(t))$. Then, the cost of route $r \in R$ in state $s \in S$ can be obtained as: $c_r^s(q(t)) = \sum_{e \in r} c_e^s(w_e(t))$. Finally, the expected cost of route r for population $i \in I$ can be expressed as follows:

$$\mathbb{E}[c_r(q)|t^i] = \sum_{s \in S} \sum_{t^{-i} \in T^{-i}} \sum_{e \in r} \beta^i(s, t^{-i}|t^i) c_e^s(w_e(t^i, t^{-i})), \quad \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I, \tag{2.8}$$

The equilibrium concept for our game $\Gamma(\lambda)$ is Bayesian Wardrop equilibrium.

Definition 2.1 (Bayesian Wardrop Equilibrium). *A strategy profile $q^* \in Q(\lambda)$ is a Bayesian Wardrop equilibrium if for any $i \in I$ and any $t^i \in T^i$:*

$$\forall r \in R, \quad q_r^{i*}(t^i) > 0 \quad \Rightarrow \quad \mathbb{E}[c_r(q^*)|t^i] \le \mathbb{E}[c_{r'}(q^*)|t^i], \quad \forall r' \in R. \tag{2.9}$$

That is, in a Bayesian Wardrop equilibrium, travelers in population i with type t^i only take routes that have the smallest expected cost based on their interim belief $\beta^i(s, t^{-i}|t^i)$.

We define the *equilibrium population cost*, denoted $C^{i*}(\lambda)$, as the expected cost incurred by a traveler of a given population across all types and network states in equilibrium:

$C^{i*}(\lambda) \triangleq \frac{1}{\lambda^i D} \sum_{t^i \in T^i} \Pr(t^i) \sum_{r \in R} \mathbb{E}[c_r(q^*)|t^i] q_r^{i*}(t^i)$. In fact, from (2.9), we can write:

$$C^{i*}(\lambda) \overset{(2.9)}{=} \frac{1}{\lambda^i D} \sum_{t^i \in T^i} \Pr(t^i) \left(\sum_{r \in R} q_r^{i*}(t^i) \right) \min_{r \in R} \mathbb{E}[c_r(q^*)|t^i] \overset{(2.4a)}{=} \sum_{t^i \in T^i} \Pr(t^i) \min_{r \in R} \mathbb{E}[c_r(q^*)|t^i]. \tag{2.10}$$

Note that $\lambda^i = 0$ is a degenerate case for population i as its size approaches 0. In this case, the cost $C^{i*}(\lambda)$ can be viewed as the expected cost faced by an individual (non-atomic) traveler who subscribes to the information platform i.

2.4 Equilibrium Characterization

In this section, we show that the game $\Gamma(\lambda)$ is a weighted potential game. This property enables us to express the sets of equilibrium strategy profiles and route flows as optimal solution sets of certain convex optimization problems.

2.4.1 Equilibrium Strategy Profiles

Following Sandholm [2001], the game $\Gamma(\lambda)$ is a weighted potential game if there exists a continuously differentiable function $\Phi : Q(\lambda) \to \mathbb{R}$ and a set of positive, type-specific weights $\{\gamma(t^i)\}_{t^i \in T^i, i \in I}$ such that:

$$\frac{\partial \Phi(q(t))}{\partial q_r^i(t^i)} = \gamma(t^i) \mathbb{E}[c_r(q)|t^i], \quad \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I. \tag{2.11}$$

We show that our game $\Gamma(\lambda)$ is a weighted potential game.

Lemma 2.1. *Game $\Gamma(\lambda)$ is a weighted potential game with the potential function Φ as follows:*

$$\Phi\left(q\right) \triangleq \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi\left(s, t\right) \int_0^{\sum_{r \ni e} \sum_{i \in I} q_r^i(t^i)} c_e^s(z) dz, \tag{2.12}$$

and the positive type-specific weight is $\gamma(t^i) = \Pr(t^i)$ for any $t^i \in T^i$ and any $i \in I$.

Using (2.6) and (2.7), Φ can be equivalently expressed as a function of the route flow f

or the edge load w induced by a strategy profile $q \in Q(\lambda)$:

$$\widehat{\Phi}(f) \triangleq \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s,t) \int_0^{\sum_{r \ni e} f_r(t)} c_e^s(z) dz \tag{2.13}$$

$$\check{\Phi}(w) \triangleq \sum_{s \in S} \sum_{e \in E} \sum_{t \in T} \pi(s,t) \int_0^{w_e(t)} c_e^s(z) dz. \tag{2.14}$$

Thus, for any feasible strategy profile $q \in Q(\lambda)$, we can write $\Phi(q) \equiv \widehat{\Phi}(f) \equiv \check{\Phi}(w)$, where f and w are the route flow and edge loads induced by the strategy profile q. In addition, $\check{\Phi}(w)$ satisfies the following property:

Lemma 2.2. *The function $\check{\Phi}(w)$ is twice continuously differentiable and strictly convex in w.*

Our first result provides a characterization of the set of equilibrium strategy profiles:

Theorem 2.1. *A strategy profile $q \in Q(\lambda)$ is a Bayesian Wardrop equilibrium if and only if it is an optimal solution of the following convex optimization problem:*

$$\min \quad \Phi(q), \quad s.t. \quad q \in Q(\lambda), \tag{OPT-Q}$$

where $Q(\lambda)$ is the set of feasible strategy profiles. The equilibrium edge load vector $w^*(\lambda)$ is unique.

The existence of Bayesian Wardrop equilibrium follows directly from Theorem 2.1. For any size vector λ, we denote the set of Bayesian Wardrop equilibria for the game $\Gamma(\lambda)$ as $Q^*(\lambda)$. Importantly, since the equilibrium edge load $w^*(\lambda)$ is unique, the equilibrium population cost $C^{i*}(\lambda)$ for each population $i \in I$ in (2.10) must also be unique for any λ. Thus, the equilibria of $\Gamma(\lambda)$ can be viewed as *essentially unique*. We denote the optimal value of (OPT-Q), i.e. the value of the weighted potential function $\Phi(q)$ in equilibrium as $\Psi(\lambda)$.

The Lagrangian of (OPT-Q) that we use in proving Theorem 2.1 is given as follows:

$$\mathcal{L}(q, \mu, \nu, \lambda) = \Phi(q) + \sum_{i \in I} \sum_{t^i \in T^i} \mu^{t^i} \left(\lambda^i D - \sum_{r \in R} q_r^i(t^i) \right) - \sum_{r \in R} \sum_{i \in I} \sum_{t^i \in T^i} \nu_r^{t^i} q_r^i(t^i), \tag{2.15}$$

where $\mu = (\mu^{t^i})_{t^i \in T^i, i \in I}$ and $\nu = (\nu_r^{t^i})_{r \in R, t^i \in T^i, i \in I}$ are Lagrange multipliers associated with the constraints (2.4a) and (2.4b), respectively. The next lemma shows that for any equilibrium $q^* \in Q^*(\lambda)$, the optimal Lagrange multipliers μ^* and ν^* in (2.15) associated with q^* are unique.

Lemma 2.3. *The Lagrange multipliers μ^* and ν^* at the optimum of* (OPT-Q) *are unique:*

$$\mu^{t^i *} = \min_{r \in R} \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i], \quad \forall t^i \in T^i, \quad \forall i \in I \tag{2.16a}$$

$$\nu_r^{t^i *} = \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \mu^{t^i *}, \quad \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I. \tag{2.16b}$$

This result follows from the fact that (OPT-Q) satisfies the *Linear Independence Constraint Qualification (LICQ)* condition (Wachsmuth [2013]), which ensures the uniqueness of Lagrange multipliers at the optimum of (OPT-Q); see Lemma A.1 for the statement of LICQ condition.

The value of $\mu^{t^i *}$ relates the expected route costs for each type t^i in equilibrium with the sensitivity analysis of the Lagrangian with respect to population sizes at the optimum of (OPT-Q). We will use this result in Section 2.5 for studying the relative ordering of equilibrium population costs.

2.4.2 Equilibrium Route Flows

Our main question of interest is how the equilibria set $Q^*(\lambda)$, i.e. optimal solution set of (OPT-Q), and more importantly, the equilibrium edge load $w^*(\lambda)$, change with the perturbations in the size vector λ. However, characterizing the effect of λ directly from (OPT-Q) is not so straightforward. Recall that in the simple routing game in Section 2.2, the effects of perturbations in λ on the equilibrium route flow are relatively easier to describe in comparison to the effects on the set of equilibrium strategy profile — the equilibrium route flow remains fixed in a certain range of λ, whereas the set of equilibrium strategy profiles do not. Thus, our approach involves studying how λ affects the set of equilibrium route flows. We show two results in this regard: (i) The set of feasible route flows and the set of feasible strategy profiles that induces a particular route flow can be both expressed as polytopes

(Proposition 2.1); (ii) The set of equilibrium route flows is the optimal solution set of a convex optimization problem (Proposition 2.2). These results enable us to evaluate how the equilibrium edge load and population costs change with perturbations in λ.

Let us start by introducing the set of route flows

$$F(\lambda) \triangleq \{f \in \mathbb{R}^{|R|\times|T|} \,|\, f \text{ satisfies (2.17a)-(2.17d)}\},$$

where the constraints are given by:

$$f_r(t^i, t^{-i}) - f_r(\tilde{t}^i, t^{-i}) = f_r(t^i, \tilde{t}^{-i}) - f_r(\tilde{t}^i, \tilde{t}^{-i}), \quad \forall r \in R,\ \forall t^i, \tilde{t}^i \in T^i, \text{and } \forall t^{-i}, \tilde{t}^{-i} \in T^{-i}, \forall i \in I, \tag{2.17a}$$

$$\sum_{r\in R} f_r(t) = D, \quad \forall t \in T, \tag{2.17b}$$

$$f_r(t) \geq 0, \quad \forall r \in R, \quad \forall t \in T, \tag{2.17c}$$

$$D - \sum_{r\in R} \min_{t^i \in T^i} f_r(t^i, t^{-i}) \leq \lambda^i D, \quad \forall t^{-i} \in T^{-i}, \quad \forall i \in I. \tag{2.17d}$$

The constraints (2.17a)-(2.17c) do not depend on the size vector λ and can be understood as follows: (2.17a) captures the fact that the change in the flow through any route resulting from change in the type of population $i \in I$ does not depend on the particular types of the remaining populations; (2.17b) ensures that all the demand D is routed through the network; and (2.17c) guarantees that the demand assigned to any route is nonnegative.

On the other hand, the constraints in (2.17d) depend on the size vector λ, wherein the size of each population $i \in I$, λ^i, appears linearly in the constraint corresponding to that population. To further interpret (2.17d), we define the "impact of information" for any given population as the maximum extent to which the signal received from its information platform can influence the routing behavior of travelers within the population. Specifically, for any strategy profile $q \in Q(\lambda)$ and population $i \in I$, we define the *impact of information* on population $i \in I$ as follows:

$$J^i(q) \triangleq \lambda^i D - \sum_{r\in R} \min_{t^i \in T^i} q_r^i(t^i). \tag{2.18}$$

Using (2.4a), we can re-write (2.18) as: $J^i(q) = \sum_{r\in R} \max_{t^i\in T^i} \left(q_r^i(\widehat{t^i}) - q_r^i(t^i)\right)$, where $\widehat{t^i}$ is an arbitrary type in T^i. That is, for each population $i \in I$, $J^i(q)$ is the summation over all routes of the maximum difference between the demands assigned to each route r by the type $\widehat{t^i}$ and any other type $t^i \in T^i$. Using (2.6), we can alternatively express this metric in terms of the flow f induced by q:

$$\widehat{J^i}(f) \equiv J^i(q) \overset{(2.6)}{=} \sum_{r\in R} \max_{t^i\in T^i}(f_r(\widehat{t^i}, \widehat{t^{-i}}) - f_r(t^i, \widehat{t^{-i}})) \overset{(2.17b)}{=} D - \sum_{r\in R} \min_{t^i\in T^i} f_r(t^i, \widehat{t^{-i}}), \tag{2.19}$$

where $(\widehat{t^i}, \widehat{t^{-i}})$ is any type profile in T. Now the constraints (2.17d) can be equivalently stated as:

$$\widehat{J^i}(f) \leq \lambda^i D, \quad \forall i \in I. \tag{IIC}$$

These constraints ensure that the impact of signals on any population's strategy is bounded by its size. We will refer to them as information impact constraints (IIC). We use (IIC) and (2.17d) interchangeably, and refer the constraint in (IIC) corresponding to population $i \in I$ as (IIC$_i$). Also, it is easy to see that for each $i \in I$, (IIC$_i$) can be written as a set of affine inequalities:

$$D - \sum_{r\in R} f_r(t^i_r, \widehat{t^{-i}}) \leq \lambda^i D, \quad \forall t^i_1 \in T^i, \ldots, \forall t^i_{|R|} \in T^i. \tag{2.21}$$

Thus, $F(\lambda)$ is a convex polytope. The following proposition relates the set of feasible strategy profiles and the induced route flows.

Proposition 2.1. *The set of feasible route flows is the convex polytope $F(\lambda)$. Furthermore, for a given route flow $f \in F(\lambda)$, any feasible strategy profile $q \in Q(\lambda)$ that induces f can be expressed as:*

$$q^i_r(t^i) = f_r(t^i, \widehat{t^{-i}}) - f_r(\widehat{t^i}, \widehat{t^{-i}}) + \chi^i_r, \quad \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I, \tag{2.22}$$

where $\widehat{t} = \left(\widehat{t^i}\right)_{i\in I}$ is any type profile in T, and $\chi = (\chi^i_r)_{r\in R, i\in I}$ satisfies the following con-

straints:

$$\sum_{r \in R} \chi_r^i = \lambda^i D, \quad \forall i \in I, \tag{2.23a}$$

$$\sum_{i \in I} \chi_r^i = f_r(\widehat{t}), \quad \forall r \in R, \tag{2.23b}$$

$$\chi_r^i \geq \max_{t^i \in T^i} \left(f_r(\widehat{t^i}, \widehat{t}^{-i}) - f_r(t^i, \widehat{t}^{-i}) \right), \quad \forall r \in R, \quad \forall i \in I. \tag{2.23c}$$

The proof of this proposition is comprised of three steps: In Step I, we show that any route flow that is induced by a feasible strategy profile must satisfy constraints (2.17a)-(2.17d). In Step II, we show the converse: for any given f, any feasible strategy profile q that induces it must be given by (2.22), where χ is a vector satisfying (2.23). In Step III, we prove that if f satisfies (2.17a)-(2.17d), then we can indeed construct a vector χ that satisfies (2.23), and the corresponding q in (2.22) is a feasible strategy profile that induces f. By combining Steps II and III, we can conclude that any f satisfying (2.17a)-(2.17d) can be induced by at least one feasible strategy profile, and thus is a feasible route flow.

The next proposition provides a characterization of the set of equilibrium route flows, and is analogous to Theorem 2.1 which characterizes the set of equilibrium strategy profiles.

Proposition 2.2. *A feasible route flow $f \in F(\lambda)$ is an equilibrium route flow if and only if f is an optimal solution of the following convex optimization problem:*

$$\min \quad \widehat{\Phi}(f), \quad s.t. \quad f \in F(\lambda), \tag{OPT-F}$$

where $F(\lambda)$ is the set of feasible route flow vectors, which satisfy constraints (2.17a) – (2.17d).

We denote the set of equilibrium route flows f^* in the game $\Gamma(\lambda)$ as $F^*(\lambda)$. From Theorem 2.1, equations (2.7) and (2.12), we know that for any size vector λ, and any $q^* \in Q^*(\lambda)$, any $f^* \in F^*(\lambda)$,

$$\Psi(\lambda) = \Phi(q^*) = \widehat{\Phi}(f^*) = \breve{\Phi}(w^*(\lambda)). \tag{2.24}$$

Propositions 2.1 and 2.2 form the basis of our analysis of how the perturbations of size vector effects the equilibrium structure and population costs.

2.5 Pairwise Comparison of Populations

In this section, we first analyze the effects of perturbations in the relative sizes of any two populations on the equilibrium structure. Next, we study how the cost difference between any two populations depends on the population sizes.

2.5.1 Equilibrium Regimes

To study the effects of perturbations in the relative sizes of any two populations, we employ the notion of directional perturbation of size vector λ. In particular, for any two populations i and j, we consider the $|I|$-dimensional direction vector $z^{ij} \triangleq (\dots 0 \dots, 1, \dots 0 \dots, -1, \dots 0 \dots)$ with 1 in the i-th position and -1 in the j-th position. When λ is perturbed in the direction of z^{ij}, the size of population i (resp. population j) increases (resp. decreases), and the sizes of the remaining populations do not change.

For any size vector λ and any two populations i and j, let the vector of the remaining populations' sizes be denoted $\lambda^{-ij} \triangleq \left(\lambda^k\right)_{k \in I \setminus \{i,j\}}$. The total size of the remaining populations is $|\lambda^{-ij}| \triangleq \sum_{k \in I \setminus \{i,j\}} \lambda^k$. For pairwise comparison, we only consider the case when the sizes of both populations are strictly positive so that $|\lambda^{-ij}| < 1$, and the range of the perturbations in the population i's size is $(0, 1 - |\lambda^{-ij}|)$. We denote the set of admissible λ^{-ij} as Λ^{-ij}.

Now consider an optimization problem that is similar to (OPT-F), except that the two constraints in the (IIC) set corresponding to the populations i and j are replaced by a single constraint:

$$\min \quad \widehat{\Phi}(f), \quad s.t. \quad (2.17a), (2.17b), (2.17c), (\mathrm{IIC}) \setminus \{i,j\}, (\mathrm{IIC}_{ij}), \qquad (\text{OPT-}F^{ij})$$

where the constraints (IIC)$\setminus\{i,j\}$ indicate that all but (IIC$_i$) and (IIC$_j$) from the original set (IIC) are included, and the constraint (IIC$_{ij}$) is defined as follows:

$$\widehat{J}^i(f) + \widehat{J}^j(f) \leq \left(1 - |\lambda^{-ij}|\right) D. \qquad (\mathrm{IIC}_{ij})$$

The constraint (IIC$_{ij}$) ensures that the total impact of information on population i and j does not exceed their total demand. We denote the set of optimal solutions for (OPT-F^{ij})

as $F^{ij,\dagger}$. Analogously to Theorem 2.1, we can show that any $f^{ij,\dagger} \in F^{ij,\dagger}$ induces a unique edge load $w^{ij,\dagger}$, which can be obtained by (2.7); see Lemma A.2. Then, the optimal solution set of (OPT-F^{ij}) can be written as the following polytope:

$$F^{ij,\dagger} = \left\{ f \;\middle|\; \begin{array}{l} f \text{ satisfies (2.17a), (2.17b), (2.17c), (IIC)}\backslash\{i,j\}, \text{ and } (\text{IIC}_{ij}), \\ \sum_{r \ni e} f_r(t) = w_e^{ij,\dagger}(t), \quad \forall e \in E, \quad \forall t \in T \end{array} \right\}. \tag{2.25}$$

Note that both $F^{ij,\dagger}$ and $w^{ij,\dagger}$ depend on λ^{-ij} but do not depend on λ^i or λ^j.

Before proceeding further, we need to define two thresholds for the size of one of the two perturbed populations (say, population i):

$$\underline{\lambda}^i \triangleq \frac{1}{D} \min_{f^{ij,\dagger} \in F^{ij,\dagger}} \left\{ \widehat{J}^i(f^{ij,\dagger}) \right\}, \quad \bar{\lambda}^i \triangleq \frac{1}{D} \max_{f^{ij,\dagger} \in F^{ij,\dagger}} \left\{ \left(1 - |\lambda^{-ij}|\right) D - \widehat{J}^j(f^{ij,\dagger}) \right\}, \tag{2.26}$$

where $\widehat{J}^i(f^{ij,\dagger})$ and $\widehat{J}^j(f^{ij,\dagger})$ are the impact of information metrics for the population i and j, respectively. We can check that $\underline{\lambda}^i$ and $\bar{\lambda}^i$ are admissible thresholds:

Lemma 2.4. $0 \leq \underline{\lambda}^i \leq \bar{\lambda}^i \leq 1 - |\lambda^{-ij}|$.

Additionally, (2.26) can be expressed as linear programming problems, see (A.7)-(A.8). These two thresholds play a crucial role in our subsequent analysis.

We are now ready to introduce the equilibrium regimes that are induced by the relative change in the sizes of populations i and j with fixed sizes of other populations $\lambda^{-ij} \in \Lambda^{-ij}$. These regimes are defined by the following sets:

$$\Lambda_1^{ij} \triangleq \{\left(\lambda^i, \lambda^j, \lambda^{-ij}\right) \big| \lambda^i \in (0, \underline{\lambda}^i)\}, \tag{2.27a}$$

$$\Lambda_2^{ij} \triangleq \{\left(\lambda^i, \lambda^j, \lambda^{-ij}\right) \big| \lambda^i \in [\underline{\lambda}^i, \bar{\lambda}^i] \setminus \{0, 1 - |\lambda^{-ij}|\}\}, \tag{2.27b}$$

$$\Lambda_3^{ij} \triangleq \left\{\left(\lambda^i, \lambda^j, \lambda^{-ij}\right) \big| \lambda^i \in (\bar{\lambda}^i, 1 - |\lambda^{-ij}|)\right\}. \tag{2.27c}$$

We say that the population i (resp. population j) is a "minor population" in regime Λ_1^{ij} (resp. regime Λ_3^{ij}) because $\lambda^i < \underline{\lambda}^i$ (resp. $\lambda^j < 1 - |\lambda^{-ij}| - \bar{\lambda}^i$). Moreover, neither population is minor in regime Λ_2^{ij}. Note that degenerate situations are possible. In particular, if either one or both of the thresholds $\underline{\lambda}^i$ and $\bar{\lambda}^i$ take values in the set $\{0, 1 - |\lambda^{-ij}|\}$, then from the regime definitions (2.27a)-(2.27c), the number of regimes are reduced to two (for example, in the the simple routing game in Section 2.2) or even one regime (see Example A.2). The following theorem describes the properties of equilibrium route flows in the regimes under the directional perturbations in the size vector λ.

Theorem 2.2. *For any two populations $i, j \in I$, and any given $\lambda^{-ij} \in \Lambda^{-ij}$, the set of equilibrium route flows $F^*(\lambda)$ when λ is in regime Λ_1^{ij} or regime Λ_3^{ij} can be expressed as follows:*

$$F^*(\lambda) = \left\{ \arg\min \widehat{\Phi}(f) \;\middle|\; \begin{array}{lll} s.t. & (2.17a), (2.17b), (2.17c), (\mathrm{IIC}_{ij}) \textit{ and } (\mathrm{IIC})\backslash\{j\} & \textit{if } \lambda \in \Lambda_1^{ij} \\ s.t. & (2.17a), (2.17b), (2.17c), (\mathrm{IIC}_{ij}) \textit{ and } (\mathrm{IIC})\backslash\{i\} & \textit{if } \lambda \in \Lambda_3^{ij} \end{array} \right\} \tag{2.28}$$

In regime Λ_1^{ij} (resp. regime Λ_3^{ij}), the constraint (IIC_i) (resp. (IIC_j)) is tight in equilibrium. Additionally, in regime Λ_2^{ij}, we have $F^*(\lambda) \subseteq F^{ij,\dagger}$.

Essentially, this result is based on how the impact of information on each perturbed population compares with its size; i.e. whether or not the constraint (IIC_i) (resp. (IIC_j)) for the population i (resp. populatiom j) is tight in equilibrium. In the first side regime Λ_1^{ij}, the constraint (IIC_i) is tight at optimum of (OPT-F). This implies that the impact of information extends to the entire demand of the minor population i. In fact, the threshold $\underline{\lambda}^i$ is the largest size of population i for which the impact of information on itself is fully attained. We can argue similarly for the other side regime Λ_3^{ij}, where population j is the minor population; i.e. (IIC_j) is tight at optimum of (OPT-F) and $(1 - |\lambda^{-ij}| - \bar{\lambda}^i)$ is the largest size of population j such that the impact of information on itself is fully attained.

In contrast to the two side regimes, in the middle regime Λ_2^{ij}, the sizes of both populations i and j are above the threshold sizes $\underline{\lambda}^i$ and $(1 - |\lambda^{-ij}| - \bar{\lambda}^i)$, respectively. We can replace the constraints (IIC_i) and (IIC_j) in the optimization problem (OPT-F) by (IIC_{ij}) without changing its optimal value, i.e. the optimal value of (OPT-F^{ij}) is equal to $\Psi(\lambda)$. However, since the set $F^{ij,\dagger}$ (as defined in (2.25)) contains all route flows that attain the optimal value $\Psi(\lambda)$ but may not necessarily satisfy the constraints (IIC_i) and (IIC_j), the equilibrium route flow set $F^*(\lambda)$ must be a subset of $F^{ij,\dagger}$. In this regime, the impact of information on neither population is fully attained.

A specialized result derived from Theorem 2.2 and Proposition 2.1 is that in routing games with two heterogeneously informed populations and a parallel-route network, the equilibrium strategy profile in the regimes Λ_1^{12} and Λ_3^{12} is unique, see Corollary A.1.

Thanks to Theorem 2.2, we can analyze the monotonicity of the value of potential function at equilibrium $\Psi(\lambda)$ with respect to perturbations of λ in the direction z^{ij}.

Proposition 2.3. *For any two populations $i, j \in I$, and any given $\lambda^{-ij} \in \Lambda^{-ij}$, under directional perturbations of λ along the direction z^{ij}, the function $\Psi(\lambda)$ monotonically decreases in regime Λ_1^{ij}, does not change in Λ_2^{ij}, and monotonically increases in Λ_3^{ij}. Furthermore, the equilibrium edge load vector $w^*(\lambda) = w^{ij,\dagger}$ if and only if $\lambda \in \Lambda_2^{ij}$.*

Following Theorem 2.2, in the side regime Λ_1^{ij} (resp. Λ_3^{ij}), the set of route flows which satisfy the constraints of the optimization problem in (2.28) increases (resp. decreases) as λ is perturbed in the direction z^{ij}. Thus, the value of the potential function in equilibrium, $\Psi(\lambda)$, is non-increasing (resp. non-decreasing) in the direction z^{ij}. In fact, since the constraint (IIC_i) (resp. (IIC_j)) is tight in equilibrium, one can argue that $\Psi(\lambda)$ strictly decreases (resp. increases) in the direction z^{ij}. In contrast, in the middle regime Λ_2^{ij}, since $F^*(\lambda) \subseteq F^{ij,\dagger}$, we can conclude that $w^*(\lambda) = w^{ij,\dagger}$. Therefore, $\Psi(\lambda) = \breve{\Phi}(w^{ij,\dagger})$, which does not change when λ is perturbed in the direction z^{ij}.

The necessary and sufficient condition for the invariance of $w^*(\lambda)$ under relative perturbations in the sizes of any two populations in Proposition 2.3 is a direct consequence of the monotonicity of $\Psi(\lambda)$ and the uniqueness of $w^*(\lambda)$. This result is useful in determining the relative ordering of population costs in equilibrium, as discussed next.

2.5.2 Relative Value of Information

We now study the difference between the equilibrium costs of any two populations under perturbations in their relative sizes. For any two populations $i, j \in I$ and size vector λ, we define the *relative value of information*, denoted $V^{ij*}(\lambda)$, as the expected travel cost saving that a traveler in population i enjoys over a traveler in population j, i.e. $V^{ij*}(\lambda) \triangleq C^{j*}(\lambda) - C^{i*}(\lambda)$. Equivalently, $V^{ij*}(\lambda)$ is the expected reduction in the cost faced by an individual traveler when her subscription unilaterally changes from platform i to platform j, while the platform subscriptions of all other travelers remain unchanged. We say that the information platform i is relatively more valuable (resp. less valuable) than platform j if $V^{ij*}(\lambda) > 0$ (resp. $V^{ij*}(\lambda) < 0$). Similarly, if $V^{ij*}(\lambda) = 0$, platform i is said to be as valuable as platform j.

It turns out that, for any given size vector λ, $V^{ij*}(\lambda)$ is closely related to the sensitivity of $\Psi(\lambda)$ (i.e. the value of the potential function in equilibrium) with respect to the perturbation in the relative sizes of populations i and j.

Lemma 2.5. *The value of the weighted potential function in equilibrium, $\Psi(\lambda)$ as defined in* (2.24)*, is convex and directionally differentiable in λ. For any $i, j \in I$,*

$$V^{ij*}(\lambda) = -\frac{1}{D}\nabla_{z^{ij}}\Psi(\lambda), \tag{2.29}$$

where $\nabla_{z^{ij}}\Psi(\lambda) \triangleq \lim_{\epsilon\to 0^+} \frac{\Psi(\lambda+\epsilon z^{ij})-\Psi(\lambda)}{\epsilon}$ is the derivative of $\Psi(\lambda)$ in the direction z^{ij}.

The proof involves applying the results on sensitivity analysis of convex optimization problems, as summarized in Lemmas A.3 and A.4; for a detailed background on these technical results, we refer the reader to Fiacco [2009], Fiacco and Kyparisis [1986], and Rockafellar [1984].

Our next theorem provides the qualitative structure of relative value of information in the three regimes (2.27a)-(2.27c).

Theorem 2.3. *For any two populations $i, j \in I$, and any $\lambda^{-ij} \in \Lambda^{-ij}$, the relative value of information $V^{ij*}(\lambda)$ is positive in regime Λ_1^{ij}, zero in regime Λ_2^{ij}, and negative in regime Λ_3^{ij}.*

Furthermore, $V^{ij*}(\lambda)$ is non-increasing in the direction z^{ij}.

Theorem 2.3 shows that one population has advantage over another population if and only if it is the minor population of the two (see Fig. 2-5). For the two side regimes, information impacts the entire demand of the minor population. As a result, in equilibrium, the travelers in the minor population do not choose the routes with a high expected cost based on the signal they receive from their platform; however, the travelers in the other population may still choose these routes. On the other hand, in the middle regime, neither population has an advantage over the other one because the information only partially impacts each population's demand. Consequently, both populations route their demand in a manner such that they face identical cost in equilibrium.

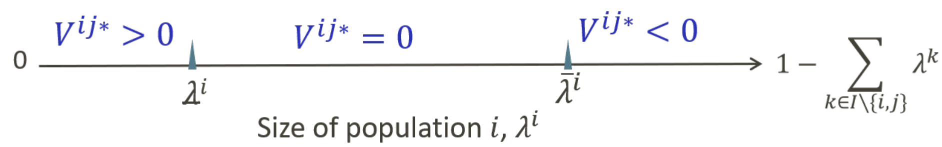


Figure 2-5: Relative value of information between population i and j in the three regimes

Additionally, the travel cost saving that population i travelers enjoy over the population j is the highest when population i has few travelers. Intuitively, in each side regime, the travelers in the non-minor population face a higher congestion externality relative to the travelers in the minor population, because all travelers within a population are routed according to the same strategy. Naturally, the difference in the equilibrium costs due to the imbalance in congestion externality decreases as the size of the minor population increases, and reduces to zero in the middle regime.

Furthermore, given any two populations $i, j \in I$ and the sizes of all other populations λ^{-ij} being fixed, Theorem 2.3 provides a computational approach to compare the equilibrium costs of populations i and j for the full range of $\lambda^i \in (0, 1 - |\lambda^{-ij}|)$ without explicit computation of equilibrium set or equilibrium route flows for each λ^i. This approach can be summarized as follows: (i) Solve (OPT-F^{ij}) to obtain an optimal solution $f^{ij,\dagger}$; (ii) Compute $w^{ij,\dagger}$ by plugging $f^{ij,\dagger}$ into (2.7); (iii) Obtain $\underline{\lambda}^i$ and $\bar{\lambda}^i$ by solving (2.26); and (iv) Find the relative ordering of equilibrium costs of population i and j by checking which of the three possible regimes the size vector λ belongs to.

Finally, we can specialize Theorem 2.3 to analyze situations when a population does not have an access to a platform, or chooses not to use it at all. Formally, we say that a population $j \in I$ is *uninformed* if its type is independent with the network state and other populations' types, i.e. $\Pr(t^j | s, t^{-j}) = \Pr(t^j)$ for any $t^{-j} \in T^{-j}$, any $t^j \in T^j$, and any $s \in S$.

Following (2.5), the uninformed population j's interim belief can be written as follows:

$$\beta^j(s, t^{-j}|t^j) \overset{(2.5)}{=} \frac{\pi(s, t^j, t^{-j})}{Pr(t^j)} = \frac{Pr(t^j|s, t^{-j}) \cdot Pr(s, t^{-j})}{Pr(t^j)} = Pr(s, t^{-j}) = \sum_{t^j \in T^j} \pi(s, t^{-j}, t^j), \tag{2.30}$$

That is, the interim belief $\beta^j(s, t^{-j}|t^j)$ is identical for any signal $t^j \in T^j$ received by population j, and is equal to the marginal probability of (s, t^{-j}) calculated from the common prior π. Therefore, the uninformed population has no further information besides the common knowledge. We show that the equilibrium cost of the uninformed travelers is no less than the cost of any other population.

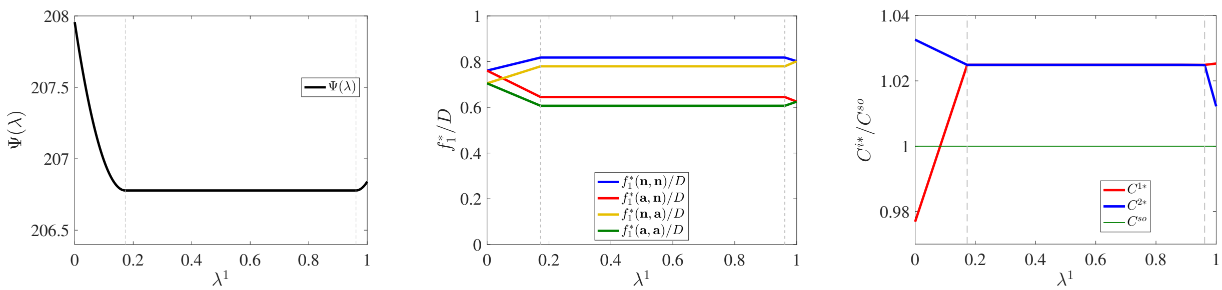
Proposition 2.4. *Consider the game $\Gamma(\lambda)$ in which population j is uninformed. Then, for any size vector λ, the equilibrium cost of population j's travelers $C^{j*}(\lambda) \geq C^{i*}(\lambda)$, where the population i is any other population (i.e. $i \in I \setminus \{j\}$).*

Indeed, if population j is uninformed, we can argue that its equilibrium routing strategy $q^{j*}(t^j)$ must be identical for any $t^j \in T^j$. Consequently, from (2.18), the impact of information metric for the population j is $\widehat{J}^j(q^{j*}) = 0$, and perturbing the relative sizes of population j and any other population $i \in I \setminus \{j\}$ never results in a regime in which population j is the minor population. Applying Theorem 2.3, we can conclude that the equilibrium cost of population j cannot be less than that of any other population.

We illustrate the results on equilibrium structure and relative value of information in the following two examples:

Example 2.1. We consider a game with two populations on two parallel routes (r_1 and r_2) with following parameters: $\theta(a) = 0.2$, $D = 10$, $c_1^{\mathbf{n}}(f_1) = f_1 + 15$, $c_1^{\mathbf{a}}(f_1) = 3f_1 + 15$, $c_2(f_2) = 2f_2 + 20$. Types t^1 and t^2 are independent conditional on the state, i.e. $\Pr(t^1, t^2|s) = \Pr(t^1|s) \cdot \Pr(t^2|s)$. Population 1 has 0.8 chance of getting accurate information of the state, and population 2 has 0.6 chance, i.e. $\Pr(t^1 = s|s) = 0.8$, and $\Pr(t^2 = s|s) = 0.6$. The value of the potential function in equilibrium, equilibrium route flows and population costs are shown in Fig. 2-6.

In this example, population 1 travelers receive more accurate state information than population 2 travelers. However, population 1 faces a higher cost than population 2 when its size is sufficiently large, i.e., when λ is in regime Λ_3^{12}; see Figure 2-6c. This is due to the fact that in regime Λ_3^{12}, the population 1's advantage of receiving more accurate information is dominated by the congestion externality it faces due to its relatively large size, in comparison to population 2.

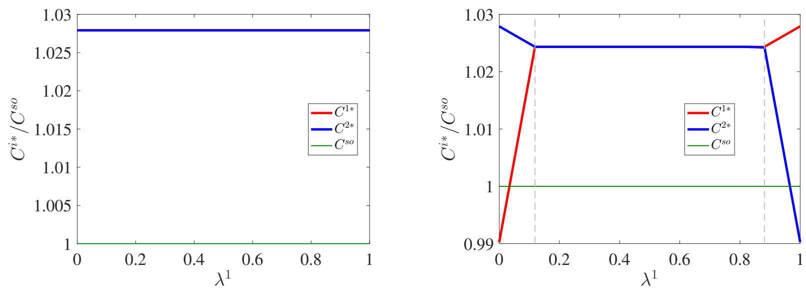


(a) (b) (c)

Figure 2-6: Effects of varying population sizes for Example 2.1: (a) Weighted potential function in equilibrium, (b) Equilibrium route flows on r_1, and (c) Equilibrium population costs.

Example 2.2. Let us now consider the game with two populations on two parallel routes (r_1 and r_2) with the same cost functions, prior distribution θ and total demand D as that in Example 2.1. Both populations 1 and 2 have 0.75 chance of getting accurate information about the state, i.e. $\Pr(t^i = s|s) = 0.75$ for any $i \in I$ and any $s \in S$. In Fig. 2-7, we illustrate the equilibrium population costs in two cases: (i) Types t^1 and t^2 are perfectly correlated, i.e. $t^1 = t^2$; (ii) Types t^1 and t^2 are independent conditional on the state, i.e. $\Pr(t^1, t^2|s) = \Pr(t^1|s) \cdot \Pr(t^2|s)$.

This example illustrates how the correlation among received signals (or lack thereof) affects the equilibrium structure. Note that case (i) can be viewed as a single-population game. This is because when t^1 and t^2 are perfectly correlated, there is no information asymmetry among travelers. Thus, λ^1 has no impact on the equilibrium outcome, and $\underline{\lambda}^1 = 0$, $\bar{\lambda}^1 = 1$ (Fig. 2-7a). However, case (ii) is not equivalent to a single-population game. Although both populations have identical chance of getting accurate information about the state, there is information heterogeneity among travelers of the two populations, i.e. travelers in one population do not know the signals received by travelers in the other population, and thus the equilibrium outcome changes with the size λ^1 (Fig. 2-7b).



(a) (b)

Figure 2-7: Effects of varying population sizes on equilibrium population costs for Example 2.2: (a) Perfectly correlated types; (b) Conditional independent types.

We include two additional examples in Appendix A.2. Example A.1 shows that the regime Λ_3^{12} can be empty even when population 2 is not an uninformed population. Thus, an uninformed population j is sufficient but not necessary for $\bar{\lambda}^i = 1 - |\lambda^{-ij}|$. In Example A.2, we present a situation when only single regime exists in equilibrium.

Our results so far focus on how equilibrium properties and population costs change with the directional perturbation of the size vector λ. We emphasize that given any $i, j \in I$, the thresholds $\underline{\lambda}^i$ and $\bar{\lambda}^i$, as defined in (2.26), depend on the sizes of the remaining populations λ^{-ij}, and the populations' interim beliefs $(\beta^i)_{i \in I}$ derived from the common prior π. Importantly, the qualitative structure of the equilibrium regimes resulting from perturbations in the sizes of any two populations is applicable for any size vector λ and any common prior. The main property that drives these results is that the equilibrium regimes only depend on whether or not the impact of information on each population is fully attained.

2.6 General Properties of Equilibrium Outcome

In this section, we first extend our approach of pairwise comparison of populations to study how the equilibrium outcome depends on population sizes in general. Then, we analyze the adoption rates of information platforms in situations where travelers can choose platform subscription.

2.6.1 Size-Independence of Edge Load Vector

Our analysis in Section 2.5 showed that if perturbations in the relative sizes of any two populations $i, j \in I$ induce a middle regime Λ_2^{ij}, then the equilibrium outcome in this regime is independent of the sizes of the perturbed populations i and j. A natural question to ask is whether this result can be generalized; i.e., can we find a set of size vectors for which the equilibrium edge load does not depend on the size of *any* population? The answer is affirmative.

We now explicitly characterize the set of size vectors, denoted $\Lambda^\dagger$, for which the edge load is size-independent. Since (OPT-F) is a convex optimization problem, and (IIC) are the only size-dependent constraints, we can equivalently view $\Lambda^\dagger$ as the set of size vectors for which all the IICs can be dropped from (OPT-F) without changing its optimal value. Hence, for any $\lambda \in \Lambda^\dagger$, the optimal value of (OPT-F) is identical to that of the following convex optimization problem:

$$\min \quad \widehat{\Phi}(f), \quad s.t. \quad \text{(2.17a), (2.17b) and (2.17c).} \tag{2.31}$$

Let us denote the optimal solution set of (2.31) as $F^\dagger$. Analogous to Theorem 2.1, one can argue that any optimal solution $f^\dagger \in F^\dagger$ induces a unique edge load $w^\dagger$, obtained from (2.7). Thus, $F^\dagger$ can be written as the convex polytope:

$$F^\dagger = \left\{ f \;\middle|\; \begin{array}{l} f \text{ satisfies (2.17a), (2.17b), (2.17c),} \\ \text{and } \sum_{r \ni e} f_r(t) = w_e^\dagger(t), \quad \forall e \in E, \quad \forall t \in T \end{array} \right\}. \tag{2.32}$$

Furthermore, since any route flow in the set $F^\dagger$ satisfies the constraints (2.17a)-(2.17c) – but not necessarily (IIC) constraints – and also attains the optimal value of (OPT-F), we must have that for any $\lambda \in \Lambda^\dagger$, $F^*(\lambda) \subseteq F^\dagger$. Therefore, for each $\lambda \in \Lambda^\dagger$, there must exist a $f^\dagger \in F^\dagger$ that is an equilibrium route flow, i.e. at least one $f^\dagger \in F^\dagger$ satisfies the (IIC) constraints corresponding to λ:

$$\Lambda^\dagger \triangleq \left\{ \lambda \;\middle|\; \begin{array}{l} \sum_{i \in I} \lambda^i = 1; \quad \lambda^i \geq 0, \ \forall i \in I; \\ \exists f^\dagger \in F^\dagger \text{ s.t. } \widehat{J}^i(f^\dagger) \leq \lambda^i D, \quad \forall i \in I \end{array} \right\} \tag{2.33}$$

Figure 2-8: Choice of information platforms

The following proposition shows the properties of equilibrium edge load and the value of potential function in the set $\Lambda^\dagger$:

Proposition 2.5. *The set $\Lambda^\dagger$ is convex, and attains the minimum of $\Psi(\lambda)$, i.e. $\Lambda^\dagger = \arg\min_\lambda \Psi(\lambda)$. The equilibrium edge load vector $w^*(\lambda)$ is size-independent, and is equal to $w^\dagger$ if and only if $\lambda \in \Lambda^\dagger$.*

This result shows that some of the properties of $\Psi(\lambda)$ and the change of equilibrium edge load vector under pairwise perturbation (Proposition 2.3) also hold for the more general case of perturbation in sizes of multiple populations.

2.6.2 Adoption Rates under Choice of Information Platforms

Our analysis so far has focused on the equilibrium properties with fixed population sizes. We now extend our results on the relative value of information (Section 2.5) and the size independence of the equilibrium edge load vector (Section 2.6.1) to analyze travelers' choice of information subscription when they can choose to subscribe to any information platform in the set I.

We model travelers' choice of information platforms and the choice of routes as a two-stage game (Fig. 2-8): In the first stage, travelers choose to subscribe to one information platform from the set I. The induced size vector is $\lambda = (\lambda^i)_{i \in I}$, where λ^i is the fraction of travelers who choose platform i. In the second stage, travelers play the Bayesian routing game $\Gamma(\lambda)$. Note that the size vector λ here is determined by the travelers' choices of platforms in the first stage, as opposed to being a parameter.

In equilibrium, a traveler who chooses platform i experiences the expected cost $C^{i*}(\lambda)$. The travelers has no ex-ante incentive to unilaterally change her platform subscription if and only if $C^{i*}(\lambda)$ is the lowest across all $i \in I$. Therefore, no traveler has the incentive to change her platform subscription if and only if:

$$\lambda^i > 0 \quad \Rightarrow \quad C^{i*}(\lambda) = \min_{j \in I} C^{j*}(\lambda), \quad \forall i \in I. \tag{2.34}$$

Such population size vector λ can be viewed as the vector of equilibrium adoption rates, one for each platform. For any size vector that satisfies (2.34), all travelers experience identical expected costs, and no traveler has the incentive to change her platform subscriptions.

Our next theorem shows that all size vectors $\lambda \in \Lambda^\dagger$ are equilibrium adoption rates of information platforms.

Theorem 2.4. *The set of equilibrium adoption rates under the choice of information platform is* $\Lambda^\dagger$.

Note that the set $\Lambda^\dagger$ is not a singleton set in general. Recall from Example 2.1, the set $\Lambda^\dagger$ is the range Λ_2^{12}, in which both platforms are chosen. Therefore, the equilibrium adoption rate of each platform is not unique. However, since $\Lambda^\dagger$ is a convex set, the equilibrium adoption rate of each platform $i \in I$ is in a continuous range, denoted $[\lambda^{i\dagger}_{min}, \lambda^{i\dagger}_{max}]$, where $\lambda^{i\dagger}_{min} = \min_{\lambda^\dagger \in \Lambda^\dagger} \lambda^{i\dagger}$ (resp. $\lambda^{i\dagger}_{max} = max_{\lambda^\dagger \in \Lambda^\dagger} \lambda^{i\dagger}$) is the minimum (resp. maximum) equilibrium adoption rate. Furthermore, since the set $\Lambda^\dagger$ is determined by the heterogeneous information environment created by all information platforms, the equilibrium adoption rate of each platform i is not only determined by the distribution of its own signal, but is also related to the distribution of other platform signals, and the possible correlations between signals of different platforms.

Finally, Theorem 2.4 can be used to assess whether or not a set of information platforms can induce a heterogeneous information environment. For any $\lambda^\dagger \in \Lambda^\dagger$, the support set of $\lambda^\dagger$, denoted $\bar{I}(\lambda^\dagger) \triangleq \{i \in I | \lambda^{i\dagger} > 0\}$, represents the set of platforms chosen by travelers. In particular, if $|\bar{I}(\lambda^\dagger)| = 1$, then all travelers choose to subscribe to a single platform even though multiple platforms are available. Thus, the resulting information environment is homogeneous; see Example A.2, where $\lambda^1 = 1$ and $\lambda^2 = 0$ is the only equilibrium adoption rate. However, if $|\bar{I}(\lambda^\dagger)| > 1$, then more than one platforms are chosen, i.e., the heterogeneous information environment is sustained. Moreover, if platform $i \notin \bar{I}(\lambda^\dagger)$ for any $\lambda^\dagger \in \Lambda^\dagger$, then this platform is redundant in that it is not chosen in equilibrium even if it is available to travelers.

2.7 Discussion

In this chapter, we study the equilibrium route choices and costs in a heterogeneous information environment, in which each population receives a private signal from their traffic information platforms. Each population maintains a belief about the unknown network state and about the signals received by other traveler populations. We focus on analyzing the equilibrium structure under perturbations of population sizes, the relative value of information between any pair of populations, as well as the equilibrium adoption rates when travelers can choose their platform subscription.

The main ideas behind our analysis approach are: (i) Identification of qualitatively distinct equilibrium regimes based on whether or not the impact of information is fully attained; (ii) Sensitivity analysis of the weighted potential function in equilibrium with respect to the population size vector; and (iii) Characterization of adoption rates under the choice of information platforms. Our approach can be easily extended to games where the edge costs are non-decreasing (rather than strictly increasing) in the edge loads. In particular, such a game still admits a weighted potential function, although now the essential uniqueness only applies to the equilibrium edge costs, rather than the loads. The qualitative properties of equilibrium structure, results about the relative ordering of population costs, and adoption rates can be extended as well. However, the characterization of regime thresholds in this case is more complicated from a computational viewpoint due to the non-uniqueness of edge load vector.

One future research question of interest is to analyze how the travelers' expected cost and platform adoption rates change when one or more information platforms make technological changes to their service (for example, improving accuracy levels), or when a new information platform is introduced. Addressing this problem would involve applying our results to evaluate the value of information for each traveler population as well as the adoption rates under the new information environment, and comparing them with that of the current environment.

Chapter 3

Information Design in Routing Games

3.1 Introduction

In Chapter 2, we developed a new game-theoretic approach to analyze the impact of heterogeneous information environment on travelers' self-interested route choices and congestion costs of the traffic network. This chapter extends Chapter 2 for designing an "optimal" information structure that can be used to regulate traffic flows in a network with uncertain state. We adopt the viewpoint of Bayesian persuasion and focus on identifying the distribution of information signals (conditional on the state) to induce an equilibrium outcome that is close to a target flow pattern which reflects the preference of a central authority (information designer).

Practically, our setup is motivated by several concerns raised by city authorities and residents of areas that have witnessed significant increase in traffic congestion in their neighborhood streets due to route recommendations provided by the navigation apps (Bliss [2015], Bagby [2016], and Foderaro [2017]). In some cases, these routes pass through school areas, evacuation zones, construction sites, or active incidents. Increased traffic through these regions naturally raises noise and safety concerns. In other cases, recommended routes are often comprised of secondary streets that were not designed for heavy and prolonged rush hour traffic. As a result, these streets can witness deterioration in infrastructure condition, and further decrease in their traffic carrying capacity.

To address these concerns, cities and transportation agencies – henceforth jointly referred as *central authority* – are now playing an active role in communicating their preferred limits on the usage of neighborhood streets to traffic information providers (Geha [2016], and Barragan [2015]). This raises the question: *How can the central authority reduce traffic spillover (average flow exceeding a specific threshold) on certain routes by designing the information environment faced by travelers?*

We present a stylized model to address this question. Our model captures an important (and practically relevant) feature of traffic information design: not all travelers can or choose to receive the signal sent by the central authority. For example, some travelers may not have access to or choose not to use the information signal. We capture this feature in a Bayesian routing game, where the heterogeneous information environment is determined not only by the the signal sent by the central authority, but also due to the amount of travelers with access to this signal. This game allows us to formulate and solve the information design problem, in which the distribution of information signal is chosen by the central authority to regulate the induced equilibrium traffic flows.

We consider the two-route transportation network presented in Sec. 2.2, where one route is prone to a random capacity-reducing event (incident), and the other is not. Incident state results in increased travel cost on the first route. The cost of each route is an increasing (affine) function of the flow on that route. The central authority (information designer) knows the true state (i.e., whether or not the incident happened), and chooses an information structure that is used to send the signal to a fraction of travelers. Travelers are strategic in that they choose routes with the minimum expected cost based on their information of the state. The induced route flow is a Bayesian Wardrop equilibrium corresponding to the information structure chosen by central authority. The objective of the central authority is to minimize the average spillover (in Bayesian Wardrop equilibrium) on a pre-selected route beyond a threshold flow (Fig. 3-1).

Our work contributes to the growing literature of information design. Previous work includes optimal information design that sends public signals to *all* travelers in order to minimize the overall traffic congestion (Das et al. [2017] and Tavafoghi and Teneketzis [2017]), and private information design that incentives information sharing in repeated routing games (Meigs et al. [2020]). The ideas used in our approach are built on the broader literature

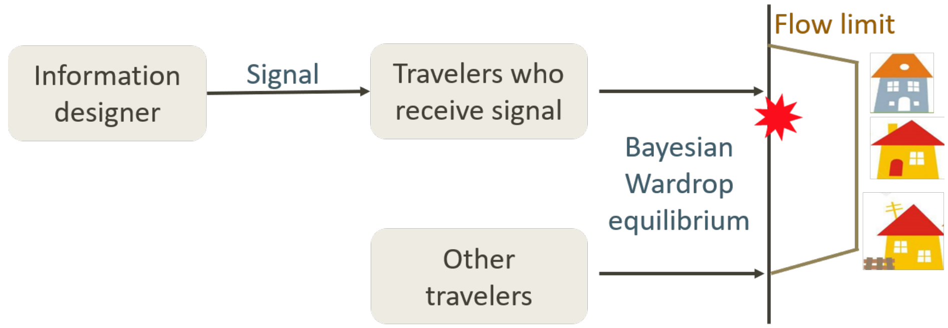


Figure 3-1: Information design in transportation networks with uncertain state.

on Bayesian persuasion; see Kamenica [2018] for a comprehensive review. These literature studied Bayesian persuasion with multiple receivers (Bergemann and Morris [2016], Mathevet et al. [2017]), and persuasion with private information (Kolotilin et al. [2017]).

Our information design model has two distinguishing features: Firstly, the objective of the central authority in our setting is to minimize the average traffic spillover on chosen route, instead of minimizing the average travel time; Secondly, travelers have the flexibility to choose if they want to receive the public signals sent by the authority. Their choices of information induce the heterogeneous information environment, and the equilibrium route flows are evaluated in Bayesian Wardrop equilibrium following Chapter 2. In this chapter, we characterize the optimal information structure for *any* fraction of travelers who receive the signal. We show that our optimal information design can achieve the minimum traffic spillover given travelers' equilibrium choice of information. Under this optimal information structure, all travelers (the ones who choose to receive the signal and the ones who do not) experience the same expected travel time costs in equilibrium.

Rest of the chapter is organized as follows: In Sec. 3.2, we present the information design problem. Sec. 3.3 characterizes the equilibrium flows under any information structure designed by the traffic authority. Sec. 3.4 presents the optimal information structure for any given threshold and any fraction of travelers with access to information signal, and Sec. 3.5 analyzes the impact of optimal information design on travelers' costs and travelers' choices of information.

3.2 Information Design Problem

We consider the same two route network as in Sec. 2.2. A single origin-destination pair is connected by two parallel routes $R = \{r_1, r_2\}$, where the uncertain state is $S = \{\mathbf{a}, \mathbf{n}\}$ and the cost function of each route is given by (2.1). The state is realized from a prior $\theta = (\theta(\mathbf{a}), \theta(\mathbf{n}))$. A set of non-atomic travelers with total demand of D make route choices in the network, and D satisfies (2.2).

We introduce a central authority (city or transportation agency) who has complete knowledge of the realized network state. This authority is an "information designer" in that she has the ability to shape the travelers' information about the state by way of sending them a (noisy) signal $t^1 \in T = \{\mathbf{a}, \mathbf{n}\}$ of the state. For example, in the context of transportation systems, the authority can influence travelers' knowledge of route conditions through advanced traveler information platforms.

Furthermore, in practice, it is reasonable to expect that some travelers may not have access to or choose not to use the signal sent by the authority. We refer the mass of travelers who receive the signals as population 1, and the remaining travelers who do not have access to signals as population 2. In our problem of optimal information design (Sec. 3.4), the fraction of population 1, denoted $\lambda^1 \in [0, 1]$, is taken as an exogenous parameter.

The authority chooses an *information structure* $p = (p(t^1|s))_{t^1 \in T^1, s \in S}$, where $p(t^1|s)$ is the probability of sending signal t^1 when the state is s. Let $\mathcal{P}$ be the set of feasible information structures satisfying the following constraints:

$$p(t^1|s) \geq 0, \quad \forall t^1 \in T^1, \text{ and } \forall s \in S, \tag{3.1a}$$

$$\sum_{t^1 \in T^1} p(t^1|s) = 1, \quad \forall s \in S, \tag{3.1b}$$

$$p(\mathbf{n}|\mathbf{n}) \geq p(\mathbf{n}|\mathbf{a}), \tag{3.1c}$$

Constraints (3.1a) - (3.1b) ensure that $(p(\mathbf{a}|s), p(\mathbf{n}|s))$ is a feasible probability vector for any $s \in S$. Constraint (3.1c) ensures that the signal $\mathbf{n}$ is more likely to be sent in state $\mathbf{n}$ than in state $\mathbf{a}$. Constraints (3.1b) and (3.1c) also imply that signal $\mathbf{a}$ is more likely to be sent in state $\mathbf{a}$ than in state $\mathbf{n}$, i.e. $p(\mathbf{a}|\mathbf{a}) \geq p(\mathbf{a}|\mathbf{n})$. We use (3.1c) to avoid duplication of equivalent information structures. Note that one information structure becomes another equivalent structure by switching the signals **a** and **n**.

The signal sent by the authority creates a heterogeneous information environment for travelers to make routing decisions. Following Sec. 2.3, we model travelers' routing decisions under the heterogeneous information environment as a Bayesian routing game. The common prior of the game is $\pi = (\pi(s,t^1))_{s \in S, t^1 \in T}$, where $\pi(s,t^1) = \theta(s)p(t^1|s)$. Additionally, the marginal probability of each signal $t^1 \in T^1$ is $\psi(t^1) = \sum_{s \in S} \theta(s)p(t^1|s)$.

The Bayesian equilibrium routing strategy profile defined in Definition 2.1 is $q^* = (q_r^{1*}(\mathbf{a}), q_r^{1*}(\mathbf{n}), q_r^{2*})_{r \in R}$, which is dependent on the information structure p designed by the authority and the population size λ. Then, $f^* = (f_r^*(t^1))_{r \in R, t^1 \in T}$ is the induced route flow vector, where $f_r^*(t^1) = q_r^{1*}(t^1) + q_r^{2*}$.

The goal of the central authority is to reduce the average amount of traffic that exceeds a given threshold (traffic spillover) on one of the two routes. The authority may select the route(s) and threshold(s) for regulating traffic flows based on factors such as desirable or enforced capacity limits and/or maximum admissible flow through various routes, to limit direct impact (e.g., congestion) or indirect impact (e.g., noise or safety concerns) of traffic flow. For ease of exposition, we assume that the authority chooses an information structure that minimizes the average spillover on r_2 given a fixed threshold τ on the route flow.[1] Therefore, the design of optimal information structure p^* can be formulated as the following optimization problem:

$$\begin{aligned} \min_{p} \quad & L(p, f^*) \triangleq \sum_{t^1 \in T^1} \psi(t^1)(f_2^*(t^1) - \tau)_+ \\ s.t. \quad & p \text{ satisfies (3.1)}, \\ & f^* \text{ is an equilibrium route flow vector in corresponding to } p \text{ and } \lambda. \end{aligned} \tag{OPT-Info}$$

where $(f_2^*(t^1) - \tau)_+ = \max\{f_2^*(t^1) - \tau, 0\}$ is the amount of traffic that exceeds the threshold τ on r_2 when the signal is t^1.

[1]Our subsequent analysis can also be applied to address the spillover on r_1 and/or arrive at a trade-off between desirable flows on r_1 and r_2.

3.3 Equilibrium Flows

To solve for the optimal information structure in (OPT-Info), we first parametrically characterizes the unique equilibrium route flow vector f^* for any information structure p and any population size $\lambda^1 \in [0,1]$. This result follows from our approach of Bayesian Wardrop equilibrium characterization developed in Chapter 2. We find that given any information structure p, the properties of f^* depends on the relative size between λ^1 and the value of a function $g : p \to [0,1]$ defined as follows:

$$g(p) \triangleq \frac{\alpha_2 D + b_2 - b_1}{\left(\bar{\alpha_1}(\beta(\mathbf{n})) + \alpha_2\right) D} - \frac{\alpha_2 D + b_2 - b_1}{\left(\bar{\alpha_1}(\beta(\mathbf{a})) + \alpha_2\right) D}, \tag{3.2}$$

where $\bar{\alpha_1}(\beta(t^1)) = \alpha_1^{\mathbf{a}}\beta(\mathbf{a}|t^1) + \alpha_1^{\mathbf{n}}\beta(\mathbf{n}|t^1)$, and $\beta(s|t^1)$ is the posterior belief of state s when receiving signal t^1. This posterior belief is obtained from the common prior π using Bayes' rule:

$$\beta(s|t^1) = \frac{\theta(s)p(t^1|s)}{\theta(\mathbf{a})p(t^1|\mathbf{a}) + \theta(\mathbf{n})p(t^1|\mathbf{n})}. \tag{3.3}$$

Since p satisfies (3.1c), we can check that $\beta(\mathbf{a}|\mathbf{a}) \geq \beta(\mathbf{a}|\mathbf{n})$ and $\beta(\mathbf{n}|\mathbf{n}) \geq \beta(\mathbf{n}|\mathbf{a})$, i.e. the belief of state s when receiving signal $t^1 = s$ is higher than that with the other signal. Since we consider two routes, and the central authority aims at minimizing the traffic spillover on r_2, we only present the equilibrium route flow on r_2. The flow on r_1 is $f_1^*(t^1) = D - f_2^*(t^1)$ for any $t^1 \in T^1$.

Proposition 3.1. *For any p and λ^1, the equilibrium route flow is unique and satisfies the following properties:*

- *[$g(p) \geq \lambda^1$.] Population 1 exclusively takes r_1 when they receive signal $\mathbf{n}$, and r_2 with signal $\mathbf{a}$; population 2 splits on the two routes. The flow on r_2 is:*

$$f_2^*(\mathbf{n}) = D - \frac{\alpha_2 D + b_2 - b_1 + \lambda^1 D\psi(\mathbf{a})\left(\bar{\alpha_1}(\beta(\mathbf{a})) + \alpha_2\right)}{\bar{\alpha_1}(\theta) + \alpha_2} \tag{3.4a}$$

$$f_2^*(\mathbf{a}) = f_2^*(\mathbf{n}) + \lambda^1 D, \tag{3.4b}$$

where $\bar{\alpha_1}(\theta) = \theta(\mathbf{a})\alpha_1^{\mathbf{a}} + \theta(\mathbf{n})\alpha_1^{\mathbf{n}}$.

- *[$g(p) < \lambda^1$.] Both populations split on the two routes. The flow on r_2 is:*

$$f_2^*(\mathbf{n}) = D - \frac{\alpha_2 D + b_2 - b_1}{\bar{\alpha_1}(\beta(\mathbf{n})) + \alpha_2}, \tag{3.5a}$$

$$f_2^*(\mathbf{a}) = D - \frac{\alpha_2 D + b_2 - b_1}{\bar{\alpha_1}(\beta(\mathbf{a})) + \alpha_2}. \tag{3.5b}$$

The proof of this result follows from the discussion of the two route example in Sec. 2.2, and Theorems 2.1 – 2.2 in Chapter 2. From Proposition 3.1, we know that for any population size λ^1, the set of feasible information structures can be partitioned into two sets as follows:

$$\mathcal{P}^1 \triangleq \left\{\mathcal{P} | g(p) \geq \lambda^1\right\}, \quad \mathcal{P}^2 \triangleq \left\{\mathcal{P} | g(p) < \lambda^1\right\}, \tag{3.6}$$

and the impact of p on f^* for the case when $p \in \mathcal{P}^1$ is distinct from $p \in \mathcal{P}^2$:

For $p \in \mathcal{P}^1$, all travelers in population 1 deviate from choosing r_1 to r_2 when the received signal changes from $\mathbf{n}$ to $\mathbf{a}$. From (3.4), the change of flow on r_2 induced by the change of signal is $f_2^*(\mathbf{a}) - f_2^*(\mathbf{n}) = \lambda^1 D$, which does not depend on the information structure p. Moreover, as λ^1 increases, $f_2^*(\mathbf{n})$ decreases and $f_2^*(\mathbf{a})$ increases.

For $p \in \mathcal{P}^2$, both populations split on the two routes. From (3.5), the change of flow on r_2 induced by the change of signal is $f_2^*(\mathbf{a}) - f_2^*(\mathbf{n}) = g(p)$. The value of $g(p)$ in (3.2) increases in $\beta(\mathbf{a}|\mathbf{a}) - \beta(\mathbf{a}|\mathbf{n})$, which evaluates the relative difference between the beliefs of state $\mathbf{a}$ given the two signals.

Furthermore, for any $p \in \mathcal{P}^2$, the equilibrium route flows in (3.5) do not change with λ^1. This implies that any equilibrium outcome when only λ^1 fraction of travelers receive signal t^1 according to $p \in \mathcal{P}^2$ is equivalent to the case where *all* travelers receive the signal. This property will be used in Sec. 3.4 for identifying an interval of λ^1, for which the optimal information structure and the equilibrium outcome do not depend on λ^1.

Finally, note that the information structure p affects the value of $g(p)$ in (3.2), and the equilibrium route flows in (3.4) - (3.5) through the posterior beliefs and the marginal probability of signals defined as follows:

$$\beta \triangleq (\beta(\mathbf{a}), \beta(\mathbf{n})), \quad \psi \triangleq (\psi(\mathbf{a}), \psi(\mathbf{n})),$$

Then, we can re-write $L(p, f^*)$ in (OPT-Info) as a function of (β, ψ), denoted $\bar{L}(\beta, \psi)$, and $g(p)$ as a a function of β, denoted $\bar{g}(\beta)$. The characterization of how the equilibrium route flow depends on (β, ψ) and the fraction λ^1 is crucial for our approach for solving the optimal information design problem in the next section.

3.4 Optimal Information Design

In this section, we present the optimal information structure p^*, and analyze how p^* changes with the fraction λ^1 and the flow threshold τ.

Due to the space limit, we only present the optimal information structure for cases where the threshold τ satisfies the following constraint:

$$D - \frac{\alpha_2 D + b_2 - b_1}{\alpha_1^{\mathbf{n}} + \alpha_2} \leq \tau \leq D - \frac{\alpha_2 D + b_2 - b_1}{\alpha_1^{\mathbf{a}} + \alpha_2}. \tag{3.7}$$

The lower (resp. upper) bound of τ is the equilibrium route flow on r_2 when all travelers have complete information of the state $\mathbf{n}$ (resp. $\mathbf{a}$). Therefore, (3.7) means that in complete information environment, the spillover is positive in state $\mathbf{a}$, but zero in state $\mathbf{n}$. Our solution approach can be easily extended to the cases where τ is outside of this range.

We first obtain that if the prior probability of state $\mathbf{a}$ is low, then the optimal information structure is to provide no information of the state.

Proposition 3.2. *If $\theta(\mathbf{a}) \leq \eta$, where*

$$\eta = \frac{1}{\alpha_1^{\mathbf{a}} - \alpha_1^{\mathbf{n}}} \left(\frac{\alpha_2 D + b_2 - b_1}{D - \tau} - \alpha_2 - \alpha_1^{\mathbf{n}} \right),$$

then the optimal information structure is to provide no information of the state, i.e. $p^*(t^1|s) = \psi(t^1)$ for any $t^1 \in T^1$ and $s \in S$. The average traffic spillover is $L(p^*, f^*) = 0$.

From Proposition 3.1, we know that if p^* does not provide state information, then $g(p^*) = 0$, and the equilibrium route flow is as follows:

$$f_2^*(t^1) = D - \frac{\alpha_2 D + b_2 - b_1}{\bar{\alpha_1}(\theta) + \alpha_2}, \quad \forall t^1 \in T^1. \tag{3.8}$$

The value of η is the threshold of p such that $f_2^*(t^1)$ in (3.8) equals to τ. For any $\theta(\mathbf{a}) \leq \eta$, $f_2^*(t^1) \leq \tau$. Therefore, the objective function in (OPT-Info) is zero (attains the minimum) when the information structure provides no information of the state. From (3.7), we know that $\eta \in [0, 1]$.

For $\theta(\mathbf{a}) > \eta$, based on Proposition 3.1, we can restate the optimal information design problem (OPT-Info) as the following optimization problem:

$$
\begin{aligned}
\min_{p} \quad & L(p, f^*) = \sum_{t^1 \in T^1} \psi(t^1)(f_2^*(t^1) - \tau)_+, \\
s.t. \quad & \begin{array}{l} f_2^* \text{ is in (3.4)}, \quad \text{if } g(p) \geq \lambda^1, \\ f_2^* \text{ is in (3.5)}, \quad \text{if } g(p) < \lambda^1, \end{array} \\
& p \text{ satisfies (3.1)},
\end{aligned} \tag{3.9}
$$

where $g(p)$ is given by (3.2).

The optimization problem (3.9) is non-linear and non-convex in the information structure p. The key difficulties in solving (3.9) are: (i) the value of $g(p)$, and the equilibrium route flows f_2^* are nonlinear functions of (β, ψ), which are again nonlinear functions of p; (ii) the expressions of the equilibrium route flows are different for information structures in $\mathcal{P}^1$ and $\mathcal{P}^2$; (iii) the objective function is a piece-wise linear (instead of linear) function of the equilibrium route flows.

We develop an approach to tackle these difficulties and solve the optimal information structure analytically: First, we characterize the set of (β, ψ) induced by information structure p satisfying (3.1), which can be used to construct another optimization problem to solve the optimal (β^*, ψ^*) directly (Lemma 3.1). Second, we identify the range of λ^1 in which the optimal information structure satisfies $p^* \in \mathcal{P}^1$, and the equilibrium route flow is given by (3.4) (Lemma 3.2). Third, we prove that the equilibrium flow on r_2 under optimal information structure is no less than τ; thus $L(p, f^*)$ is equivalent to a linear function of f^* (Lemma 3.3).

Lemma 3.1. *A tuple (β, ψ) can be induced by a feasible information structure p if and only* *if* (β, ψ) *satisfies:*

$$\beta(\mathbf{a}|\mathbf{a}) \cdot \Pr(\mathbf{a}) + \beta(\mathbf{a}|\mathbf{n}) \cdot \Pr(\mathbf{n}) = p, \tag{3.10a}$$

$$\beta(\mathbf{a}|\mathbf{a}) \geq \beta(\mathbf{a}|\mathbf{n}), \tag{3.10b}$$

$$\beta(\mathbf{n}|\mathbf{a}) + \beta(\mathbf{a}|\mathbf{a}) = 1, \beta(\mathbf{a}|\mathbf{a}), \beta(\mathbf{n}|\mathbf{a}) \geq 0, \tag{3.10c}$$

$$\beta(\mathbf{n}|\mathbf{n}) + \beta(\mathbf{a}|\mathbf{n}) = 1, \beta(\mathbf{n}|\mathbf{n}), \beta(\mathbf{a}|\mathbf{n}) \geq 0, \tag{3.10d}$$

$$\Pr(\mathbf{a}) + \Pr(\mathbf{n}) = 1, \quad \Pr(\mathbf{a}), \Pr(\mathbf{n}) \geq 0. \tag{3.10e}$$

The idea of the proof follows Proposition 1 in Kamenica and Gentzkow [2011].

Constraint (3.10a) ensures that β is derived from θ and p as in (3.3). Constraint (3.10b) results from (3.1c) to exclude beliefs that are induced by equivalent information structures. Constraints (3.10c) – (3.10e) ensure that β and ψ are feasible probability vectors.

Based on Lemma 3.1 and following (3.9), we can solve for the optimal (β^*, ψ^*) from the following optimization problem:

$$\begin{aligned} \min_{\beta,\psi} \quad & \bar{L}(\beta, \psi) = \sum_{t^1 \in T^1} \Pr(t^1)(f_2^*(t^1) - \tau)_+, \\ s.t. \quad & \begin{array}{l} f_2^* \text{ is in (3.4)}, \quad \text{if } \bar{g}(\beta) \geq \lambda^1, \\ f_2^* \text{ is in (3.5)}, \quad \text{if } \bar{g}(\beta) < \lambda^1, \end{array} \\ & (\beta, \psi) \text{ satisfies (3.10)}, \end{aligned} \tag{3.11}$$

where $\bar{g}(\beta)$ is a function of β as in (3.2). We use both (3.9) and (3.11) for designing the optimal information structure.

Next, we identify a threshold $\widetilde{\lambda}^1 \in (0, 1)$ as follows:

$$\widetilde{\lambda}^1 = 1 - \frac{\alpha_2 D + b_2 - b_1}{(\alpha_1^{\mathbf{a}} + \alpha_2)\, D} - \frac{\tau}{D}. \tag{3.12}$$

Lemma 3.2. *For any* $\theta(\mathbf{a}) > \eta$*, and any* $\lambda^1 < \widetilde{\lambda}^1$*, the optimal information structure* $p^* \in \mathcal{P}^1$*, i.e.* $g(p^*) \geq \lambda^1$*, where* $g(p^*)$ *is in* (3.2)*. The equilibrium route flow is given by* (3.4)*.*

Furthermore, we show that given the optimal information structure, $f_2^*(t^1)$ is no less than the threshold τ for any $t^1 \in T^1$.

Lemma 3.3. *For any $\theta(\mathbf{a}) > \eta$ and any $\lambda^1 \in [0, 1]$, the equilibrium route flows induced by the optimal information structure must satisfy $f_2^*(t^1) \geq \tau$ for any $t^1 \in T^1$.*

Lemma 3.3 shows that the objective function in (3.9) can be simplified as a linear function of f^*:

$$L(p, f^*) = \psi^{\mathbf{a}}(f_2^*(\mathbf{a}) - \tau) + \psi^{\mathbf{n}}(f_2^*(\mathbf{n}) - \tau)$$
$$= \psi^{\mathbf{a}} f_2^*(\mathbf{a}) + \psi^{\mathbf{n}} f_2^*(\mathbf{n}) - \tau. \tag{3.13}$$

We are now ready to derive the optimal information structure p^*. We find that p^* is different for λ^1 in three regimes: $\Lambda_1^* : \lambda^1 \in [0, \underset{\sim}{\lambda}^1)$; $\Lambda_2^* : \lambda^1 \in [\underset{\sim}{\lambda}^1, \widetilde{\lambda}^1)$, and $\Lambda_3^* : \lambda \in [\widetilde{\lambda}^1, 1]$. The threshold $\underset{\sim}{\lambda}^1$ is given by:

$$\underset{\sim}{\lambda}^1 = \frac{(D - \tau)\left(\bar{\alpha_1}(\theta) + \alpha_2\right) - \alpha_2 D - b_2 + b_1}{D\theta(\mathbf{a})(\alpha_1^{\mathbf{a}} + \alpha_2)}, \tag{3.14}$$

and $\widetilde{\lambda}^1$ is given by (3.12).[2] Since τ satisfies (3.7) and $\theta(\mathbf{a}) > \eta$, we can check that $0 < \underset{\sim}{\lambda}^1 < \widetilde{\lambda}^1 < 1$. Therefore, the three regimes are well-defined intervals of λ^1.

Based on Proposition 3.1 and Lemmas 3.1 – 3.3, we characterize the optimal information structure in each regime. We also present the equilibrium route flow and the average traffic spillover in each regime.

Theorem 3.1. *For any $\theta(\mathbf{a}) > \eta$, in regime Λ_1^*, the optimal information structure is:*

$$p^*(\mathbf{a}|\mathbf{n}) = 0, \quad p^*(\mathbf{n}|\mathbf{n}) = 1, \tag{3.15a}$$
$$p^*(\mathbf{a}|\mathbf{a}) = 1, \quad p^*(\mathbf{n}|\mathbf{a}) = 0. \tag{3.15b}$$

The equilibrium route flow is:

$$f_2^*(\mathbf{n}) = D - \frac{\alpha_2 D + b_2 - b_1 + \lambda^1 D\theta(\mathbf{a})(\alpha_1^{\mathbf{a}} + \alpha_2)}{\bar{\alpha_1}(\theta) + \alpha_2}$$

[2] The thresholds $\underset{\sim}{\lambda}^1$, $\widetilde{\lambda}^1$ and the regimes Λ_1^*, Λ_2^*, Λ_3^* are defined to distinguish different qualitative properties of the optimal information structure p^*. These thresholds and regimes are different from that in Chapter 2.

$$f_2^*(\mathbf{a}) = f_2^*(\mathbf{n}) + \lambda^1 D.$$

The average traffic spillover decreases in λ^1:

$$L(p^*, f^*) = D - \tau - \frac{\alpha_2 D + b_2 - b_1}{\bar{\alpha_1}(\theta) + \alpha_2} - \frac{\theta(\mathbf{a})\theta(\mathbf{n})(\alpha_1^{\mathbf{a}} - \alpha_1^{\mathbf{n}})\lambda^1 D}{\bar{\alpha_1}(\theta) + \alpha_2}.$$

In regime Λ_2^*, the optimal information structure is:

$$p^*(\mathbf{a}|\mathbf{n}) = 0, \quad p^*(\mathbf{n}|\mathbf{n}) = 1, \tag{3.16a}$$

$$p^*(\mathbf{a}|\mathbf{a}) = \frac{(D-\tau)(\bar{\alpha_1}(\theta) + \alpha_2) - \alpha_2 D - b_2 + b_1}{\lambda^1 D(\alpha_1^{\mathbf{a}} + \alpha_2)\theta(\mathbf{a})}, \tag{3.16b}$$

$$p^*(\mathbf{n}|\mathbf{a}) = 1 - p^*(\mathbf{a}|\mathbf{a}). \tag{3.16c}$$

The equilibrium route flow is:

$$f_2^*(\mathbf{n}) = \tau, \quad f_2^*(\mathbf{a}) = \tau + \lambda^1 D.$$

The average traffic spillover does not change with λ^1:

$$L(p^*, f^*) = \frac{(D-\tau)(\bar{\alpha_1}(\theta) + \alpha_2) - \alpha_2 D - b_2 + b_1}{\alpha_1^{\mathbf{a}} + \alpha_2}. \tag{3.17}$$

In regime Λ_3^*, the optimal information structure is:

$$p^*(\mathbf{a}|\mathbf{n}) = 0, \quad p^*(\mathbf{n}|\mathbf{n}) = 1, \tag{3.18a}$$

$$p^*(\mathbf{a}|\mathbf{a}) = \frac{(D-\tau)(\bar{\alpha_1}(\theta) + \alpha_2) - \alpha_2 D - b_2 + b_1}{((D-\tau)(\alpha_1^{\mathbf{a}} + \alpha_2) - \alpha_2 D - b_2 + b_1)\,\theta(\mathbf{a})}, \tag{3.18b}$$

$$p^*(\mathbf{n}|\mathbf{a}) = 1 - p^*(\mathbf{a}|\mathbf{a}). \tag{3.18c}$$

The equilibrium route flow is:

$$f_2^*(\mathbf{n}) = \tau, \quad f_2^*(\mathbf{a}) = D - \frac{\alpha_2 D + b_2 - b_1}{\alpha_1^{\mathbf{a}} + \alpha_2}. \tag{3.19a}$$

The average traffic spillover $L(p^*, f^*)$ *is as in* (3.17).

Now we discuss the properties of optimal information structure in detail. Firstly, in state $\mathbf{n}$, the signal provides complete state information, i.e. $p^*(\mathbf{n}|\mathbf{n}) = 1$. This is because when the cost on r_1 is low in state $\mathbf{n}$, sending signal $\mathbf{a}$ will unnecessarily increase the traffic spillover on r_2. In state $\mathbf{a}$, the signal provides complete state information when the fraction of population 1 is smaller than $\underset{\sim}{\lambda}^1$, but only provides partial state information ($p^*(\mathbf{a}|\mathbf{a}) < 1$) if $\lambda^1 > \underset{\sim}{\lambda}^1$ to avoid sending large flow to r_2.

Secondly, the average spillover decreases with λ^1 in regime Λ_1^* ($\lambda^1 < \underset{\sim}{\lambda}^1$), and does not change with λ^1 in regimes Λ_2^* and Λ_3^* ($\lambda^1 \geq \underset{\sim}{\lambda}^1$). This implies that the minimum average traffic spillover can be achieved by the optimal information structure as long as the fraction of travelers receiving the signal exceeds the threshold $\underset{\sim}{\lambda}^1$, which is smaller than 1. Moreover, if $\lambda^1 \geq \underset{\sim}{\lambda}^1$, then the spillover is only positive in state $\mathbf{a}$, i.e. traffic flow on r_2 only exceeds the threshold flow τ if there is an incident. The probability of positive spillover with optimal information structure ($p^*(\mathbf{a}|\mathbf{a}) \cdot \theta(\mathbf{a})$) is smaller than that in the case where travelers have no state information (the spillover probability is 1) and the case where travelers have complete state information (the spillover probability is $\theta(\mathbf{a})$).

Thirdly, $\widetilde{\lambda}^1$ is the threshold fraction beyond which the optimal information structure p^* and the equilibrium route flow f^* do not depend on λ^1. Additionally, $\widetilde{\lambda}^1$ is the maximum impact of the signal on route flows, i.e. the maximum fraction of travelers who change routing decisions with the received signals. For any $\lambda^1 \leq \widetilde{\lambda}^1$, $p^* \in \mathcal{P}^1$ and the signal influences the routing decisions of all travelers in population 1 ($\lambda^1 < \widetilde{\lambda}^1$ fraction). On the other hand, for any $\lambda^1 > \widetilde{\lambda}^1$, $p^* \in \mathcal{P}^2$. Then, regardless of the fraction of population 1, $\widetilde{\lambda}^1$ fraction of travelers change their routing decisions with the signal.

Finally, we find that the regime boundaries $\underset{\sim}{\lambda}^1$ and $\widetilde{\lambda}^1$ in (3.14) and (3.12) decrease as τ increases. Practically, this implies that if more traffic can be routed on r_2 (i.e. τ is larger), then the minimum average spillover can be achieved by sending signals to a smaller fraction of travelers ($\underset{\sim}{\lambda}^1$) according to the optimal information structure. Additionally, the maximum fraction of travelers who change routing decisions with the received signals ($\widetilde{\lambda}^1$) is also smaller.

3.5 Impact of Information Design on Travel Costs

We now analyze how the optimal information structure designed for minimizing the spillover affects the travelers' cost in equilibrium. Recall that the equilibrium population cost $C^{i*}(\lambda^1)$ is the average travel time costs experienced by travelers in each population in equilibrium as in (2.10). Then, the equilibrium average cost of all travelers as $C^*(\lambda^1) = \lambda^1 C^{1*}(\lambda^1) + (1 - \lambda^1)C^{2*}(\lambda^1)$.

For any $\theta(\mathbf{a}) \leq \eta$, we know from Proposition 3.2 that the optimal information structure provides no state information to travelers. Hence, the information structure has no impact on the travelers' equilibrium costs.

For any $\theta(\mathbf{a}) > \eta$, we can compute the equilibrium population costs and the equilibrium average cost based on Proposition 3.1 and Theorem 3.1.

Proposition 3.3. *Given the optimal information structure p^*, $C^{1*}(\lambda^1) \leq C^{2*}(\lambda^1)$ if $\lambda^1 \in [0, \widetilde{\lambda}^1)$, and $C^{1*}(\lambda^1) = C^{2*}(\lambda^1)$ if $\lambda^1 \in [\widetilde{\lambda}^1, 1]$. Furthermore, as λ^1 increases, $C^*(\lambda^1)$ monotonically decreases in regime Λ_1^*, increases in regime Λ_2^*, does not change in regime Λ_3^*.*

Proposition 3.3 shows that the signal sent by the central authority gives travelers in population 1 an advantage over population 2 in terms of the average costs if $\lambda^1 < \widetilde{\lambda}^1$, and the two populations experience the same cost if $\lambda^1 \geq \widetilde{\lambda}^1$. Furthermore, Theorem 3.1 and Proposition 3.3 show that in regime Λ_1^*, increasing λ^1 reduces both the traffic spillover on r_2 and the equilibrium average cost. However, if λ^1 increases beyond $\underset{\sim}{\lambda}^1$, then the average cost increases, while the average spillover does not change.

One can interpret these insights in the context of two practical situations – the fraction λ^1 is induced by travelers' choice of accessing to the signal versus chosen by the designer versus. In both situations, the average traffic spillover is the same.

If travelers can choose whether or not to get access to signals – the information, then the fraction of population 1 will be higher or equal to $\widetilde{\lambda}^1$. This is because travelers in population 2 will continue to switch to population 1 until the costs of two populations are the same. Hence, the optimal information structure p^* is given by (3.18).

On the other hand, if the designer can choose the fraction λ^1 as well as the information structure p, then it is optimal for the authority to provide complete state information as in

(3.15) to $\underset{\sim}{\lambda}^1$ fraction of travelers in order to achieve the minimum average spillover and the minimum average cost. However, in this case, $C^{1*}(\lambda^1) < C^{2*}(\lambda^1)$. Therefore, the resulting information structure favors the set of travelers who receive the signals.

We illustrate our results in the following example.

Example 3.1. The cost functions of the network are $c_1^{\mathbf{a}}(f_1) = 3f_1 + 15$, $c_1^n(f_1) = f_1 + 15$, and $c_2(f_2) = 2f_2 + 20$. The total demand $D = 10$, and the threshold $\tau = 2.5$. The probability of state $\mathbf{a}$ is $\theta(\mathbf{a}) = 0.3$. From (3.14) and (3.12), the thresholds are $\underset{\sim}{\lambda}^1 = 0.133$, and $\widetilde{\lambda}^1 = 0.25$.

Fig. 3-2a shows $p^*(\mathbf{a}|\mathbf{a})$ for $\lambda^1 \in [0, 1]$. Fig. 3-2b shows the resulting equilibrium cost of each population. Fig. 3-2c and Fig. 3-2d compare the average traffic spillover and the equilibrium average cost under the optimal information structure with the corresponding costs in two situations: (1) the central authority provides no information to travelers; (2) complete information of the state is provided to λ^1 fraction of travelers.

In this example, the minimum average spillover can be achieved as long as more than 13.3% of travelers have access to the signal (Fig. 3-2c). If the fraction increases over 25%, then the optimal information design does not depend on λ^1 (Fig. 3-2a). Moreover, the optimal information structure achieves 18% lower spillover in comparison to the case of no information, and 47% lower spillover in comparison to providing complete state information to all travelers. This demonstrates that the central authority achieves non-trivial reduction in average spillover by optimal information design even when a high fraction of travelers do not have access to the information signal.

Additionally, Fig. 3-2b shows that in this example population 1 enjoys reduction of cost by 7% compared with population 2 if $\lambda^1 = \underset{\sim}{\lambda}^1$, which is the fraction that minimizes the average spillover and the average cost. Finally, from Fig. (3-2d), we see that the equilibrium average cost under optimal information structure is lower than that in the case with no state information, but the minimum cost under optimal information structure (when $\lambda^1 = \underset{\sim}{\lambda}^1$) is slightly (1%) higher than the minimum cost in the case where the signal provides complete state information.

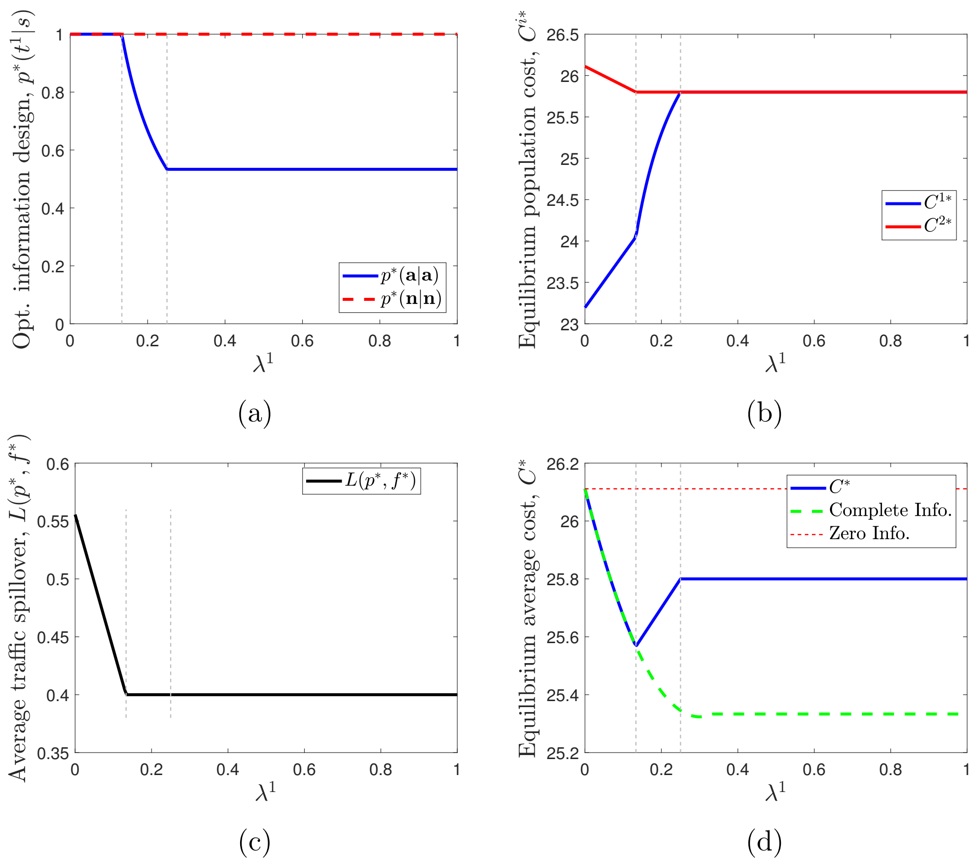


Figure 3-2: Congestion costs under optimal information structure: (a) Probability of accident signal in state **a**; (b) Equilibrium population costs; (c) Average spillover on r_2; (d) Equilibrium average cost.

Chapter 4

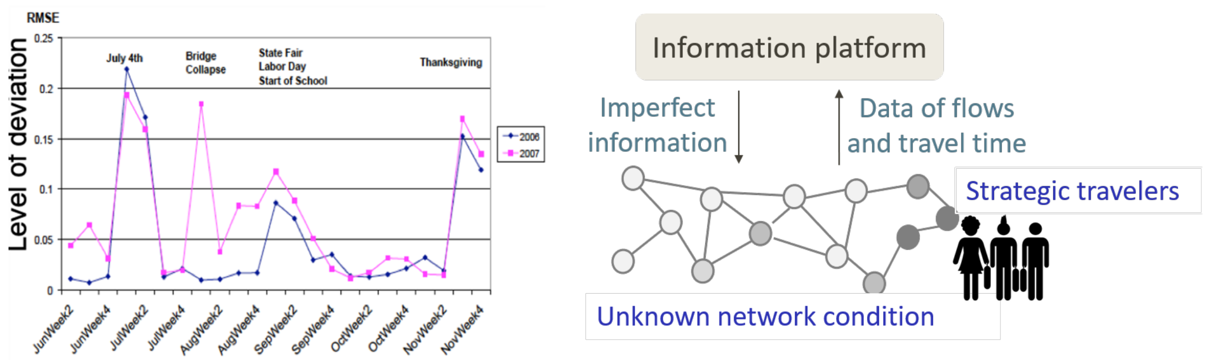
Multi-agent Bayesian Learning with Best Response Strategies

4.1 Introduction

In Chapters 2 and 3, we have studied the impact of information platforms on players' strategic decision making in static settings. In practice, strategic players often need to engage in repeated interactions with each other while learning an unknown environment that impacts their payoffs. The goal of this chapter is to build a stochastic learning dynamics that analyzes the role of information platforms in the process of learning the unknown environment and adjusting strategies.

Major disruptions in transportation networks such as random infrastructure breakdowns, natural disasters and security attacks often lead to a sudden change in the latent network condition that influences travel costs on one or more network edges. After the 2007 collapse of I-35W bridge over Mississippi River in Minneapolis, data collected from the loop detectors showed that the flow patterns in the surrounding area experienced high fluctuation for several weeks (Fig. 4-1a, Zhu et al. [2010]). This suggests that the change of the network condition triggered a learning process of the new condition, and adjustment of travel decisions. In this process, information platforms repeatedly provide estimates of the new network condition to travelers based on the collect data on traffic flows and travel time costs (Fig. 4-1b).

Similar situations also arise in other settings. For example, buyers and sellers repeatedly



(a) (b)

Figure 4-1: (a) Comparison of flow fluctuations before and after the collapse of the Mississippi River Bridge in 2007 (Source: Zhu et al. [2010]); (b) Learning unknown network condition with strategic travelers.

make their transaction decisions while learning the latent market condition on online market platforms. The market condition is unknown, and it governs the price distribution. The price distribution is updated based on the previous transactions and buyer reviews on platforms such as Amazon, eBay, and Airbnb (Moe and Fader [2004], Acemoglu et al. [2017]). The users' decisions and realized prices drive the learning of the overall latent market condition, which further impacts the subsequent transactions.

Our work is motivated by the need of establishing a learning foundation that captures how self-interested players adaptively adjust their strategies while learning the uncertain environment through an information platform. The distinguishing feature of the learning process is that players' strategic decisions (route choices in transportation networks or purchases and sales on online platforms) influence the learning of the unknown environment (latent network or market condition), which then impact the players' future decisions. Therefore, the long-run outcome of strategic interactions among players is governed by the joint evolution of stage-wise decisions made by the players and learning of the unknown environment.

In this chapter, we first present a generic learning dynamics that captures this joint evolution in a game-theoretic setting. In this dynamics, strategic agents (players) repeatedly play a game with an unknown payoff-relevant parameter vector belonging to a finite set. A public information platform updates and broadcasts a Bayesian estimate of the payoff parameter based on stage-wise game outcomes (i.e. strategies and randomly realized payoffs) to all players. Players update their strategies by incorporating a best response strategy based on the updated belief and the opponents' play.

We develop a new approach to analyze the long-run outcomes – convergence and stability properties (both local and global) – of the beliefs and strategies induced by the interplay of Bayesian updates and best response dynamics. We show that, with probability 1, beliefs and strategies converge to a fixed point, where the belief consistently estimates the payoff distribution for the strategy, and the strategy is an equilibrium corresponding to the belief. However, learning may not always identify the unknown parameter because the belief estimate relies on the game outcomes that are endogenously generated by players' strategies. We obtain sufficient and necessary conditions, under which learning leads to a globally stable fixed point that is a complete information Nash equilibrium. We also provide sufficient conditions that guarantee local stability of fixed point beliefs and strategies. Our technical results are useful to study other types of learning dynamics, such as learning with two timescales, and learning under non-Bayesian estimates of the unknown parameter.

Next, we apply our analysis to repeated routing games, in which travelers adjust their route choices based on the belief estimate of unknown edge cost parameters provided by a traffic information platform. The information platform repeatedly updates belief estimates of edge cost distributions based on traffic flows and realized costs. We show that travelers eventually form consistent estimates of costs on edges with positive flows, but may overestimate the costs on unused edges. Therefore, learning does not necessarily lead to a complete information equilibrium unless certain conditions are satisfied. We also show that the long-run average travel cost is higher when learning fails to converge to complete information equilibrium.

Our model and analysis extend the results on learning in games with complete information to situations when long-run outcomes depend on learning of an unknown parameter. Past literature has addressed convergence analysis of discrete and continuous time best response dynamics (Milgrom and Roberts [1990], Monderer and Shapley [1996b], Hofbauer and Sorin [2006]), fictitious play (Fudenberg and Kreps [1993], Monderer and Shapley [1996a]) and stochastic fictitious play (Benaim and Hirsch [1999], Hofbauer and Sandholm [2002]) in complete information environment.

The key distinction between our learning dynamics and classical best response dynamics is that, in our model, players are imperfectly informed about the payoff-relevant parameter, and their strategy updates in each stage rely on the updated Bayesian belief. Our approach to studying the convergence and stability properties of the coupled belief and strategy dynamics combines ideas from Bayesian learning and analysis of best response dynamics in complete information games. Furthermore, our conditions that guarantee fixed point stability can be viewed as natural extensions of the stability conditions for both best response dynamics and evolutionary dynamics in games with complete information (Smith and Price [1973], Taylor and Jonker [1978], Samuelson and Zhang [1992], Matsui [1992], Hofbauer and Sandholm [2009], Sandholm [2010]).

Additionally, the notion of fixed point in our learning dynamics is similar to the self-confirming equilibrium introduced in Fudenberg and Levine [1993] for extensive-form games.[1] At a self-confirming equilibrium, players maintain consistent beliefs of their opponents' strategies at information sets that are reached, but the beliefs of strategies can be incorrect at unreached information sets. In our model, a fixed point can be different from a complete information equilibrium due to the incorrect estimates on the unobserved game outcomes formed by the beliefs. In general, these incorrect estimates may never be corrected by the learning dynamics because information of game outcomes is endogenously acquired based on the chosen strategies in each stage.[2]

Finally, our application in routing games also contributes to the extensive literature on other types of learning dynamics: log-linear learning (Blume et al. [1993], Marden and Shamma [2012], Alós-Ferrer and Netzer [2010]), regret-based learning (Hart and Mas-Colell [2003], Foster and Young [2006], Marden et al. [2007], Daskalakis et al. [2011]), payoff-based learning (Cominetti et al. [2010], Marden et al. [2009]), and replicator dynamics (Beggs [2005], Hopkins [2002]). These dynamics typically prescribe the manner in which the players adjust their strategies based on the randomly realized payoffs in each stage. On the other hand, the strategy updates in our learning dynamics capture a rational behavioral adjustment

[1]Similar concepts include conjectural equilibrium in Hahn [1978] and subjective equilibrium in Kalai and Lehrer [1993] and Kalai and Lehrer [1995].

[2]The phenomenon that endogenous information acquisition leads to incomplete learning is also central to multi-arm bandit problems Rothschild [1974], Easley and Kiefer [1988] and endogenous social learning Duffie et al. [2009], Acemoglu et al. [2014], Ali [2018].

of self-interested players in an imperfect information environment.

This chapter is organized as follows: Section 4.2 describes the generic learning model for continuous games, and Section 4.3 details the convergence and stability properties. In Section 4.4, we discuss the extensions of our main results to other types of learning dynamics such as two-timescale learning, and learning with maximum a posteriori or least square estimates. We present the application of repeated routing games in Sec. 4.5. All proofs are included in Appendix B.

4.2 Model of Learning Dynamics in Continuous Games

Our learning dynamics is induced by strategic players in a finite set I who repeatedly play a game G for an infinite number of stages. The players' payoffs in game G depend on an *unknown* parameter vector s belonging to a finite set S. The true parameter is denoted $s^* \in S$. Learning is mediated by a public information platform (or an aggregator) that repeatedly updates and broadcasts a belief estimate $\theta = (\theta(s))_{s \in S} \in \Delta(S)$ to all players, where $\theta(s)$ denotes the estimated probability of parameter s.

In game G, the strategy of each player $i \in I$ is a finite dimensional vector q_i in a convex and continuous strategy set Q_i. The players' strategy profile is denoted $q = (q_i)_{i \in I} \in Q \triangleq \prod_{i \in I} Q_i$. The payoff of each player is realized randomly according to a probability distribution. Specifically, the distribution of players' payoffs $y = (y_i)_{i \in I}$ for any strategy profile $q \in Q$ and any parameter $s \in S$ is represented by the probability density function $\phi^s(y|q)$. We assume that $\phi^s(y|q)$ is continuous in q for all $s \in S$. Without loss of generality, we write the player i's payoff y_i for any $s \in S$ as the sum of an average payoff $u_i^s(q)$ that is a continuous function of q and a noise term $\epsilon_i^s(q)$ with zero mean:

$$y_i = u_i^s(q) + \epsilon_i^s(q). \tag{4.1}$$

The noise terms $(\epsilon_i^s(q))_{i \in I}$ can be correlated across players.

In game G with belief θ, each player i's best response correspondence given their opponents' strategies $q_{-i} = (q^j)_{j \in I \setminus \{i\}}$ is the set of strategies that maximize their expected utility, i.e. $\mathrm{BR}_i(\theta, q_{-i}) \triangleq \arg\max_{q_i \in Q_i} \mathbb{E}_\theta \left[u_i^s(q_i, q_{-i})\right] = \arg\max_{q_i \in Q_i} \sum_{s \in S} \theta(s) u_i^s(q_i, q_{-i})$. Additionally, the set of equilibrium strategies for any belief θ is a non-empty set $\mathrm{EQ}(\theta)$.

Our learning model can be specified as a discrete-time stochastic dynamics, with state comprising of the belief estimate of unknown parameter and the players' strategies:[3] In each stage $k \in \mathbb{N}_+$, the information platform broadcasts the current belief estimate θ^k; the players act according to a strategy profile $q^k = \left(q_i^k\right)_{i \in I}$; and the payoffs $y^k = \left(y_i^k\right)_{i \in I}$ are realized according to $\phi^s(y^k|q^k)$ when the parameter is $s \in S$. The state of learning dynamics in stage k is $\left(\theta^k, q^k\right) \in \Delta(S) \times Q$.

The initial belief θ^1 in our learning dynamics does not exclude any possible parameter, i.e. $\theta^1(s) > 0$ for all $s \in S$, and the initial strategy $q^1 \in Q$ is feasible. The evolution of states $\left(\theta^k, q^k\right)_{k=1}^\infty$ is jointly governed by belief and strategy updates, which we introduce next.

Belief update. In our model, the belief is updated intermittently and infinitely. The stages at which the information platform updates the belief can be deterministic or random, denoted by the subsequence $(k_t)_{t=1}^\infty$. In update stage k_{t+1}, the previous belief estimate θ^{k_t} is updated using the observed players' strategy profile $\left(q^k\right)_{k=k_t}^{k_{t+1}-1}$ and realized payoffs $\left(y^k\right)_{k=k_t}^{k_{t+1}-1}$ between the stages k_t and k_{t+1} according to the Bayes' rule:

$$\theta^{k_{t+1}}(s) = \frac{\theta^{k_t}(s) \prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k)}{\sum_{s' \in S} \theta^{k_t}(s') \prod_{k=k_t}^{k_{t+1}-1} \phi^{s'}(y^k|q^k)}, \quad \forall s \in S. \tag{θ-update}$$

Strategy update. Players update their strategies in each stage based on the updated belief and the current strategies played by their opponents. Given any θ^{k+1} and any $q_{-i}^k = \left(q_j^k\right)_{j \in I \setminus \{i\}}$, we generically denote the strategy update for each $i \in I$ as a set-valued function $F_i\left(\theta^{k+1}, q_{-i}^k\right) : \Delta(S) \times Q_{-i} \rightrightarrows Q_i$:

$$q_i^{k+1} \in F_i\left(\theta^{k+1}, q_{-i}^k\right), \quad \forall i \in I. \tag{q-update}$$

In particular, we consider the following three types of best response update rules for F_i:

[3]In this chapter, we follow the terminology of stochastic dynamical systems and refer the state as the stage-wise belief and strategy. This terminology is different from that in Bayesian routing game adopted by Chapters 2 – 3, where the state is the unknown network cost parameter.

1. *Simultaneous best response dynamics.* Each player chooses a strategy that is in the best response correspondence given their opponents' strategies and the updated belief:

$$F_i(\theta^{k+1}, q_{-i}^k) = \mathrm{BR}_i(\theta^{k+1}, q_{-i}^k), \quad \forall i \in I. \tag{Simultaneous-BR}$$

2. *Sequential best response dynamics.* In each stage, exactly one player updates their strategy as the best response strategy given the new belief. Players sequentially updates their strategies:

$$F_i(\theta^{k+1}, q_{-i}^k) = \begin{cases} \mathrm{BR}_i(\theta^{k+1}, q_{-i}^k), & \text{if } k \bmod |I| = i, \\ \left\{q_i^k\right\}, & \text{otherwise.} \end{cases} \tag{Sequential-BR}$$

3. *Linear best response dynamics.* Each player updates their strategy as a linear combination of their current strategy and a best response strategy given the updated belief:

$$F_i(\theta^{k+1}, q_{-i}^k) = (1-\alpha^k)q_i^k + \alpha^k \mathrm{BR}_i(\theta^{k+1}, q_{-i}^k), \quad \forall i \in I, \quad \forall k, \tag{Linear-BR}$$

where $\alpha^k \in [0, 1]$ is the rate of strategy update in stage k.

Next, we present few remarks about our learning dynamics: Firstly, players are strategic in that their strategy updates utilize a best response strategy that maximizes their expected utilities given the latest belief estimate of the unknown parameter and the strategies played by their opponents. If all players know the true parameter s^*, then the three strategy updates reduce to the classical best response dynamics in the corresponding game with complete information.

Secondly, the three types of strategy updates differ in the timing and the extent at which best response is incorporated in the updated strategy: All players update their strategies in every stage in (Simultaneous-BR) and (Linear-BR), while only one player updates strategy in (Sequential-BR). While players entirely adopt the new best response strategy in updates of (Simultaneous-BR) and (Sequential-BR), in (Linear-BR) each player weighs their best response strategy according to the strategy update rate.

Thirdly, the belief updates can occur less frequently than the strategy updates since the subsequence of belief update stages satisfy $k_{t+1} - k_t \geq 1$. Here, we assume that $k_{t+1} - k_t$ is finite with probability (w.p.) 1; i.e., both belief and strategy updates follow the same timescale. In Sec. 4.4, we extend our analysis to the case when belief updates occur at a slower timescale in comparison to strategy updates, i.e. $\lim_{t\to\infty} k_{t+1} - k_t = \infty$.

Fourthly, our learning dynamics considers Bayesian estimate of the unknown parameter. In Sec. 4.4, we also argue that our results hold under other types of parameter estimates such as the maximum a posteriori (MAP) estimate, and the ordinary least square (OLS) estimate.

Finally, we show in Appendix B.3 that our convergence and stability results also apply to learning the unknown parameter in games with finite action (pure strategy) set, where players choose mixed strategies. As a special case, for games with finite strategies, the linear best response dynamics (Linear-BR) with update rates $\alpha^k = \frac{1}{k}$ for all k is equivalent to fictitious play with repeatedly updated belief estimates.

4.3 Main Results

4.3.1 Convergence

Before introducing our convergence result, we introduce two necessary definitions.

Definition 4.1 (Kullback–Leibler (KL)-divergence)**.** *For a strategy profile $q \in Q$, the KL divergence between the distributions of observed payoffs y with parameters s and $s^* \in S$ is defined as:*

$$D_{KL}\left(\phi^{s^*}(y|q)||\phi^s(y|q)\right) \triangleq \begin{cases} \int_y \phi^{s^*}(y|q)\log\left(\frac{\phi^{s^*}(y|q)}{\phi^s(y|q)}\right)dy, & \text{if } \phi^{s^*}(y|q) \ll \phi^s(y|q), \\ \infty & \text{otherwise.} \end{cases}$$

Here $\phi^{s^*}(y|q) \ll \phi^s(y|q)$ means that the distribution $\phi^{s^*}(y|q)$ is absolutely continuous with respect to $\phi^s(y|q)$, i.e. $\phi^s(y|q) = 0$ implies $\phi^{s^*}(y|q) = 0$ w.p. 1.

Definition 4.2 (Payoff-equivalent parameters)**.** *A parameter $s \in S$ is payoff-equivalent to the true parameter s^* for a strategy $q \in Q$ if $D_{KL}\left(\phi^{s^*}(y|q)||\phi^s(y|q)\right) = 0$. For a given* *strategy profile $q \in Q$, the set of parameters that are payoff-equivalent to s^* is defined as:*

$$S^*(q) \triangleq \{S | D_{KL}\left(\phi^{s^*}(y|q) || \phi^s(y|q)\right) = 0\}.$$

The KL-divergence between any two distributions is non-negative, and is equal to zero if and only if the two distributions are identical. For a given strategy profile q, if a parameter s is in the payoff-equivalent parameter set $S^*(q)$, then the distributions of the observed payoffs are identical for parameters s and s^*, i.e. $\phi^{s^*}(y|q) = \phi^s(y|q)$ for all y. In this case, the observed payoffs cannot be used by the information platform to distinguish s and s^* in the belief update (θ-update) (since the belief ratio $\frac{\theta^k(s)}{\theta^k(s^*)}$ remains unchanged w.p. 1). Also note that the set $S^*(q)$ can vary with strategy profile q, and hence a payoff-equivalent parameter for a given strategy profile may not be payoff-equivalent for another strategy profile.

We need the following assumption on the strategy updates.

Assumption 1. *For any initial strategy q^1, the sequence of strategies induced by* (q-update) *under any constant belief $\theta^k = \theta \in \Delta(S)$ for all k converges to an equilibrium strategy profile in* EQ(θ).

This assumption requires that the strategy updates converge to an equilibrium strategy when the belief is held constant (instead of being repeatedly updated). Without this assumption, strategies may fail to converge even in games with complete information (Shapley [1964]). Thus, Assumption 1 is a basic requirement to guarantee the convergence of states in our learning dynamics.

Assumption 1 is satisfied by the best response dynamics (Simultaneous-BR), (Sequential-BR) and (Linear-BR) in a variety of games with complete information, including potential games, zero sum games, and dominance solvable games (Milgrom and Roberts [1990], Monderer and Shapley [1996b], Hofbauer and Sorin [2006], Fudenberg and Kreps [1993], Monderer and Shapley [1996a]). Under Assumption 1, the sequence of states (beliefs and strategies) induced by our stochastic learning dynamics converges to a fixed point.

Theorem 4.1. *For any initial state $(\theta^1, q^1) \in \Delta(S) \times Q$, under Assumption 1, the sequence of states $(\theta^k, q^k)_{k=1}^\infty$ induced by* (θ-update) *and* (q-update) *converges to a fixed point $(\bar{\theta}, \bar{q})$*

w.p. 1, and $(\bar{\theta}, \bar{q})$ *satisfies:*

$$[\bar{\theta}] \subseteq S^*(\bar{q}), \tag{4.2a}$$

$$\bar{q} \in \mathrm{EQ}(\bar{\theta}), \tag{4.2b}$$

where $[\bar{\theta}] \triangleq \{S|\bar{\theta}(s) > 0\}$, *and* $\mathrm{EQ}(\bar{\theta})$ *is the set of equilibrium strategies corresponding to belief* $\bar{\theta}$.

Moreover, for any $s \in S \setminus S^*(\bar{q})$, *if* $\phi^{s^*}(y|\bar{q}) \ll \phi^s(y|\bar{q})$, *then* $\theta^k(s)$ *converges to 0 exponentially fast:*

$$\lim_{k\to\infty} \frac{1}{k} \log(\theta^k(s)) = -D_{KL}(\phi^{s^*}(y|\bar{q})||\phi^s(y|\bar{q})), \quad w.p.\ 1. \tag{4.3}$$

Otherwise, there exists a positive integer $K^* < \infty$ *such that* $\theta^k(s) = 0$ *for all* $k > K^*$ *w.p. 1.*

From Theorem 4.1, the following properties must hold at a fixed point $(\bar{\theta}, \bar{q})$:

(1) Belief $\bar{\theta}$ identifies the true parameter s^* in the payoff-equivalent set $S^*(\bar{q})$ corresponding to fixed point strategy $\bar{q}$. Therefore, the belief forms a consistent estimate of the payoff distribution at the fixed point. To see this, let us denote the estimated distribution of the observed payoff y as $\mu(y|\bar{\theta}, \bar{q})$. Then,

$$\mu(y|\bar{\theta}, \bar{q}) \triangleq \sum_{s\in S} \bar{\theta}(s)\phi^s(y|\bar{q}) \overset{(4.2a)}{=} \sum_{s\in S^*(\bar{q})} \bar{\theta}(s)\phi^s(y|\bar{q}) = \sum_{s\in S^*(\bar{q})} \bar{\theta}(s)\phi^{s^*}(y|\bar{q}) = \phi^{s^*}(y|\bar{q}). \tag{4.4}$$

(2) Players have no incentive to deviate from fixed point strategy profile $\bar{q}$ because it is an equilibrium of the game G with fixed point belief $\bar{\theta}$.

We prove Theorem 4.1 in three steps: Firstly, we prove that the sequence of beliefs $(\theta^k)_{k=1}^\infty$ converges to a fixed point belief $\bar{\theta} \in \Delta(S)$ w.p. 1 by applying the martingale convergence theorem (Lemma 4.1). Secondly, we show that under Assumption 1, the strategies $(q^k)_{k=1}^\infty$ in our learning dynamics with belief updates also converge. This convergent strategy is an equilibrium corresponding to the belief $\bar{\theta}$ (Lemma 4.2). Finally, we prove that the belief of any

$s \in S$ that is not payoff-equivalent to s^* given $\bar{q}$ must converge to 0 with rate of convergence governed by (4.3) (Lemma 4.3). Hence, we can conclude that beliefs and strategies induced by the learning dynamics converge to a fixed point $(\bar{\theta}, \bar{q})$ that satisfies (4.2) w.p. 1. The formal proofs of Lemmas 4.1 – 4.3 are in Appendix B.

Lemma 4.1. $\lim_{k\to\infty} \theta^k = \bar{\theta}$ *w.p. 1, where* $\bar{\theta} \in \Delta(S)$.

To prove this property, we note that the subsequences of the belief ratios $\left(\frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)}\right)_{t=1}^{\infty}$ is a martingale for all $s \in S$, and $\left(\theta^{k_t}(s^*)\right)_{t=1}^{\infty}$ is a sub-martingale. From the martingale convergence theorem, $\left(\frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)}\right)_{t=1}^{\infty}$ and $\left(\theta^{k_t}(s^*)\right)_{t=1}^{\infty}$ converge w.p. 1. Thus, the belief subsequence $\left(\theta^{k_t}\right)_{t=1}^{\infty}$ converges to a fixed point belief $\bar{\theta}$ w.p. 1. Since $\theta^k = \theta^{k_t}$ for any $k = k_t, \dots, k_{t+1}-1$, the sequence $\left(\theta^k\right)_{k=1}^{\infty}$ must also converge to $\bar{\theta}$.

Lemma 4.2. *Under Assumption 1,* $\lim_{k\to\infty} q^k = \bar{q}$ *w.p. 1, where* $\bar{q}$ *satisfies* (4.2b).

In the proof of Lemma 4.2, for each stage $K = 1, 2, \dots$, we construct an auxiliary strategy sequence $\left(\hat{q}^k\right)_{k=1}^{\infty}$ such that the strategies in this sequence are identical to that in the original sequence up to a certain stage K (i.e. $\hat{q}^k = q^k$ for all $k = 1, \dots, K$), and the remaining strategies $\left(\hat{q}^k\right)_{k=K+1}^{\infty}$ are induced by the best response update with the fixed point belief $\bar{\theta}$ (instead of the repeatedly updated belief sequence $\left(\theta^k\right)_{k=K+1}^{\infty}$). Under Assumption 1, the auxiliary strategy sequence must converge to an equilibrium $\bar{q} \in \mathrm{EQ}(\bar{\theta})$. Recall from Lemma 4.1, the beliefs converge to $\bar{\theta}$. Moreover, since the expected utility function $\mathbb{E}_\theta\left[u_i^s(q)\right]$ of each player $i \in I$ is continuous in θ and q, we know from the Berge's maximum theorem that the best response correspondence $\mathrm{BR}_i(\theta, q)$ is upper hemicontinuous in θ and q (Lemma B.1 in Appendix B). Thus, we can prove that as $K \to \infty$, the distance between the auxiliary strategy sequence and the original strategy sequence converges to zero, which implies that the original strategy sequence $\left(q^k\right)_{k=1}^{\infty}$ also converges to $\bar{q} \in \mathrm{EQ}(\bar{\theta})$ (i.e. $\bar{q}$ satisfies (4.2b)).

Lemma 4.3. *Any fix point* $(\bar{\theta}, \bar{q})$ *satisfies* (4.2a). *Furthermore, for any* $s \in S \setminus S^*(\bar{q})$*, if* $\phi^{s^*}(y|\bar{q}) \ll \phi^s(y|\bar{q})$*, then* $\theta^k(s)$ *satisfies* (4.3)*. Otherwise, there exists a finite positive integer* K^* *such that* $\theta^k(s) = 0$ *for all* $k > K^*$ *w.p. 1.*

Lemma 4.3 is based on Lemmas 4.1 and 4.2. Although the data of the realized payoffs $\left(y^k\right)_{k=1}^{\infty}$ is not independently and identically distributed (i.i.d.) due to players' strategy updates, we can show that since q^k converge to $\bar{q}$ (Lemma 4.2), the distribution of y^k converges to an i.i.d. process with $\phi^s(y^k|\bar{q})$, which is the payoff distribution given the fixed point strategy $\bar{q}$, for each parameter s as $k \to \infty$.

Finally, if the payoff distribution for the true parameter is absolutely continuous with respect to any non-payoff-equivalent parameter $s \in S \setminus S^*(\bar{q})$, then we show that the log-likelihood ratio $\log\left(\frac{\theta^k(s)}{\theta^k(s^*)}\right)$ converges to $-\infty$ with an exponential rate given by the (non-zero) KL-divergence between the distributions of realized payoff under parameters s and s^*. Thus, the belief $\theta^k(s)$ converges to zero exponentially fast as in (4.3). On the other hand, if $\phi^s(y|\bar{q})$ is not absolutely continuous with the true distribution $\phi^{s^*}(y|\bar{q})$, then we can find a small neighborhood of $\bar{q}$ such that, with positive probability, the realized payoff y satisfies $\phi^s(y|q) = 0$ but $\phi^{s*}(y|q) > 0$ for q in this neighborhood. In this case, the belief update (θ-update) will assign probability 0 to the parameter s. From the Borel–Cantelli lemma, the probability that $\theta^k(s)$ remains positive infinitely often is zero. Hence, there must exist a finite stage K^*, after which $\theta^k(s)$ remains to be zero with probability 1.

Complete information fixed points. From (4.2), we define the set of fixed points Ω as follows:

$$\Omega \triangleq \left\{ \left(\bar{\theta}, \bar{q}\right) \middle| [\theta] \subseteq S^*\left(\bar{q}\right), \ \bar{q} \in \mathrm{EQ}(\bar{\theta}) \right\}. \tag{4.5}$$

We denote the belief vector θ^* with $\theta^*(s^*) = 1$ as the *complete information belief*, and any strategy $q^* \in \mathrm{EQ}(\theta^*)$ as a *complete information equilibrium*. Since $[\theta^*] = \{s^*\} \subseteq S^*(q^*)$, the state (θ^*, q^*) is always a fixed point (i.e. $(\theta^*, q^*) \in \Omega$), and has the property that all players have complete information of the true parameter s^* and choose a complete information equilibrium. Therefore, we refer to (θ^*, q^*) as a *complete information fixed point*.

Additionally, the set Ω may contain other fixed points $\left(\bar{\theta}, \bar{q}\right)$ that are not equivalent to the complete information environment, i.e. $\bar{\theta} \neq \theta^*$. Such belief $\bar{\theta}$ must assign positive probability to at least one parameter $s \neq s^*$. The property (4.2a) ensures that s is payoff-equivalent to s^* given the fixed point strategy profile $\bar{q}$, and hence the average payoff function in (4.1) satisfies $u_i^s(\bar{q}) = u_i^{s^*}(\bar{q})$ for all $i \in I$. However, for other strategies $q \neq \bar{q}$, the value of $u_i^s(q)$ may be different from $u_i^{s^*}(q)$ for one or more players $i \in I$. That is, belief $\bar{\theta}$ consistently estimates the payoff at a fixed point strategy $\bar{q}$ but not necessarily at all $q \in Q$. Consequently, if one or more players had access to complete information of the true parameter s^*, they may have an incentive to deviate from the fixed point strategy; such a fixed point strategy profile $\bar{q}$ is not a complete information equilibrium.

We next present the sufficient and necessary condition under which all fixed points are complete information fixed points. Besides, we derive a sufficient condition on the set of fixed points Ω and the average payoff functions to ensure that the strategy played in the fixed point is equivalent to a complete information equilibrium, although the fixed point belief may not be a complete information belief.

Proposition 4.1. *The fixed point set* $\Omega \equiv \{(\theta^*, q^*) \,|q^* \in \mathrm{EQ}(\theta^*)\}$ *if and only if* $[\theta] \setminus S^*(q)$ *is a non-empty set for any* $\theta \in \Delta(S) \setminus \{\theta^*\}$ *and any* $q \in \mathrm{EQ}(\theta)$.

Furthermore, for a fixed point $\left(\bar{\theta}, \bar{q}\right)$, $\bar{q} = q^*$ *if (i) There exists a positive number* $\xi > 0$ *such that* $[\bar{\theta}] \subseteq S^*(q)$ *for any* $\|q - \bar{q}\| < \xi$*; and (ii) The payoff function* $u_i^s(q_i, q_{-i})$ *is concave in* q_i *for all* $i \in I$ *and all* $s \in [\bar{\theta}]$.

Proposition 4.1 is intuitive: By definition (4.5), if all fixed points in Ω are complete information fixed points, then any imperfect information belief $\theta \in \Delta(S)$ other than the complete information belief θ^* cannot be a fixed point belief. Therefore, the set Ω is comprised of only the complete information fixed points if and only if the support set of any $\theta \in \Delta(S) \setminus \{\theta^*\}$ has at least one parameter that can be distinguished from the true parameter s^* with an equilibrium corresponding to θ.

Besides, for any fixed point $\left(\bar{\theta}, \bar{q}\right)$, since $\bar{q}_i$ is a best response strategy of $\bar{q}_{-i}$, $\bar{q}_i$ is a local maximizer of the expected payoff function $\mathbb{E}_{\bar{\theta}}[u_i^s(q_i, \bar{q}_{-i})]$. Condition (i) in Proposition 4.1 ensures that the value of the expected payoff function is identical to that with the true parameter s^* for any q_i belonging to a small neighborhood of $\bar{q}_i$. Therefore, $\bar{q}_i$ must be a local maximizer of the payoff function with the true parameter $u_i^{s^*}(q_i, \bar{q}_{-i})$. Moreover, since condition (ii) provides that payoffs are concave functions of q_i, $\bar{q}_i$ must also be a global maximizer of $u_i^{s^*}(q_i, \bar{q}_{-i})$. Thus, any fixed point strategy $\bar{q}$ is an equilibrium of the game with complete information of s^*.

Next we present three illustrative examples to further discuss the properties of fixed points in our learning dynamics.

Example 4.1 (Cournot game). A set of I firms produce an identical product and compete in a market. In each stage k, firm i's strategy is their production level $q_i^k \in [0,3]$. The price of the product is $p^k = \alpha^s - \beta^s \left(\sum_{i \in I} q_i^k\right) + \epsilon^s$, where $s = (\alpha^s, \beta^s)$ is the unknown parameter vector in the price function, and ϵ is a random variable with zero mean. The set of parameter vectors is $S = \{s_1, s_2\}$, where $s_1 = (2,1)$ and $s_2 = (4,3)$. The true parameter is $s^* = s_1$. The marginal cost of each firm is 0. Therefore, the payoff of firm i in stage k is $y_i^k = q_i^k \left(\alpha^s - \beta^s \left(\sum_{k \in K} q_i^k\right) + \epsilon^s\right)$ for each $s \in S$. The information platform updates belief θ^k based on the total production $\sum_{i \in I} q_i^k$ and the realized price p^k.

This game has a potential function, and the best response correspondence $\mathrm{BR}(q, \theta)$ is a contraction mapping for all $q \in Q$ and all $\theta \in \Delta(S)$. Thus, all three types of best response update rules satisfy Assumption 1.[4] Thus, from Theorem 4.1, the states of the learning dynamics converge to a fixed point with probability 1 with all three types of strategy updates. The complete information fixed point is $\theta^* = (1,0)$ and $q^* = (2/3, 2/3)$. Additionally, $\theta^\dagger = (0.5, 0.5)$ and $q^\dagger = (0.5, 0.5) \in \mathrm{EQ}(\theta^\dagger)$ is also a fixed point since $[\theta^\dagger] \subseteq S^*(q^\dagger) = \{s_1, s_2\}$. Thus, $\left(\theta^\dagger, q^\dagger\right)$ is another fixed point. Note that at $q^\dagger$, the two parameters s_1 and s_2 lead to identical price distribution, and thus cannot be distinguished.

In fact, since any $\theta \neq \theta^*$ must include s_2 in the support set, one can show that $q^\dagger = (0.5, 0.5)$ is the only strategy profile for which s_1 and s_2 are payoff-equivalent. Thus, there does not exist any other fixed points apart from (θ^*, q^*) and $\left(\theta^\dagger, q^\dagger\right)$; i.e. $\Omega = \left\{(\theta^*, q^*), (\theta^\dagger, q^\dagger)\right\}$.

Example 4.2 (Zero sum game). Two players $i \in \{1,2\}$ repeatedly play a zero-sum game with identical convex and closed strategy sets $Q_1 = Q_2 = [0,6]$. For any strategy profile q, the payoff of each player is $y_1 = -y_2 = v^s(q) + \epsilon^s$, where

$$v^s(q) = \left(\max\left(|q_1^k - q_2^k|, s\right) - s\right)^2 - 2(q_1^k)^2,$$

and $s \in S = \{1,3,5\}$ is the unknown parameter. The true parameter $s^* = 3$. Belief is

[4] For any $\theta \in \Delta(S)$ and any $q \in Q$, the best response strategy is $\mathrm{BR}_i(\theta, q) = \{\frac{\mathbb{E}_\theta[\alpha^s]}{2\mathbb{E}_\theta[\beta^s]} - \frac{1}{2}\sum_{j \neq i} q_j\}$, where $\mathbb{E}_\theta[\alpha^s] = \sum_{s \in S} \theta(s)\alpha^s$ and $\mathbb{E}_\theta[\beta^s] = \sum_{s \in S} \theta(s)\beta^s$. Then, for any $q, q' \in Q$, we have $\|\mathrm{BR}(\theta, q) - \mathrm{BR}(\theta, q')\| < \frac{1}{4}\|q - q'\|$, i.e. $\mathrm{BR}(\theta, q)$ is a contraction mapping. Thus, for each of the three best response dynamics, $F(\theta, q)$ is also a contraction mapping. Therefore, the sequence of strategies converges to the equilibrium strategy in $\mathrm{EQ}(\theta)$ under all three best response dynamics.

updated by an information platform based on the observed strategy and the realized payoffs.

All three types of best response update rules satisfy Assumption 1 in this zero sum game.[5] From Theorem 4.1, the learning dynamics under (Linear-BR) converges to a fixed point w.p. 1.

The set of complete information fixed points is $\theta^* = (0, 1, 0)$ and $\mathrm{EQ}(\theta^*) = \{(q_1^*, q_2^*) \,|q_1^* = 0,\ q_2^* \leq 3\}$. Apart from the complete information fixed points, any $\theta^\dagger \in \Delta(S) \setminus \{(0, 0, 1)\}$ and any $q^\dagger \in \{(\bar{q}_1, \bar{q}_2) \,|q_1^\dagger = 0,\ q_2^\dagger \leq \min\{[\theta^\dagger]\}\}$ is also a fixed point. This is because for any belief $\theta^\dagger$ that assigns positive probability on $s = 1$ or $s = 3$, $q_1^\dagger = 0$ and $q_2^\dagger$ such that $|q_2^\dagger - q_2^\dagger| \leq \min\{[\theta^\dagger]\}$ is an equilibrium corresponding to $\theta^\dagger$, and the two parameters $s = 1$ and $s = 3$ are payoff equivalent at $q^\dagger$.

Moreover, we can check that conditions (i) and (ii) in Proposition 4.1 are satisfied by any fixed point $\left(\theta^\dagger, q^\dagger\right)$. Thus, any fixed point strategy in the set $\{\left(q_1^\dagger, q_2^\dagger\right) |q_1^\dagger = 0,\ q_2^\dagger \leq \min\{[\theta^\dagger]\}\}$ is a complete information equilibrium although $\theta^\dagger$ is not a complete information belief.

Example 4.3 (Investment game)**.** Two players repeatedly play an investment game. In each stage k, the strategy $q_i^k \in [0, 1]$ is the non-negative level of investment of player i. Given the strategy profile $q^k = \left(q_1^k, q_2^k\right)$, the return of a unit investment is randomly realized according to $r^k = s + q_1^k + q_2^k + \epsilon^s$, where $s \in S = \{0, 1, 2\}$ is the unknown parameter that represents the average baseline return and ϵ^s is the noise term. The true parameter is $s^* = 1$. The stage cost of investment for each player is $3\left(q_i^k\right)^2$. Therefore, the payoff of each player $i \in I$ is $y_i^k = q_i^k(s + q_1^k + q_2^k + \epsilon^s) - 3\left(q_i^k\right)^2 = q_i^k(s - 2q_i^k + q_{-i}^k + \epsilon^s)$ for all $s \in S$. In each stage k, the information platform updates belief θ^k based on the total investment $q_1^k + q_2^k$ and the unit investment return r^k.

This game is a supermodular game, and it is also dominance solvable. All three best response dynamics satisfy Assumption 1.[6] Thus, states in learning with converge to a fixed

[5]For any $\theta \in \Delta(S)$, $q_1 = 0$ maximizes the expected utility of player 1. Thus, regardless of the sequence of player 2's strategies, the sequence of player 1's strategy converges to 0 under all three best response dynamics. Additionally, the sequence of player 2's strategies converges to a best response strategy in $\mathrm{BR}_2(\theta, 0) = \{Q_2 | q_2 \leq \min\{[\theta]\}\}$. Since $\mathrm{EQ}(\theta) = \{(q_1, q_2) \,|q_1 = 0, q_2 \leq \min\{[\theta]\}\}$, the sequence of strategies converges to an equilibrium strategy under all three best response dynamics.

[6]For any $\theta \in \Delta(S)$ and any $q \in Q$, the best response strategy is $\mathrm{BR}(\theta, q) = \{\frac{\mathbb{E}_\theta[s]+q_2}{4}, \frac{\mathbb{E}_\theta[s]+q_1}{4}\}$, where $\mathbb{E}_\theta[s] = \sum_{s \in S} \theta(s)s$. Same as Example 1, for any $q, q' \in Q$, we have $\|\mathrm{BR}(\theta, q) - \mathrm{BR}(\theta, q')\| = \frac{1}{4}\|q - q'\| <$

point with probability 1. In this game, since $S^*(q) = \{s^* = 1\}$ for any $q \in Q$, the unique fixed point is the complete information fixed point, i.e. $\Omega = \{(\theta^*, q^*) = ((0, 1, 0), (1/3, 1/3))\}$.

4.3.2 Stability

In this section, we analyze both global and local stability properties of fixed point belief $\bar{\theta}$ and the associated equilibrium set $\mathrm{EQ}(\bar{\theta})$.

Definition 4.3 (Global stability). *A fixed point belief $\bar{\theta} \in \Delta(S)$ and the associated equilibrium set $\mathrm{EQ}(\bar{\theta})$ are* globally stable *if for any initial state (θ^1, q^1), the beliefs of the learning dynamics $\left(\theta^k\right)_{k=1}^{\infty}$ converge to $\bar{\theta}$ and the strategies $\left(q^k\right)_{k=1}^{\infty}$ converge to $\mathrm{EQ}(\bar{\theta})$ with probability 1.*

Thus, global stability requires that that the convergent fixed point belief and the corresponding equilibrium set do not depend on the initial state.

We next introduce the definition of local stability. For any $\epsilon > 0$, we define an ϵ-neighborhood of belief $\bar{\theta}$ as $N_\epsilon(\bar{\theta}) \triangleq \left\{\theta \,\middle|\, \|\theta - \bar{\theta}\| < \epsilon\right\}$. For any $\delta > 0$, we define the δ-neighborhood of equilibrium set as $N_\delta(\mathrm{EQ}(\bar{\theta})) \triangleq \left\{q \,\middle|\, D\left(q, \mathrm{EQ}(\bar{\theta})\right) < \delta\right\}$, where $D\left(q, \mathrm{EQ}(\bar{\theta})\right) = \min_{q' \in \mathrm{EQ}(\bar{\theta})} \|q - q'\|$ is the Euclidean distance between q and the set $\mathrm{EQ}(\bar{\theta})$.

Definition 4.4 (Local stability). *A fixed point belief $\bar{\theta} \in \Delta(S)$ and the associated equilibrium set $\mathrm{EQ}(\bar{\theta})$ are* locally stable *if for any $\gamma \in (0, 1)$ and any $\bar{\epsilon}, \bar{\delta} > 0$, there exist $\epsilon^1, \delta^1 > 0$ such that for the learning dynamics that starts with $\theta^1 \in N_{\epsilon^1}(\bar{\theta})$ and $q^1 \in N_{\delta^1}(\mathrm{EQ}(\bar{\theta}))$, the following holds:*

$$\lim_{k \to \infty} \Pr\left(\theta^k \in N_{\bar{\epsilon}}(\bar{\theta}),\ q^k \in N_{\bar{\delta}}(\mathrm{EQ}(\bar{\theta}))\right) > \gamma. \tag{4.6}$$

Thus, local stability requires that when the learning starts with an initial state that is sufficiently close to a fixed point belief $\bar{\theta}$ and the associated equilibrium set $\mathrm{EQ}(\bar{\theta})$, then the sequence of beliefs (resp. sequence of strategies) is guaranteed to be arbitrarily close to $\bar{\theta}$ (resp. $\mathrm{EQ}(\bar{\theta})$), with arbitrarily high probability. In other words, when the belief $\bar{\theta}$ and the

$\|q - q'\|$, i.e. $\mathrm{BR}(\theta, q)$ is a contraction mapping. Thus, under any one of the three best response dynamics, $F(\theta, q)$ is also a contraction mapping, and the sequence of strategies converges to an equilibrium strategy in $\mathrm{EQ}(\theta)$.

equilibrium strategy set $\mathrm{EQ}(\bar{\theta})$ are locally stable, the learning dynamics is robust to small perturbations around $\bar{\theta}$ and $\mathrm{EQ}(\bar{\theta})$. On the other hand, if $\bar{\theta}$ and $\mathrm{EQ}(\bar{\theta})$ are locally unstable, then there exists a non-zero probability $\iota > 0$ such that the state of learning dynamics can leave the neighborhood of $\bar{\theta}$ and $\mathrm{EQ}(\bar{\theta})$ with probability at least ι even when the initial belief θ^1 (resp. strategy q^1) is arbitrarily close to $\bar{\theta}$ (resp. $\mathrm{EQ}(\bar{\theta})$).

Note that both global and local stability notions are not defined for individual fixed points, but rather for the tuple $\left(\bar{\theta}, \mathrm{EQ}(\bar{\theta})\right)$, i.e. the set of fixed points with an identical belief $\bar{\theta}$. This is important when the game has multiple equilibria; i.e., $\mathrm{EQ}(\theta)$ is not a singleton set for some belief $\theta \in \Delta(S)$. That is, our stability notions do not hinge on the convergence to a particular equilibrium in the fixed point equilibrium set $\mathrm{EQ}(\bar{\theta})$.

we provide a necessary and sufficient condition for global stability:

Proposition 4.2. *There exists a globally stable fixed point if and only if all fixed points are complete information fixed points, i.e.* $\Omega = \{(\theta^*, \mathrm{EQ}(\theta^*))\}$. *In fact, in this case, all fixed points in* $(\theta^*, \mathrm{EQ}(\theta^*))$ *are globally stable.*

This result is quite intuitive: If the set Ω contains another fixed point that is not a complete information fixed point, then whether the states of learning dynamics converge to the complete information fixed point or another fixed point depends on the initial state; hence no fixed point in the set can be globally stable. Also recall from Proposition 4.1 that all fixed points being complete information fixed points is equivalent to the condition that any parameter other than the true parameter s^* can be distinguished from s^* at the equilibrium. From Proposition 4.2, we know that this condition is also equivalent to the existence of globally stable fixed points.

To prove local stability, we assume that the following set of conditions hold:

Assumption 2. *For a fixed point belief* $\bar{\theta}$ *and the associated equilibrium set* $\mathrm{EQ}(\bar{\theta})$, $\exists \epsilon,\ \delta > 0$ *such that the neighborhoods* $N_\epsilon\left(\bar{\theta}\right)$ *and* $N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$ *satisfy*

(A2a) Local upper hemicontinuity*:* $\mathrm{EQ}(\theta)$ *is upper-hemicontinuous in* θ *for any* $\theta \in N_\epsilon\left(\bar{\theta}\right)$.

(A2b) Local invariance*: Neighborhood* $N_\delta(\mathrm{EQ}(\bar{\theta}))$ *is a locally invariant set of the best response correspondence, i.e.* $\mathrm{BR}(\theta, q) \subseteq N_\delta(\mathrm{EQ}(\bar{\theta}))$ *for any* $q \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$ *and any* $\theta \in N_\epsilon\left(\bar{\theta}\right)$.

(A2c) Local consistency*: Fixed point belief* $\bar{\theta}$ *forms a consistent payoff estimate in the local*

neighborhood $N_\delta(EQ(\bar{\theta}))$, i.e. $[\bar{\theta}] \subseteq S^*(q)$ for any $q \in N_\delta(EQ(\bar{\theta}))$.

Theorem 4.2. *A fixed point belief $\bar{\theta} \in \Delta(S)$ and the associated equilibrium set $EQ(\bar{\theta})$ is locally stable under the learning dynamics* (θ-update) *and* (q-update) *if Assumptions 1 and 2 are satisfied.*

From Theorem 4.1, we know that Assumption 1 ensures the convergence of beliefs and strategies under local perturbations. We now discuss the role of each of the three conditions in Assumption 2 towards local stability. Firstly, the local upper hemicontinuity condition *(A2a)* guarantees that the convergent equilibrium strategy remains close to the original fixed point equilibrium when the belief is locally perturbed. Secondly, the local invariance condition *(A2b)* guarantees that the strategy sequence resulting from the strategy updates remains within the local invariant neighborhood of the fixed point equilibrium. We remark that for games with complete information, local invariance reduces to the standard condition on the existence of invariant set for best response strategy updates under no parameter uncertainty, and this property is sufficient to ensure the local stability of complete information equilibrium. Hence, the conditions of local upper hemicontinuity and local invariance conditions together ensure that the strategy sequence in our learning dynamics does not leave the local neighborhood of $\mathrm{EQ}(\bar{\theta})$ so long as the perturbed beliefs remain close to $\bar{\theta}$.

Finally, the local consistency condition *(A2c)* ensures that (θ-update) keeps the beliefs close to $\bar{\theta}$. Under this condition, any parameter in the support of $\bar{\theta}$ remains to be payoff equivalent to s^* for any strategy in a local neighborhood of $\mathrm{EQ}(\bar{\theta})$. That is, $\bar{\theta}$ forms a consistent estimate of players' payoffs not just at fixed point strategy $\bar{q}$, but also when the strategy is locally perturbed around $\bar{q}$. Therefore, the Bayesian belief update keeps the beliefs of all parameters in $[\bar{\theta}]$ close to their respective probabilities in $\bar{\theta}$ when the strategies are in the local neighborhood, and eventually any parameters that are not in $[\bar{\theta}]$ are excluded by the learning dynamics.

We now detail the proof ideas of Theorem 4.2 (the formal proof is given in Appendix B). From Definition 4.4, to prove local stability, we need to characterize the local neighborhoods $N_{\epsilon^1}\left(\bar{\theta}\right)$ and $N_{\delta^1}\left(\mathrm{EQ}(\bar{\theta})\right)$ of the initial state (θ^1, q^1) such that (4.6) is satisfied. In our proof, we first show via Lemma 4.4 that (4.6) is satisfied if the sequence of states – beliefs and strategies – remain with probability higher than γ in the specifically constructed neighborhoods $N_{\hat{\epsilon}}\left(\bar{\theta}\right)$ and $N_{\delta}\left(\mathrm{EQ}(\bar{\theta})\right)$, respectively; here, $\hat{\epsilon} \in (0, \epsilon)$ and ϵ, and δ are chosen according to Assumption 2. Subsequently, in Lemmas 4.5 and 4.6, we precisely characterize the neighborhoods $N_{\epsilon^1}\left(\bar{\theta}\right)$ and $N_{\delta^1}\left(\mathrm{EQ}(\bar{\theta})\right)$ such that the sequence of beliefs and strategies starting from initial state $(\theta^1, q^1) \in (N_{\epsilon^1}(\bar{\theta}) \times N_{\delta^1}(\mathrm{EQ}(\bar{\theta}))$ remain in the respective neighborhoods $N_{\hat{\epsilon}}(\bar{\theta})$ and $N_{\delta}(\mathrm{EQ}(\bar{\theta}))$ that we specifically construct in Lemma 4.4 with probability higher than γ.

In Lemma 4.4, parts *(i)* and *(ii)* show that under Assumption *(A2a)* – *(A2b)*, the properties of local upper-hemicontinuity and local invariance hold in the neighborhoods $N_{\hat{\epsilon}}\left(\bar{\theta}\right)$ and $N_{\delta}\left(\mathrm{EQ}(\bar{\theta})\right)$. Additionally, part *(iii)* shows that if the belief sequence and strategy sequence are in respective sets $N_{\hat{\epsilon}}(\bar{\theta})$ and $N_{\delta}(EQ(\bar{\theta}))$, then the convergent state must be in $N_{\bar{\epsilon}}(\bar{\theta})$ and $N_{\bar{\delta}}\left(\mathrm{EQ}(\bar{\theta})\right)$.

Lemma 4.4. *Under Assumptions 1 and (A2a) – (A2b),*

(i) *For any* $\bar{\delta} > 0$, $\exists \epsilon' \in (0, \epsilon)$ *such that any* $\theta \in N_{\epsilon'}(\bar{\theta})$ *satisfies* $\mathrm{EQ}(\theta) \subseteq N_{\bar{\delta}}(\mathrm{EQ}(\bar{\theta}))$.

(ii) *For any* $\bar{\epsilon} > 0$, $\mathrm{BR}(\theta, q) \subseteq N_{\delta}(\mathrm{EQ}(\bar{\theta}))$ *for all* $q \in N_{\delta}(\mathrm{EQ}(\bar{\theta}))$ *and all* $\theta \in N_{\hat{\epsilon}}\left(\mathrm{EQ}(\bar{\theta})\right)$, *where* $\hat{\epsilon} = \min\{\epsilon, \epsilon', \bar{\epsilon}\}$.

(iii) $\lim_{k\to\infty} \Pr\left(\theta^k \in N_{\bar{\epsilon}}(\bar{\theta}),\ \ q^k \in N_{\bar{\delta}}(\mathrm{EQ}(\bar{\theta}))\right) \geq \Pr\left(\theta^k \in N_{\hat{\epsilon}}(\bar{\theta}),\ q^k \in N_{\delta}(\mathrm{EQ}(\bar{\theta})),\ \forall k\right)$.

In Lemma 4.4, *(i)* follows from Assumption *(A2a)* that $\mathrm{EQ}(\theta)$ is upper-hemicontinuous in θ in the local neighborhood $N_{\epsilon}\left(\bar{\theta}\right)$. Then, we obtain *(ii)* from Assumption *(A2b)* that $N_{\delta}\left(\mathrm{EQ}(\bar{\theta})\right)$ is an invariant set of the best response correspondence. Furthermore, if beliefs are in $N_{\hat{\epsilon}}(\bar{\theta})$ for all stages, then the convergent belief must also be in $N_{\hat{\epsilon}}(\bar{\theta}) \subseteq N_{\bar{\epsilon}}(\bar{\theta})$. Based on Theorem 4.1, the sequence of strategies converges. Since $N_{\hat{\epsilon}}(\bar{\theta}) \subseteq N_{\epsilon'}(\bar{\theta})$, we know from *(i)* in Lemma 4.4 that the convergent strategy is an equilibrium in $N_{\bar{\delta}}(\mathrm{EQ}(\bar{\theta}))$. Thus, *(iii)* holds.

Thanks to Lemma 4.4 *(iii)*, to prove local stability as in (4.6), it remains to be established that there exist $N_{\epsilon^1}(\bar{\theta})$ and $N_{\delta^1}\left(\mathrm{EQ}(\bar{\theta})\right)$ for the initial belief θ^1 and strategy q^1 such that $\Pr\left(\theta^k \in N_{\hat{\epsilon}}(\bar{\theta}),\ \ q^k \in N_{\delta}(\mathrm{EQ}(\bar{\theta})),\ \forall k\right) > \gamma$. In particular, $\theta^k \in N_{\hat{\epsilon}}\left(\bar{\theta}\right)$ is guaranteed if $|\theta^k(s) - \bar{\theta}(s)| \leq \frac{\hat{\epsilon}}{|S|}$ for all $s \in S$. We separately analyze the beliefs of all $s \in S \setminus [\bar{\theta}]$ (i.e. the set of parameters with zero probability in $\bar{\theta}$) in Lemma 4.5, and that of $s \in [\bar{\theta}]$ in Lemma 4.6. Additionally, parts *(a)* and *(b)* in Lemma 4.4 are useful in Lemmas 4.5 and 4.6 for constructing ϵ^1 and δ^1.

Before proceeding, we need to define the following thresholds:

$$\rho^1 \triangleq \min_{s \in [\bar{\theta}]} \left\{ \frac{(1-\gamma)\bar{\theta}(s)\hat{\epsilon}}{(1-\gamma+|S \setminus [\bar{\theta}]|)(|S \setminus [\bar{\theta}]|+1)|S| + (1-\gamma)\hat{\epsilon}} \right\}, \tag{4.7a}$$

$$\rho^2 \triangleq \frac{\hat{\epsilon}}{(|S \setminus [\bar{\theta}]|+1)|S|}, \tag{4.7b}$$

$$\rho^3 \triangleq \min_{s \in [\bar{\theta}]} \left\{ \frac{\hat{\epsilon} - |S \setminus [\bar{\theta}]||S|\rho^2\bar{\theta}(s)}{|S| - |S \setminus [\bar{\theta}]||S|\rho^2}, \ \frac{\hat{\epsilon}}{|S| + |S \setminus [\bar{\theta}]| \left(\bar{\theta}(s)|S| + \hat{\epsilon}\right)}, \ \bar{\theta}(s) \right\}. \tag{4.7c}$$

Lemma 4.5 below shows that if the initial belief θ^1 is in the neighborhood $N_{\rho^1}(\bar{\theta})$, then $\theta^k(s) \leq \rho^2$ for all $s \in S \setminus [\bar{\theta}]$ in all stages of the learning dynamics with probability higher than γ. Note that $\theta^k(s) \leq \rho^2$ ensures $|\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}$ since $\bar{\theta}(s) = 0$ for all $s \in S \setminus [\bar{\theta}]$ and $\rho^2 < \frac{\hat{\epsilon}}{|S|}$. Additionally, the threshold ρ^2 is specifically constructed to bound the beliefs of the remaining parameters in $[\bar{\theta}]$, which will be used later in Lemma 4.6.

Lemma 4.5. *For any $\gamma \in (0,1)$, if the initial belief satisfies*

$$\theta^1(s) < \rho^1, \quad \forall s \in S \setminus [\bar{\theta}], \tag{4.8a}$$

$$\bar{\theta}(s) - \rho^1 < \theta^1(s) < \bar{\theta}(s) + \rho^1, \quad \forall s \in [\bar{\theta}], \tag{4.8b}$$

then

$$\Pr\left(\theta^k(s) \leq \rho^2, \ \forall s \in S \setminus [\bar{\theta}], \ \forall k\right) > \gamma. \tag{4.9}$$

In the proof of Lemma 4.5, we say that the belief $\theta^k(s)$ completes an upcrossing of the interval $[\rho^1, \rho^2]$ if $\theta^k(s)$ increases from less than ρ^1 to higher than ρ^2. Note that if the belief of a parameter $s \in S \setminus [\bar{\theta}]$ is initially smaller than ρ^1 but later becomes higher than ρ^2 in some stage k, then the belief sequence $(\theta^j(s))_{j=1}^k$ must have completed at least one upcrossing of $[\rho^1, \rho^2]$ before stage k. Therefore, $\theta^k(s) \leq \rho^2$ for all k is equivalent to that the number of upcrossings completed by the belief is zero.

Additionally, by bounding the initial belief of parameters $s \in [\bar{\theta}]$ as in (4.8b), we construct another interval $\left[\rho^1/\left(\bar{\theta}(s^*) - \rho^1\right), \rho^2\right]$ such that the number of upcrossings with respect to this interval completed by the sequence of belief ratios $\left(\frac{\theta^k(s)}{\theta^k(s^*)}\right)_{k=1}^{\infty}$ is no less than the number of upcrossings with respect to interval $[\rho^1, \rho^2]$ completed by $\left(\theta^k(s)\right)_{k=1}^{\infty}$. Recall that the sequence of belief ratios $\left(\frac{\theta^k(s)}{\theta^k(s^*)}\right)_{k=1}^{\infty}$ forms a martingale process (Lemma 4.1). By applying Doob's upcrossing inequality, we obtain an upper bound on the expected number of upcrossings completed by the belief ratio corresponding to each parameter $s \in S \setminus [\bar{\theta}]$, which is also an upper bound on the expected number of upcrossings made by the belief of s. Using Markov's inequality and the upper bound of the expected number of upcrossings, we show that with probability higher than γ, no belief $\theta^k(s)$ of any parameter $s \in S \setminus [\bar{\theta}]$ can ever complete a single upcrossing with respect to the interval $[\rho^1, \rho^2]$ characterized by (4.7a) – (4.7b). Hence, $\theta^k(s)$ remains lower than the threshold ρ^2 for all $s \in S \setminus [\bar{\theta}]$ and all k with probability higher than γ.

Furthermore, Lemma 4.6 utilizes another set of conditions on the initial belief and strategy; these conditions ensure that the beliefs of the remaining parameters $s \in [\bar{\theta}]$ satisfy $|\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}$, and the strategy $q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$ for all k so long as $\theta^k(s) < \rho^2$ for any parameter $s \in S \setminus [\bar{\theta}]$. Recall that $\theta^k(s) < \rho^2$ for all $s \in S \setminus [\bar{\theta}]$ is satisfied with probability higher than γ under the conditions provided in Lemma 4.5.

Lemma 4.6. *Under Assumption (A2b) – (A2c), if* $|\theta^1(s) - \bar{\theta}(s)| < \rho^3$ *for all* $s \in [\bar{\theta}]$ *and* $q^1 \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$, *then*

$$\Pr\left(\begin{array}{l} |\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}, \ \forall s \in [\bar{\theta}], \ \forall k \\ \textit{and } q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right), \ \forall k \end{array} \middle| \theta^k(s) < \rho^2, \ \forall s \in S \setminus [\bar{\theta}], \ \forall k \right) = 1. \tag{4.10}$$

We prove this lemma by mathematical induction. Since $\rho^2 < \frac{\hat{\epsilon}}{|S|}$ as in (4.7b), under the condition that $\theta^k(s) < \rho^2$ for all $s \in S \setminus [\bar{\theta}]$ and all k, we know that $\theta^k(s) < \frac{\hat{\epsilon}}{|S|}$ for all $s \in S \setminus [\bar{\theta}]$ and all k. In any stage k, assume that $|\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}$ for all $s \in [\bar{\theta}]$ and $q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$. Then, $\theta^k \in N_{\hat{\epsilon}}\left(\bar{\theta}\right)$ in stage k. Additionally, under local consistency condition in Assumption *(A2c)*, $s \in [\bar{\theta}]$ remains to be payoff equivalent at q^k. Thus, we can show that the belief of the next stage must satisfy $|\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}$ for all $s \in [\bar{\theta}]$, which ensures that $\theta^{k+1} \in N_{\hat{\epsilon}}\left(\bar{\theta}\right)$. Since $\hat{\epsilon} \leq \epsilon$ as in part *(ii)* of Lemma 4.4, we know that the updated strategy q^{k+1} is in $N_{\delta}\left(\mathrm{EQ}(\bar{\theta})\right)$. Hence, we obtain (4.10) by induction.

Finally, by setting $\epsilon^1 = \min\{\rho^1, \rho^3\}$ and $\delta^1 = \delta$, where ρ^1, ρ^3 are as in (4.7a), (4.7c) and δ is given by Assumption 2, the initial state in $N_{\epsilon^1}(\bar{\theta})$ and $N_{\delta^1}\left(\mathrm{EQ}(\bar{\theta})\right)$ satisfies the conditions in Lemmas 4.5 and 4.6. Then, by combining (4.9) and (4.10), we obtain that all beliefs and strategies are in the neighborhoods $N_{\hat{\epsilon}}\left(\bar{\theta}\right)$ and $N_{\delta}\left(\mathrm{EQ}(\bar{\theta})\right)$ respectively with probability higher than γ. From (c) in Lemma 4.4, we know that $\lim_{k\to\infty} \Pr\left(\theta^k \in N_{\bar{\epsilon}}(\bar{\theta}), q^k \in N_{\bar{\delta}}(\mathrm{EQ}(\bar{\theta}))\right) \geq \gamma$. Thus, we have constructed the local neighborhoods of the initial state that satisfy (4.6), and we conclude Theorem 4.2.

We discuss the local and global stability properties of the fixed points in Examples 1 – 3.

Example 4.4 (Cournot game continued)**.** In Example 4.1, since the complete information fixed point is not the unique fixed point, no fixed point is globally stable. We now show that the complete information fixed point $\theta^* = (1, 0)$, $q^* = (2/3, 2/3)$ is locally stable. Consider $\epsilon = 1/3$ and $\delta = 1$. We can check that all three conditions in Assumption 2 are satisfied in the neighborhoods $N_{\epsilon}\left(\theta^*\right)$ and $N_{\delta}\left(q^*\right)$, and thus this fixed point is locally stable. On the other hand, the other fixed point $\theta^\dagger = (0.5, 0.5)$ and $q^\dagger = (0.5, 0.5)$ does not satisfy the local consistency condition since the two parameters s_1 and s_2 can be distinguished when the strategy is perturbed in local neighborhood of $q^\dagger$.

Example 4.5 (Zero sum game continued)**.** In Example 4.2, since the complete information fixed point is not the unique fixed point, no fixed point is globally stable. Moreover, by setting $\epsilon = 1/2$ and $\delta = 6$, we can check that all fixed points in Ω satisfy the three conditions in Assumption 2, and thus are locally stable.

Example 4.6 (Investment game continued)**.** The unique fixed point of the investment game in Example 4.3 is the complete information fixed point $(\theta^*, q^*) = \left((0, 1, 0), (1/2, 1/3)\right)$. From Proposition 4.2, the complete information fixed point is globally stable.

4.4 Extensions

In this section, we consider three types of extensions of the basic learning model introduced in Sec. 4.2: (1) Learning with two timescales; (2) Learning with maximum a posteriori probability (MAP) or ordinary least squares (OLS) estimates.

(1) Learning with two timescales. Consider the case where strategy update is at a faster timescale compared with the belief updates, i.e. $\lim_{t\to\infty} k_{t+1} - k_t = \infty$ with probability 1. Under Assumption 1, as $t \to \infty$, the strategies between two belief updates k_t and k_{t+1} converge to an equilibrium strategy profile in $\mathrm{EQ}\left(\theta^{k_t}\right)$ before the next belief update in stage k_{t+1}. Then, the updated belief $\theta^{k_{t+1}}$ forms an accurate payoff estimate given the equilibrium strategy. Our convergence result (Theorem 4.1) holds for this two timescale dynamics. The local and global stability results in Theorem 4.2 and Proposition 4.2 also hold in an analogous manner.

(2) MAP and OLS estimates. Now consider a continuous and bounded parameter set S, and that the initial belief $\theta^1(s)$ is a probability density function of s on the set S, and $\theta^1(s) > 0$ for all $s \in S$. Since the unknown parameter s is continuous, Bayesian belief update in (θ-update) at stage k_{t+1} is as follows:

$$\theta^{k_{t+1}}(s) = \frac{\theta^{k_t}(s) \prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k)}{\int_{s\in S} \theta^{k_t}(s) \prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k) ds}, \quad \forall s \in S.$$

Instead of computing the full posterior belief in each stage (which entails computing the continuous integration in the denominator of the Bayesian update), we consider learning with maximum a posteriori (MAP) estimator that maximizes the posterior belief of the unknown parameter:

$$\theta_M^{k_{t+1}}(s) = \arg\max_{s\in S} \theta^{k_{t+1}}(s) = \arg\max_{s\in S} \theta^{k_t}(s) \prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k). \tag{θ_M-update}$$

Note that if the initial belief θ^1 is a uniform distribution of all parameters, then the MAP estimate is also a maximum likelihood estimate (MLE).

Our result on convergence of state (Theorem 1) can be directly extended to this case of learning with MAP estimate. In particular, under Assumption 1, the sequence of MAP estimates converges to a payoff equivalent parameter $\bar{\theta}_M \in S^*(\bar{q})$ given the fixed point strategy profile, and the strategies converge to an equilibrium strategy $\bar{q} \in \mathrm{EQ}(\bar{\theta}_M)$ of game G with parameter $\bar{\theta}_M$.

Moreover, under Assumptions 1 and 2, we can check that if the initial belief θ^1 is in a small local neighborhood of the belief vector that assigns probability 1 to a fixed point MAP estimate $\bar{\theta}_M$ and the strategy profile is in a small local neighborhood of the equilibrium $\mathrm{EQ}(\bar{\theta}_M)$, then the convergent belief remains in a small neighborhood of the singleton belief vector so that the MAP estimate remains to be $\bar{\theta}_M$ and the equilibrium set is $\mathrm{EQ}(\bar{\theta}_M)$. Therefore, analogous to Theorem 4.2, we can conclude that $\left(\bar{\theta}_M, \mathrm{EQ}(\bar{\theta}_M)\right)$ is locally stable under conditions given by Assumptions 1 and 2.

Finally, we consider a special case, where the average payoff functions are affine in strategies:

$$y_i = (q, 1) \cdot s_i + \epsilon_i^s, \quad \forall i \in I. \tag{4.11}$$

The unknown parameter vector is $s = (s_i)_{i \in I}$, where s_i has $|q| + 1$ dimensions. The noise term ϵ_i^s is realized from a normal distribution with zero mean and finite variance.

From stage 1 to k_t, player i's realized payoff $\left(y_i^k\right)_{k=1}^{k_t}$ can be written as a linear function of the strategies $\left(q^k\right)_{k=1}^{k_t}$ in the following matrix form:

$$\underbrace{\begin{pmatrix} y_i^1 \\ y_i^2 \\ \vdots \\ y_i^{k_t} \end{pmatrix}}_{Y_i^{k_t}} = \underbrace{\begin{pmatrix} q^1, & 1 \\ q^2, & 1 \\ \vdots & \vdots \\ q^{k_t}, & 1 \end{pmatrix}}_{\tilde{Q}^{k_t}} s_i + \begin{pmatrix} \epsilon_i^1 \\ \epsilon_i^2 \\ \vdots \\ \epsilon_i^{k_t} \end{pmatrix}.$$

The OLS estimate is $\hat{s}^{k_t} = \left(\hat{s}_i^{k_t}\right)_{i \in I}$ where

$$\hat{s}_i^{k_t} = \left(\left(\tilde{Q}^{k_t}\right)' \tilde{Q}^{k_t}\right)^{-1} \left(\tilde{Q}^{k_t}\right)' Y_i^{k_t}, \quad \forall i \in I, \quad \forall k_t. \tag{$\hat{s}$ - update}$$

In learning dynamics with OLS estimate, the convergence of the OLS estimates can be viewed as a special case of learning with MAP estimator because $\hat{s}^{k_t}$ is identical to the MLE estimator $\theta_M^{k_t}$ when each player's payoff as in (4.11) is an affine function of the strategy profile plus a noise term with Normal distribution. Therefore, we obtain the same convergence result in the learning with OLS estimate as in learning with MAP estimate. That is, the OLS estimates converge to an estimate $\bar{s} \in S$ such that $u_i^{\bar{s}}(\bar{q}) = u_i^{s^*}(\bar{q})$ for all $i \in I$, and strategies converge to $\bar{q} \in \mathrm{EQ}(\bar{s})$ with probability 1. Furthermore, as we have shown in Example 1, when payoff functions are linear in players' strategies, only the complete information fixed point satisfies the locally consistency condition Assumption *(A2c)*. Thus, no other fixed point satisfies the sufficient conditions for local stability.

4.5 Applications in Routing Games

In this section, we apply our results to study the learning dynamics in which travelers repeatedly play a routing game in a transportation network.

The setup of the network and travelers' routing strategies follow from that in Chapter 2. A set of non-atomic travelers with a total demand of D repeatedly make routing decisions in a network. The set of edges in the network is E, and the set of routes is R. In each stage k, travelers' routing strategy is $q^k = \left(q_r^k\right)_{r\in R}$, where q_r^k is the demand of travelers who take route r. Given q^k, the aggregate load of each edge e is $w_e^k = \sum_{r \ni e} q_r^k$.

The distribution of cost (travel time) on edges depends on an unknown parameter $s \in S$ that represents the latent condition of the traffic network. In stage k, the cost of each edge $e \in E$ is $y_e^k = \ell_e^s(w_e^k) + \epsilon_e^s(w_e^k)$, where $\ell_e^s(w_e^k)$ is an increasing function of the edge load w_e^k, and the noise term $\epsilon_e^s(w_e^k)$ has zero mean. In each stage k, a public information platform observes the edge load vector $w^k = \left(w_e^k\right)_{e\in E}$ and the realized costs of edges that are taken by travelers. That is, the observed edge cost vector is $y^k = \left(y_e^k\right)_{e\in E^k}$, where $E^k = \{E | w_e^k > 0\}$. However, the costs are unknown on edges that are not taken by travelers.

Learning starts with an initial belief θ^1 and a routing strategy q^1. The public traffic information platform updates the belief based on the observed edge load vector w^k, and the realized cost vector y^k according to the Bayes' rule, and broadcasts the updated belief to all travelers. Then, travelers update their routing strategy in stage k as a Wardrop equilibrium $q^k \in \mathrm{EQ}(\theta^k)$ corresponding to the belief θ^k. In this equilibrium, travelers take routes with the minimum expected cost based on θ^k. Since the edge cost functions are increasing in the edge loads, the equilibrium is essentially unique in that the induced edge load vector w^k is unique for all $q^k \in \mathrm{EQ}(\theta^k)$. This strategy update can be viewed as a two-timescale learning, where individual travelers in the population change their own routing decisions at a faster timescale compared with the belief update. Therefore, the traveler population's routing strategy arrives at an equilibrium in each stage before the belief is updated by the information platform for the next stage.

Following Theorem 4.1, we know that the state of the learning dynamics – the belief θ^k and the routing strategy q^k – converge to a fixed point $\left(\bar{\theta}, \bar{q}\right)$ w.p. 1. At a fixed point, the belief $\bar{\theta}$ consistently estimates the cost distribution of edges in $\bar{E} = \{E|\bar{w}_e = \sum_{r \ni e} \bar{q}_r > 0\}$ that are taken by travelers in $\bar{q}$, and the strategy $\bar{q}$ is a Wardrop equilibrium corresponding to $\bar{\theta}$. Since $\bar{w}$ is unique for each $\bar{\theta}$, we also equivalently represent a fixed point as the tuple $\left(\bar{\theta}, \bar{w}\right)$. The fixed point belief $\bar{\theta}$ may not consistently estimates the costs on edges that are not taken by any travelers. Thus, the fixed point strategy $\bar{q}$ may not be a complete information Wardrop equilibrium.

We next provide a set of conditions, under each of which the sequence of strategies converges to a complete information Wardrop equilibrium with probability 1. We denote the edge load vector induced by a complete information equilibrium as w^*.

Proposition 4.3. *For any true parameter $s^* \in S$, the learning is complete, i.e.* $\lim_{k \to \infty} w^{k*} = w^*$ *with probability 1, if any of the following conditions is satisfied:*

(1) Fully distinguishable parameters: *For any w, any $s \in S \setminus \{s^*\}$ is distinguishable from s^* given the realized costs.*

(2) Constant free flow travel time: *For any $e \in E$ and any $s, s' \in S$, $\ell_e^s(0) = \ell_e^{s'}(0)$.*

(3) All edges are utilized: *$w_e^* > 0$ for any $e \in E$.*

Each condition ensures that travelers repeatedly use the set of edges that should be taken in complete information equilibrium. Then, travelers will eventually learn the costs on these edges, and choose routes as if they know the true cost functions.

In practice, condition *(1)* is relevant if the unknown network condition impacts the costs on all edges. Condition *(2)* requires that the unknown condition only impacts the costs when there is congestion ($w_e > 0$). For example, lane closure does not change the cost when there is no traffic, but significantly aggravates congestion due to the loss of capacity. Condition *(3)* requires that all edges are utilized regardless of the network condition. This will hold when the traffic demand is high.

The average cost of the traveler population at a fixed point with $\bar{q}$ is $C(\bar{q}) \triangleq \sum_{e \in E} \bar{w}_e \ell_e^{s^*}(\bar{w}_e)$, where $\bar{w}$ is the edge load vector induced by $\bar{q}$. Additionally, the average cost in the complete information equilibrium q^* is $C(w^*) \triangleq \sum_{e \in E} w_e^* \ell_e^{s^*}(w_e^*)$. The next proposition shows that if the network is series-parallel (i.e. the network does not have an embedded wheatstone network, see Milchtaich [2006]), then the average cost at any fixed point is no less than that in complete information equilibrium.

Proposition 4.4. *If the network is series-parallel, then* $C(\bar{w}) \geq C(w^*)$ *at any fixed point* $(\bar{\theta}, \bar{w}) \in \Omega$.

In the proof of this result, we find that the edge load at any fixed point is equivalent to the complete information equilibrium in a routing game on a subnetwork with edges in $\bar{E}$. In other words, travelers make route choice as if they only know a subset of the available edges in the original network. The result (Theorem 1) in Milchtaich [2006] shows that if the network is series-parallel, then the equilibrium average cost on any subnetwork is no less than that on the whole original network. Therefore, the equilibrium cost of any fixed point must be higher or equal to the cost of the complete information equilibrium.

Furthermore, we note that any local perturbation of the fixed point strategy would enable travelers to learn the realized costs on edges that are not taken (i.e. $E \setminus \bar{E}$), and to correct the wrong estimates of costs on those edges. Therefore, for any fixed point that is not a complete information fixed point, the belief does not satisfy Assumption *(A2c)* as presented in Sec. 4.3.2, and thus is not robust to local perturbations on the strategies. This implies that only the complete information fixed point satisfies the sufficient conditions that guarantee local stability.

Finally, we illustrate the learning dynamics in repeated routing games in the following example.

Example 4.7 (Routing game)**.** Consider the three-edge series-parallel network with $S = \{e_1, e_2, e_3, \emptyset\}$, where $s = e$ represents that edge e is compromised, and $s = \emptyset$ represents that no edge is compromised. The cost of edge e is $\ell_e(w_e)$ if $s \neq e$, and $\ell_e^{\otimes}(w_e)$ if $s = e$. See Fig. 4-2 for the network and cost functions. In this example, we assume that the noise term in each stage is $\epsilon^k = \left(\epsilon_e^k\right)_{e \in E} \sim \mathcal{N}(0, \Sigma)$, where Σ is a three-dimensional identity matrix. The total demand is 1.

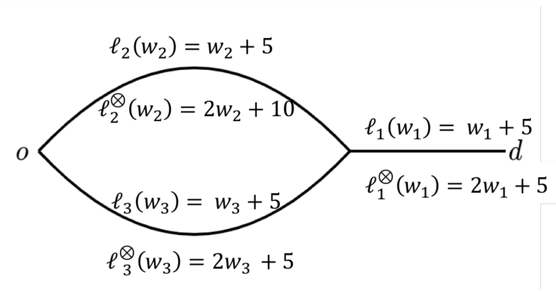


Figure 4-2: Three-edge network

Let the true parameter be $s^* = \emptyset$. The set of fixed points is as follows:

$$\Omega = \left\{ \begin{array}{l} \theta^* = (0, 0, 0, 1), \\ w^* = (1, 0.5, 0.5) \end{array} \right\} \cup \left\{ \begin{array}{l} \theta^\dagger = (0, x, 0, 1-x) \; s.t. \; x \geq 0.2, \\ w^\dagger = (1, 0, 1) \end{array} \right\}$$

That is, apart from the complete equilibrium edge load w^*, travelers may exclusively choose the route e_3 - e_1 if they believe that the probability of $s = e_2$ is no less than 0.2. In this case, the public information platform cannot distinguish parameter $s = e_2$ from the true parameter $s^* = \emptyset$ based on the realized costs. We can check that $C(w^*) = 11.5 < C(w^\dagger) = 12$.

We simulate the learning dynamics with the initial belief $\theta^1 = (1/4, 1/4, 1/4, 1/4)$. Figures 4-3a - 4-3b demonstrate the belief θ^k and the edge load w^k in each stage for a realized dynamics that converges to (θ^*, w^*), i.e. learning converges to a complete information equilibrium. Figures 4-3c - 4-3d illustrate a realized dynamics that converges to $(\theta^\dagger, w^\dagger)$, where $\theta^\dagger = (0, 1/2, 0, 1/2)$ and $w^\dagger = (1, 0, 1)$, i.e. travelers exclusively take e_3-e_1, and do not learn the true parameter.

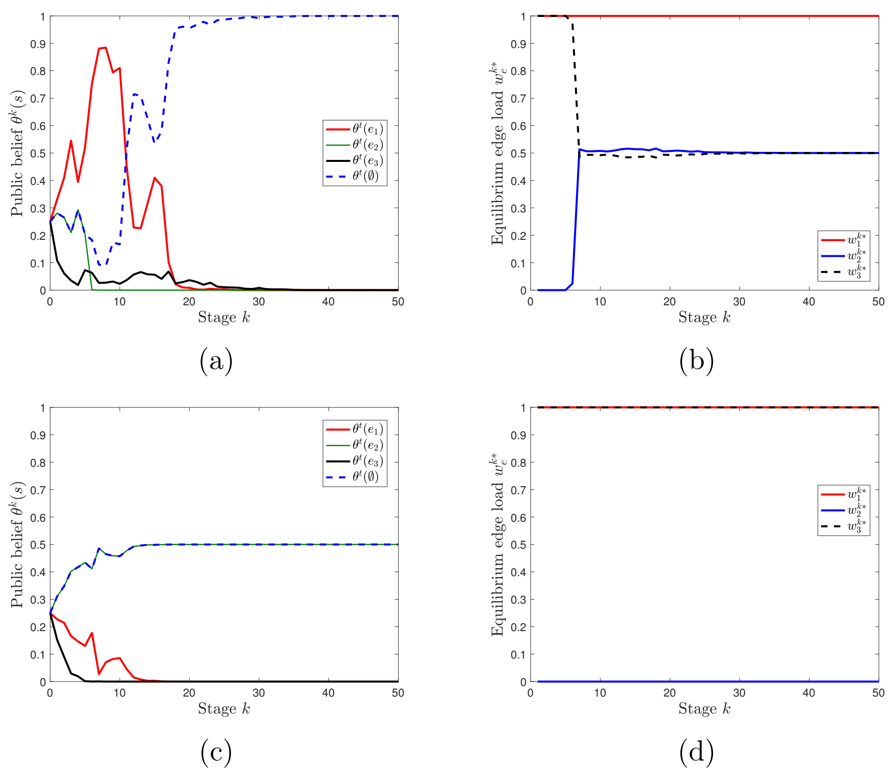


Figure 4-3: Beliefs and edge load vectors in learning dynamics: (a) - (b) Learning converges to a complete information fixed point (θ^*, w^*); (c) - (d) Learning converges to another fixed point $(\theta^\dagger, w^\dagger)$.

4.6 Discussion

In this chapter, we study stochastic learning dynamics induced by a set of strategic players who repeatedly play a game with an unknown parameter. We analyze the convergence of beliefs and strategies induced by the stochastic dynamics, and derived conditions for local and global stability of fixed points. We also apply the learning dynamics to study how strategic travelers learn the uncertain network condition after infrastructure disruptions and adjust their route choices with the access of a public information platform. We compare the average cost at any fixed point routing strategy with that of a complete information equilibrium, and provide conditions that guarantee complete learning in traffic networks.

A future research question of interest is to analyze the learning dynamics when players seek to efficiently learn the true parameter by choosing off-equilibrium strategies. When there are one or more parameters that are payoff equivalent to the true parameter at fixed point, complete learning requires players to take strategies that may reduce their individual payoffs in some stages. In our setup, if a player were to chooses a non-equilibrium strategy, the information resulting from that player's realized payoff would be incorporated into the belief update, and the new belief is known to all players. Under what scenarios the utility-maximizing players will choose their strategies to engage such explorative behavior is an interesting question, and worthy of further investigation.

Another promising extension is to study multi-agent reinforcement learning problem from a Bayesian viewpoint. In such settings, the unknown parameter changes over time according to a Markovian transition process, and players may have imperfect or no knowledge of the underlying transition kernel. The ideas presented in this article are useful to analyze how players learn the belief estimates of payoffs that depend on the latent Markov state, and adaptively adjust their strategies that converges to an equilibrium.

Chapter 5

Efficient Carpooling and Toll Pricing for Autonomous Transportation

5.1 Introduction

Autonomous transportation has the potential to significantly transform urban mobility when the technology becomes mature enough for real-world deployment. A fleet of driverless cars could be utilized to organize carpooled trips at a much cheaper price and in a more flexible manner relative to the current mobility services that rely on human drivers. Whether autonomous driving technology will relieve or aggravate congestion crucially depends on how this technology would reshape the riders' incentives to make trips and share cars (i.e. carpooling). Thus, to fully exploit the potential of self-driving cars, we need to incentivize efficient usage of limited road capacity by leveraging the flexibility of carpooling.

In Chapters 2 – 4, we have studied the impact of information on travelers' routing behavior in both static and dynamic settings. We have also designed information structure for platforms to induce socially desirable travel patterns. In this chapter, we explore the role of incentive mechanisms on shaping travel behavior. Particularly, we seek to set tolls on transportation networks that incentivize travelers (riders) to take carpooled trips and split costs.

Toll pricing has been adopted worldwide as an effective way to manage traffic demand and mitigate congestion in urban transportation networks (Santora [2017], and Arnold et al.

[2010]). Previous work has demonstrated that properly designed toll prices can significantly reduce congestion, or even induce socially optimal route choices (Pigou [1932], and de Palma and Lindsey [2011]). This chapter shows that toll prices can also govern riders' incentives of participating carpooled trips.[1]

Ostrovsky and Schwarz [2019] have introduced a competitive market model to address the complementarity between efficient carpooling and optimal tolling. In this model, the transportation authority sets toll prices on edges with limited capacities. Riders with heterogeneous preferences organize carpooled trips and make payments to split the toll prices and trip costs. A market outcome is defined by organized trips, riders' payments, and edge toll prices.

Building on Ostrovsky and Schwarz [2019], we consider that riders' heterogeneous preferences are represented by their valuations of carpooled trips that depend on the travel time of the chosen route in network, and rider-specific parameters which capture their value of time and carpool disutilities. Equilibrium of this carpooling market is defined as an outcome in which no rider has an incentive to deviate from the organized trips or opt-out, and riders' payments cover the toll prices plus the trip costs. Additionally, trip organization at equilibrium, if exists, ensures social efficiency (i.e., maximum social welfare). However, such an equilibrium may not exist in general due to the integral nature of carpooled trips on general networks and riders' incentive constraints. Thus, the question of equilibrium existence becomes central to the implementation of the autonomous carpooling market.

In this chapter, we identify sufficient conditions that guarantee the existence of market equilibrium. Note that market equilibrium in our setting is challenging to analyze because both trip organization and toll pricing are crucially influenced by the network topology. Particularly, the trip organization is essentially a coalition formation problem on the network, where any trip on a certain route consumes a unit capacity of all edges in that route. Additionally, the toll price on a single edge can impact the coalition formation of all trips that use routes going through that edge. Therefore, the classical methods in mechanism design

[1] For example, when toll prices are zero on all roads, all riders will choose to take solo trips on the shortest route in the network, and the traffic load will exceed the capacity. As the toll prices of edges on this route increase, riders will be incentivized to take carpooled trips in order to split the toll prices (or switch to longer routes).

and coalition games cannot be readily applied to address the impact of network topology. We develop a new approach to analyze market equilibrium by extending the results in market design (Kelso Jr and Crawford [1982], Gul and Stacchetti [1999], De Vries and Vohra [2003], Leme [2017]) and network flow optimization (Dantzig and Fulkerson [2003]). Indeed, we find that the equilibrium existence requires certain restrictions on network topology.

Another important question of organizing this carpooling market is efficient implementation of market equilibrium. We consider the situation, where the market is facilitated by a neural mobility platform that collects riders' reported preferences, organizes trips and charge payments (Fig. 5-1). We identify a strategyproof market equilibrium under which riders truthfully report their preferences so that this equilibrium can be implemented by the platform. We find that in this equilibrium, riders' payments are equal to their externalities on other riders, and hence are equivalent to the payments in the classical Vickery-Clark-Grove mechanism. This equilibrium also has the advantage of achieving the highest rider utilities among all market equilibria, and only collecting the minimum total toll prices.

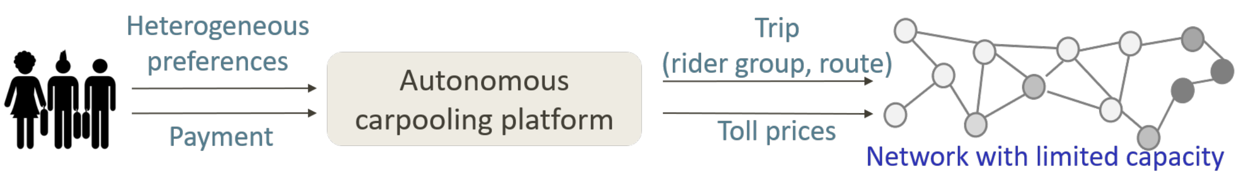


Figure 5-1: Platforms organize autonomous carpooling trips on networks with limited capacity.

Our model and results contribute to the growing literature on autonomous vehicle market design and competition. The paper Siddiq and Taylor [2019] studied the impact of competition between two ride-hailing platforms on their choices of autonomous vehicle fleet sizes, prices and wages of human drivers. The authors of Lian and van Ryzin [2020] studied the prices in ride-hailing markets, where an uncertain aggregate demand is served by a fixed fleet of autonomous vehicles and elastic supply of human drivers. They argue that the only design that unambiguously reduces the service prices corresponds to the setting when the provision of autonomous carpooled trips occurs in a competitive environment. This finding aligns well with our focus on a competitive autonomous carpooling market. We show that by exploiting the complementarity between carpooling and road pricing, we can achieve an equilibrium outcome that is socially optimal (when sufficient conditions for equilibrium existence are satisfied).

This chapter is organized as follows: Sec. 5.2 describes the model of carpooling market. Sec. 5.3 shows the primal and dual formulation of the optimal trip organization problem. We introduce the sufficient conditions for equilibrium existence in Sec. 5.4, and presents an algorithm for computing the equilibrium in Sec. 5.5. Sec. 5.6 identifies a strategyproof market equilibrium that can be implemented by platform. We conclude this chapter in Sec. 5.7. All proofs are included in Appendix C.

5.2 A Market Model

5.2.1 Network, Riders, and Trips

Consider a traffic network modeled as a directed graph with a single origin-destination pair. The set of edges in the network is E, and the capacity of each edge $e \in E$ is a positive integer $q_e \in \mathbb{N}_+$. The set of routes is R, where each route $r \in R$ is a sequence of edges that form a directed path from the origin to the destination. We denote the travel time of each edge e as $t_e > 0$, and the travel time of each route r as $t_r = \sum_{e \in r} t_e$.[2]

A finite set of riders $m = 1, \ldots, M$ want to take autonomous carpool trips to travel from the origin to the destination. A *trip* is defined as a tuple (b, r), where b is the group of riders taking route r during the trip.[3] The maximum number of riders in any group must be below the capacity of individual car, denoted A.[4] Thus, the set of rider groups is $B \triangleq \left\{2^M \mid |b| < A\right\}$, and the set of trips is $(b, r) \in B \times R$. If the group b in a trip (b, r) is a singleton set $\{m\}$, then rider m takes a solo trip on route r. Otherwise riders in b share a pooled trip. Each trip (b, r) occupies a unit capacity for all edges in route r.

[2]Thus, in our setting, each edge has an L-shaped cost function: cost is a constant when the edge load is below the edge capacity, and becomes extremely high once the load exceeds capacity. In the context of traffic congestion: when the traffic load is below the road capacity, all vehicles pass through the segment at the free-flow speed. However, when the traffic load exceeds the capacity, the travel time significantly increases due to congestion. In our market mechanism, the toll prices are set to ensure that the load of each edge does not exceed its capacity.

[3]All individuals in the set b of an autonomous carpool trip are riders. On the other hand, in human-driven carpool trips, we need to designate a driver in the set b, and match riders with drivers.

[4]For simplicity, we assume that cars are of homogeneous capacity.

The value of each trip (b, r) for a rider $m \in b$, denoted as $v_r^m(b)$, is given by:

$$v_r^m(b) = \alpha^m - \beta^m t_r - \pi^m(|b|) - \gamma^m(|b|)t_r, \quad \forall b \in \{B | b \ni m\}, \quad \forall m \in M, \quad \forall r \in R. \quad (5.1)$$

Riders' trip values equal to the value of arriving at the destination α^m nets the cost of trip time $\beta^m t_r$ and the carpool disutility $\pi^m(|b|) + \gamma^m(|b|)t_r$. In particular, the parameter α^m is rider m's value of arriving at the destination, β^m is rider m's value of time, $\pi^m(|b|)$ is rider m's fixed disutility of carpooling with rider group of size $|b|$, and $\gamma^m(|b|)$ is the disutility of sharing the vehicle with group size $|b|$ for a unit travel time. We define $(\alpha^m, \beta^m, \pi^m, \gamma^m)$ as each rider m's preference parameters, where $\pi^m = (\pi^m(d))_{d=1}^A$ and $\gamma^m = (\gamma^m(d))_{d=1}^A$.

The carpool disutility $\pi^m(|b|)+\gamma^m(|b|)t_r$ represents the rider m's inconvenience of sharing the vehicle with other riders in the carpool group, potentially due to the need to share space with others and time spent on taking detours and walking to pick-up location. Both parameters $\pi^m(|b|)$ and $\gamma^m(|b|)$ only depend on the group sizes rather than the identity of riders in the group, riders' values are identical for any two trips (b, r) and (b', r) with the same group sizes (i.e. $|b| = |b'|$) and the same route r. We consider that the carpool disutility parameters $\pi^m(|b|), \gamma^m(|b|) \geq 0$ for all $|b| = 1, \ldots, A$, and the disutility of solo trip is zero, i.e. $\pi^m(1), \gamma^m(1) = 0$ for all $m \in M$. Thus, all riders prefer to take solo trips rather than pooling with other riders, and the carpool disutility increases in the trip travel time. Additionally, the marginal disutilities $\pi^m(|b| + 1) - \pi^m(|b|)$ and $\gamma^m(|b| + 1) - \gamma^m(|b|)$ are non-decreasing in the group size $|b|$ for all $|b| = 1, \ldots, A - 1$, i.e. the extra carpool disutility of adding one rider to any trip (b, r) is non-decreasing in the original trip size $|b|$.

The cost of each trip includes the fuel charge and the cost of car's wear and tear. We simply assume that the cost of each trip $(b, r) \in B \times R$ equals to the cost of driving one rider $\sigma + \delta t_r$ multiplied with the rider group size $|b|$, i.e. $c_r(b) = (\sigma + \delta t_r)\,|b|$, and the cost parameters $\sigma, \delta \geq 0$.

The social value of each trip (b, r) is the summation of the trip values for riders in b nets the cost of trip:

$$V_r(b) = \sum_{m \in b} v_r^m(b) - c_r(b)$$

$$= \sum_{m \in b} (\alpha^m - \beta^m t_r) - \sum_{m \in b} (\pi^m + \gamma^m(|b|)t_r) - (\sigma + \delta t_r)\,|b|, \quad \forall b \in B,\ \forall r \in R. \tag{5.2}$$

5.2.2 Market Equilibrium

We now discuss how an efficient autonomous carpooling market can be organized. A transportation authority sets non-negative toll prices $\tau = (\tau_e)_{e \in E} \in \mathbb{R}^{|E|}_{\geq 0}$ on edges in the network, where τ_e is the toll price of edge e. Riders form carpool trips. The trip vector is a binary vector $x = (x_r(b))_{r \in R, b \in B} \in \{0,1\}^{|B| \times |R|}$, where $x_r(b) = 1$ if trip (b, r) is organized and $x_r(b) = 0$ if otherwise. A trip vector x must satisfy the following feasibility constraints:

$$\sum_{r \in R} \sum_{b \ni m} x_r(b) \leq 1, \quad \forall m \in M, \tag{5.3a}$$

$$\sum_{r \ni e} \sum_{b \in B} x_r(b) \leq q_e, \quad \forall e \in E, \tag{5.3b}$$

$$x_r(b) \in \{0,1\}, \quad \forall b \in B, \quad \forall r \in R, \tag{5.3c}$$

where (5.3a) ensures that no rider takes more than 1 trip, and (5.3b) ensures that the total number of trips that use any edge $e \in E$ does not exceed the edge capacity.

Additionally, each rider $m \in M$ makes a payment p^m for covering the cost of their trip and the toll prices of the taken edges. The payment vector is $p = (p^m)_{m \in M}$.

An outcome of the carpooling market is represented by the tuple (x, p, τ). Given any (x, p, τ), the utility of each rider $m \in M$ equals to the value of the trip that m takes minus the payment:

$$u^m = \sum_{r \in R} \sum_{b \ni m} v_r^m(b) x_r(b) - p^m, \quad \forall m \in M. \tag{5.4}$$

We next define four properties of the market outcomes, namely *individual rationality*, *stability*, *budget balance*, and *market clearing*. Firstly, an outcome (x, p, τ) is *individually rational* if riders' utilities are non-negative:

$$u^m \geq 0, \quad \forall m \in M. \tag{5.5}$$

That is, no rider has an incentive to opt-out of the market.

Secondly, an outcome (x, p, τ) is *stable* if there is no rider group in B that can gain higher utilities by organizing trips that are not included in x. Note that the total utility of all riders in any group b for organizing a trip (b, r) cannot exceed the value of the trip minus the toll price for route r, i.e. $V_r(b) - \sum_{e \in r} \tau_e$. Thus, a stable market outcome (x, p, τ) requires that the total utilities of riders in b obtained using (5.4) is higher or equal to the total utility that can be obtained from *any* feasible trip (b, r):[5]

$$\sum_{m \in b} u^m \geq V_r(b) - \sum_{e \in r} \tau_e, \quad \forall b \in B, \quad \forall r \in R. \tag{5.6}$$

Thirdly, an outcome (x, p, τ) is *budget balanced* if the total payments of each organized trip is equal to the sum of the toll prices and the cost of the trip; and moreover a rider's payment is zero if they are not part of any organized trip, i.e.

$$x_r(b) = 1, \quad \Rightarrow \quad \sum_{m \in b} p^m = \sum_{e \in r} \tau_e + c_r(b), \quad \forall b \in B, \quad \forall r \in R, \tag{5.7a}$$

$$x_r(b) = 0, \quad \forall r \in R, \quad \forall b \ni m, \quad \Rightarrow \quad p^m = 0, \quad \forall m \in M. \tag{5.7b}$$

Fourthly, an outcome (x, p, τ) is *market-clearing* if there are zero tolls on all edges whose capacity limits are not met:

$$\sum_{r \ni e} \sum_{b \in B} x_r(b) < q_e, \quad \Rightarrow \quad \tau_e = 0, \quad \forall e \in E. \tag{5.8}$$

We define market equilibrium as an outcome that satisfies all four properties:

Definition 5.1. *A market outcome (x^*, p^*, τ^*) is an equilibrium if it is individually rational, stable, budget balanced and market clearing.*

The autonomous carpooling market assumes a competitive environment in that riders are free to join any trip and occupies a unit capacity on any route as long as their total payments cover the trip cost and toll prices. From an implementation viewpoint, the process

[5] A stable market outcome (x, p, τ) is Pareto optimal in that no rider's utility can be improved by organizing different trips that are not in x without decreasing the utilities of other riders.

of trip organization and payment can be facilitated by introducing a market platform.[6] to the platform, and the platform assigns riders to trips according to the trip vector x^*. Then, riders make payments according to p^* to the platform, and the platform pays for the toll prices τ^* and trip costs on the riders' behalf. When the vector (x^*, p^*, τ^*) is a market equilibrium, riders follow the trip assigned by the platform, the payments cover the toll prices and trip costs, and toll prices are non-zero only on edges where the load meets the capacity. [7]

In paper Ostrovsky and Schwarz [2019], the authors argued that such a transportation market can be mapped into a standard competitive market, where the market equilibrium defined in Definition 5.1 is equivalent to the standard concept of competitive equilibrium. The key issue that we seek to investigate is that market equilibrium may not exist since the edge capacities and riders are indivisible. On the other hand, if an equilibrium (x^*, p^*, τ^*) exists, from the first welfare theorem, we conclude that the trip vector x^* necessarily maximizes the total social welfare (Theorem 1 in Ostrovsky and Schwarz [2019]); i.e., x^* is an optimal solution of the following optimal trip organization problem:

$$\begin{aligned} \max_x \quad & S(x) = \sum_{b \in B} \sum_{r \in R} V_r(b) x_r(b) \\ s.t. \quad & x \text{ satisfies (5.3a) – (5.3c)}, \end{aligned} \tag{IP}$$

where $S(x)$ is the social welfare of all trips given by x.

5.3 Primal and Dual Formulations

In this section, we show that there exists a market equilibrium if and only if the linear relaxation of the optimal trip organization problem (IP) has integer optimal solutions. We also show that the equilibrium outcomes can be derived from the optimal solutions from the linear relaxation and its dual program.[8]

[6]For simplicity, we assume that this platform is a simple non-strategic market mediator and does not charge a fee for organizing trips. However, a non-negative constant fee can be added to the model without changing the results. In such an implementation, each rider $m \in M$ reports their preference parameters $(\alpha^m, \beta^m, \pi^m, \gamma^m)$

[7]The computed market equilibrium depends on the reported preference parameters (α, β, γ). For simplicity, we drop the dependence of (x^*, p^*, τ^*) with respect to these parameters in notation.

[8]All results in this section hold for arbitrary trip values $V = (V_r(b))_{b \in B, r \in R}$.

We first introduce the linear relaxation of (IP) and its dual formulation. The primal linear program is as follows:

$$\begin{aligned}
\max_{x} \quad & S(x) = \sum_{b \in B} \sum_{r \in R} V_r(b) x_r(b), \\
s.t. \quad & \sum_{r \in R} \sum_{b \ni m} x_r(b) \leq 1, \quad \forall m \in M, && \text{(LP.a)} \\
& \sum_{r \ni e} \sum_{b \in B} x_r(b) \leq q_e, \quad \forall e \in E, && \text{(LP.b)} \\
& x_r(b) \geq 0, \quad \forall b \in B, \quad \forall r \in R. && \text{(LP.c)}
\end{aligned}$$

Note that the constraint $x_r(b) \leq 1$ is implicitly included in (LP.a), so it is omitted.

By introducing dual variables $u = (u^m)_{m \in M}$ for constraints (LP.a) and $\tau = (\tau_e)_{e \in E}$ for constraints (LP.b), the dual program of (LP) can be written as follows:

$$\begin{aligned}
\min_{u,\tau} \quad & U(u,\tau) = \sum_{m \in M} u^m + \sum_{e \in E} q_e \tau_e \\
s.t. \quad & \sum_{m \in b} u^m + \sum_{e \in r} \tau_e \geq V_r(b), \quad \forall b \in B, \quad \forall r \in R, && \text{(D.a)} \\
& u^m \geq 0, \quad \tau_e \geq 0, \quad \forall m \in M, \quad \forall e \in E. && \text{(D.b)}
\end{aligned}$$

Theorem 5.1. *A market equilibrium (x^*, p^*, τ^*) exists if and only if* (LP) *has an optimal integer solution. Any optimal integer solution x^* of* (LP) *is an equilibrium trip vector, and any optimal solution (u^*, τ^*) of* (D) *is an equilibrium utility vector and an equilibrium toll vector. The equilibrium price vector p^* is given by:*

$$p^{m*} = \sum_{r \in R} \sum_{b \ni m} x_r^*(b) v_r^m(b) - u^m, \quad \forall m \in M. \tag{5.11}$$

Thus, the question of existence of market equilibrium is equivalent to resolving whether there exists an integer optimal solution for the LP relaxation of the optimal trip problem. This result follows from the fact that the four properties of market equilibrium, namely individual rationality, stability, budget balance, and market clearing are equivalent to the constraints of (LP) and (D), and the complementary slackness conditions. From strong duality, a market equilibrium exists if and only if the optimality gap between the linear relaxation (LP) and the integer problem (IP) is zero. Hence, the linear relaxation (LP) must have an integer optimal solution, which is the equilibrium trip vector x^*.

Theorem 5.1 turns the problem of finding sufficient conditions on the existence of market equilibrium to finding conditions under which (LP) has optimal integer solutions. Moreover, it enables us to compute market equilibrium as optimal solutions of (LP) and (D).

As a consequence, we obtain that the total toll prices of shorter routes (routes with lower travel time) must be no less than that of the longer ones (routes with higher travel time).

Corollary 5.1. *In any market equilibrium* (x^*, p^*, τ^*)*, for any* $r, r' \in R$ *such that* $t_r \geq t_{r'}$*,* $\sum_{e \in r} \tau_e^* \leq \sum_{e \in r'} \tau_e^*$*.*

This result is intuitive since for all rider groups, taking a shorter route results in a higher trip value than taking a longer route. Therefore, the toll price (which is charged per unit capacity) of shorter routes must be no less than that of longer routes.

5.4 Existence of Market Equilibrium

We characterize the sufficient conditions on network topology and trip values under which the there exists a market equilibrium. We first present an example when market equilibrium does not exist on a wheatstone network.

Example 5.1. Consider the wheatstone network as in Fig. 5-2. The capacity of each edge in the set $\{e_1, e_2, e_3, e_4\}$ is 1, and the capacity of edge e_5 is 4. The travel time of each edge is given by $t_1 = 1$, $t_2 = 3$, $t_3 = 3$, $t_4 = 1$, and $t_5 = 0$.

The maximum capacity of vehicle is $A = 2$. Three riders $m = 1, 2, 3$ travel on this network. Riders have identical reference parameters: value of trip $\alpha^m = 7$, value of time $\beta^m = 1$, zero carpool disutility, i.e. $\pi^m(d) = 0$ and $\gamma^m(d) = 0$ for any $d = 1, 2$ and any $m \in M$, and zero trip cost parameters, i.e. $\sigma = 0$, $\delta = 0$.

We define the route e_1-e_2 as r_1, e_1-e_5-e_4 as r_2, and e_3-e_4 as r_3. Then, trip values are: $V_1(m) = V_3(m) = 3$, and $V_2(m) = 5$ for all $m \in M$; $V_1(m, m') = V_3(m, m') = 6$, and $V_2(m, m') = 10$ for all $m, m' \in M$. The unique optimal solution of the linear program (LP) on this network is $x_1^*(1,2) = x_2^*(2,3) = x_3^*(1,3) = 0.5$, and $S(x^*) = 11$. That is, (LP) does not have an integer optimal solution, and market equilibrium does not exist (Theorem 5.1).

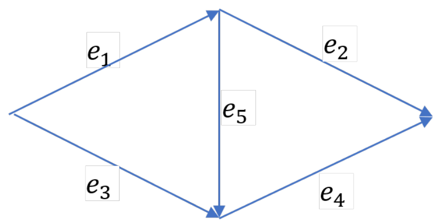


Figure 5-2: Wheatstone network

We define a network to be series-parallel if a Wheatstone structure as in Example 5.1 is not embedded.

Definition 5.2 (Series-Parallel (SP) Network Milchtaich [2006])**.** *A network is series-parallel if there do not exist two routes that pass through an edge in opposite directions. Equivalently, a network is series-parallel if and only if it is constructed by connecting two series-parallel networks either in series or in parallel for finitely many iterations.*

Our next theorem shows that market equilibrium is guaranteed to exist if the network is series-parallel (i.e. the Wheatstone structure is not embedded) and riders have homogeneous carpool disutilities.

Theorem 5.2. *Market equilibrium (x^*, p^*, τ^*) exists if the network is series-parallel and all riders have identical carpool disutility parameters, i.e.*

$$\pi^m(d) = \pi(d), \quad \gamma^m(d) = \gamma(d), \quad \forall d = 1, \dots, A, \quad \forall m \in M. \tag{5.12}$$

Theorem 5.2 shows that the sufficient conditions for equilibrium existence include (i) network is series-parallel and (ii) riders have homogeneous levels of carpool disutility. We emphasize that condition (ii) does not exclude heterogeneity in riders' trip values since they can still have different trip values and values of time. These two conditions provide useful guidelines for designers of the autonomous transportation systems. Particularly, condition (i) suggests that the transportation authority should avoid embedding a Wheatstone network structure in designating a subset of routes for the autonomous carpooling service. On the other hand, condition (ii) implies that rider groups that can be differentiated due to different levels of carpool disutilities should be allocated to differently sized autonomous cars. These guidelines ensure that the socially optimal trips can be organized through this market mechanism.

Our proof of Theorem 5.2 has three parts: Firstly, we compute an integer route capacity vector $k^* = (k^*_r)_{r \in R}$, where $R^* \triangleq \{R | k^*_r > 0\}$ is the set of routes that are assigned positive capacity and k^*_r is the integer capacity of each route r. We show that when the network is series-parallel, any optimal trip vector for the sub-network with routes R^* and capacity vector k^* is also an optimal trip vector for the original network (Lemma 5.1).

Secondly, we argue that mathematically the problem of trip organization on the sub-network with capacity vector k^* can be viewed as a problem of allocating goods in an economy with indivisible goods, and the existence of integer optimal solution is equivalent to the existence of Walrasian equilibrium in the economy (Lemmas 5.2 and 5.3).

Thirdly, we show that when riders have homogeneous carpool disutility parameters, the trip value functions satisfy gross substitutes condition. This condition is sufficient to ensure the existence of Walrasian equilibrium in the equivalent economy (Lemmas 5.4 and 5.6).

These three parts ensure that the trip organization problem on the sub-network with capacity vector k^* has an integer optimal solution, and this solution is also an integer optimal solution of (LP). From Theorem 5.1, we know that the existence of market equilibrium is equivalent to the existence of integer optimal solution in (LP). Therefore, we can conclude that a market equilibrium exists.

The rest of this section presents the lemmas corresponding to each of the three parts. The proofs of these lemmas are included in Appendix C.1.

Part 1. We first compute the route capacity vector k^* by a greedy algorithm (Algorithm 1). The algorithm begins with finding a shortest route of the network r_{min}, and sets its capacity as $k^*_{r_{min}} = \min_{e \in r_{min}} q_e$, which is the maximum possible capacity that can be allocated to r_{min}. After allocating the capacity $k^*_{r_{min}}$ to route r_{min}, the residual capacity of each edge on r_{min} is reduced by $k^*_{r_{min}}$. We then repeat the process of finding the next shortest route and allocating the maximum possible capacity to that route until there exists no route with positive residual capacity in the network.

Note that in each step of Algorithm 1, the capacity of at least one edge is fully allocated to the route that is chosen in that step. Therefore, the algorithm must terminate in less than $|E|$ number of steps. The algorithm returns the capacity vector k^*, where $R^* = \{R|k_r^* > 0\}$ is the set of routes allocated with positive capacity, and the capacity of each $r \in R^*$ is k_r^*. The remaining routes in $R \setminus R^*$ are set with zero capacity. Since the network is series-parallel, the total capacity given by the output of the greedy algorithm equals to the network capacity C (Bein et al. [1985]), i.e. $\sum_{r \in R^*} k_r^* = C$.

Moreover, the shortest path of the network in each step can be computed by Dijkstra's algorithm with time complexity of $O(|N|^2)$, where $|N|$ is the number of nodes in the network. Therefore, Algorithm 1 has time complexity of $O(|N|^2|E|)$.

ALGORITHM 1: Greedy algorithm for computing route capacity

Initialize: Set $\tilde{k}_e \leftarrow q_e, \ \forall e \in E$; $k_r \leftarrow 0, \ \forall r \in R$; $\tilde{E} \leftarrow E$;
$(t_{min}, r_{min}) \leftarrow ShortestRoute(\tilde{E})$;
while $t_{min} < \infty$ **do**
  $k^*_{r_{min}} \leftarrow \min_{e \in r_{min}} \tilde{k}_e$;
  for $e \in r_{min}$ **do**
    $\tilde{k}_e \leftarrow \tilde{k}_e - k^*_{r_{min}}$;
    if $\tilde{k}_e = 0$ **then**
      $\tilde{E} \leftarrow \tilde{E} \setminus \{e\}$;
    end
  end
  $(t_{min}, r_{min}) \leftarrow ShortestRoute(\tilde{E})$;
end
Return k^*

Next, we consider the sub-network comprised of routes in R^* with corresponding route capacities given by k^*. Analogous to (LP), the linear relaxation of optimal trip organization problem on this sub-network is given by:

$$\max_x \quad S(x) = \sum_{b \in B} \sum_{r \in R^*} V_r(b) x_r(b),$$

$$s.t. \quad \sum_{r \in R} \sum_{b \ni m} x_r(b) \leq 1, \quad \forall m \in M, \tag{LPk^*.a}$$

$$\sum_{b \in B} x_r(b) \leq k_r^*, \quad \forall r \in R^*, \tag{LPk^*.b}$$

$$x_r(b) \geq 0, \quad \forall b \in B, \quad \forall r \in R^*, \tag{LPk^*.c}$$

where (LPk^*.a) ensures that each rider is in at most one trip, and (LPk^*.b) ensures that the total number of trips in each route r does not exceed the route capacity k^*_r given by k^*.

Lemma 5.1. *If the network is series-parallel, then any optimal solution of* (LPk^*) *is an optimal solution of* (LP)*.*

To prove Lemma 5.1, we first prove that any feasible solution of (LPk^*) is also a feasible solution of (LP) by showing that the capacity vector k^* computed from Algorithm 1 satisfies $\sum_{r \ni e} k^*_r \leq q_e$ for all $e \in E$. Thus, the optimal value of (LPk^*) is no higher than that of (LP).

Next, we argue that for a series-parallel network, the optimal value of (LPk^*) is no less than that of (LP); hence, any optimal solution of (LPk^*) must also be an optimal solution of (LP). We prove this argument by showing that for any optimal solution $\hat{x}^*$ of (LP), there exists another trip vector x^* such that x^* is feasible in (LPk^*), and $S(x^*) \geq S(\hat{x}^*)$. Hence, x^* must be an optimal solution of (LP), and the optimal value of (LPk^*) is no less than that of (LP).

The key step of the proof is to construct such x^*. Given any optimal solution $\hat{x}^*$ of (LP), we denote $\hat{B} \triangleq \{B | \sum_{r \in R} \hat{x}^*_r(\hat{b}) > 0\}$ as the set of rider groups with positive weights in $\hat{x}^*$, and $f(\hat{b}) \triangleq \sum_{r \in R} \hat{x}^*_r(\hat{b})$ is the flow of each rider group $\hat{b} \in \hat{B}$ given $\hat{x}^*$. From (5.2), we re-write the trip value function as $V_r(\hat{b}) = z(\hat{b}) - g(\hat{b})t_r$, where $z(\hat{b}) = \sum_{m \in \hat{b}} (\alpha^m - \pi^m) - \sigma|\hat{b}|$, and $g(\hat{b}) = \sum_{m \in \hat{b}} \Big(\beta^m + \gamma^m(|\hat{b}|)\Big) + \delta|\hat{b}|$ can be viewed as group $\hat{b}$'s time sensitivity.

We construct x^* by re-assigning the flows of rider groups in $\hat{B}$ to routes in R^*. The re-assignment procedure selects rider groups in $\hat{B}$ one-by-one in decreasing order of time sensitivity $g(\hat{b})$. Flows of the selected rider groups are first assigned to the shortest routes in R^* until the capacity of the route given k^* is fully used. Then, we proceed to assign the flows of rider groups to the second shortest route in R^*. This procedure is repeated until the flows of all groups in $\hat{B}$ are assigned to routes in R^*.

The constructed trip vector x^* is a feasible solution of (LPk^*) because rider groups are assigned to the subnetwork with routes in R^*, and the flow assigned to each route does not exceed the capacity given by k^*. Moreover, the re-assignment procedure does not change the set of organized rider groups $\hat{B}$ given by $\hat{x}^*$ or the flow of these groups, but simply re-distributing these flows on routes in R^* in a way that enables rider groups with higher time sensitivity to take shorter routes. We prove by mathematical induction that the constructed x^* satisfies the inequality $S(x^*) \geq S(\hat{x})$ when the network is series-parallel: If the inequality holds on any two series-parallel networks, then it also holds on the network that is constructed by connecting the two sub-networks in series or in parallel. Thus, we can conclude that the optimal value of (LPk^*) and (LP) are equal, and any optimal solution of (LPk^*) is also an optimal solution of (LP).

In part 1, Lemma 5.1 ensures that if (LPk^*) has an integer optimal solution, then that solution must be an optimal integer solution of (LP). It remains to show that (LPk^*) indeed has an integer optimal solution.

<u>Part 2.</u> In this part, we first construct an augmented trip value function that is monotonic in the rider group. Then, we construct an auxiliary network comprised of parallel routes with unit capacities based on the set of routes given by k^*. We show that (LPk^*) has an integer optimal solution if and only if the linear relaxation of the trip organization problem on the auxiliary network with the augmented value function has integer optimal solution. Moreover, the trip organization problem on the auxiliary network with the augmented value function is equivalent to an allocation problem in an economy with indivisible goods. The existence of optimal integer solution is equivalent to the existence of Walrasian equilibrium in this economy.

To begin with, we introduce the definition of monotonic trip value function as follows:

Definition 5.3 (Monotonicity). *For each $r \in R$, the trip value function V_r is monotonic if for any $b, b' \in B$, $V_r(b \cup b') \geq V_r(b)$.*

Monotonicity condition requires that adding any rider group b' to a trip (b, r) does not reduce the trip's value. The monotonicity condition may not be always satisfied in general because of two reasons: First, if the size of riders $|b \cup b'| > A$, then the trip $(b \cup b', r)$ is infeasible, and the trip value is not defined. Second, even when $|b \cup b'| \leq A$, the value $V_r(b \cup b')$ may be less than $V_r(b)$ when the carpool disutility is sufficiently high.

We augment $V : B \times R \to \mathbb{N}$ to a monotonic value function $\overline{V} : \bar{B} \times R \to \mathbb{N}$, where $\bar{B} \triangleq 2^M$ is the set of all rider subsets (including the rider subsets with sizes larger than A). The value of $\overline{V}_r(\bar{b})$ can be written as follows:

$$\overline{V}_r(\bar{b}) \triangleq \max_{b \subseteq \bar{b},\ b \in B} V_r(b), \quad \forall r \in R, \quad \forall \bar{b} \in \bar{B}. \tag{5.14}$$

That is, the value of any rider group $\bar{b} \in \bar{B}$ on route r equals to the maximum value of a feasible trip (b, r) where rider group b is a subset of $\bar{b}$. The augmented value function $\overline{V}$ satisfies the monotonicity condition.

We refer $h_r(\bar{b}) \triangleq \arg\max_{b \subseteq \bar{b},\ b \in B} V_r(b)$ as the *representative rider group* of $\bar{b}$ for route r. From (5.2), we can re-write the augmented trip value function $\overline{V}$ as a linear function of travel time:

$$\overline{V}_r(\bar{b}) = \sum_{m \in h_r(\bar{b})} (\alpha^m - \beta^m t_r) - \sum_{m \in h_r(\bar{b})} \big(\pi^m + \gamma^m(|h_r(\bar{b})|)t_r\big) - (\sigma + \delta t_r)\,|h_r(\bar{b})|, \quad \forall \bar{b} \in \bar{B}, \quad \forall r \in R.$$

Next, we construct an auxiliary network given the set of routes R^* with capacity vector k^* output from Algorithm 1. Specifically, we convert each route $r \in R^*$ with integer capacity k_r^* to the same number of parallel routes each with a unit capacity in the auxiliary network. We denote the route set of the auxiliary network as $L = \cup_{r \in R^*} L_r$, where each set L_r is k_r^* number of routes converted from route r.

We now consider the trip organization problem on the auxiliary network with the augmented trip value function. For each $l \in L$ and each $\bar{b} \in \bar{B}$, we define $\big(\bar{b}, l\big)$ as an augmented trip. In this trip, the rider group $h_r(\bar{b})$ takes route l of the auxiliary network, while the remaining riders $m \in \bar{b} \setminus h_r(\bar{b})$ are not included in the trip. We denote the augmented trip vector as $y = \big(y_l(\bar{b})\big)_{\bar{b} \in \bar{B}, l \in K} \in \{0,1\}^{|\bar{B}| \times L}$, where $y_l(\bar{b}) = 1$ if the augmented trip $\big(\bar{b}, l\big)$ is organized, and $y_l(\bar{b}) = 0$ if otherwise. The value of the augmented trip is defined as $W_l(\bar{b}) = \overline{V}_r(\bar{b})$ for any $\bar{b} \in \bar{B}$, any $l \in L_r$ and any $r \in R^*$.

For any $y \in \{0,1\}^{|\bar{B}| \times L}$, we can compute a trip vector for the original optimal trip organization problem $x = \chi(y) \in \{0,1\}^{|B| \times R}$ such that the actually organized trips given by $x = \chi(y)$ are the same as that given by y. In particular, for each route $r \in R^*$, and each augmented trip $\left(\bar{b}, l\right) \in \bar{B} \times L_r$ such that $y_l(\bar{b}) = 1$, we choose a representative rider group $\hat{b} \in h_r(\bar{b})$ and set $x_r(\hat{b}) = 1$ for the original trip $\left(\hat{b}, r\right)$ that represents the organized augmented trip $\left(\bar{b}, l\right)$. We set $x_r(b) = 0$ for all other trips. Thus, the trip vector $x = \chi(y)$ can be written as follows:

$$\forall r \in R^*,\ \forall \left(\bar{b}, l\right)\ s.t.\ y_l(\bar{b}) = 1,\ \exists \hat{b} \in h_r(\bar{b}),\ s.t.\ x_r(\hat{b}) = 1,\ \text{and}\ x_r(b) = 0,\ \forall b \in B \setminus \{\hat{b}\} \tag{5.15}$$

Hence, we write the linear relaxation of optimal trip organization problem on the auxiliary network with the augmented trip value function as follows:

$$\begin{aligned} \max_{y} \quad & S(y) = \sum_{\bar{b} \in \bar{B}} \sum_{l \in L} W_l(\bar{b}) y_l(\bar{b}), \\ s.t. \quad & \sum_{l \in L} \sum_{\bar{b} \ni m} y_l(\bar{b}) \leq 1, \quad \forall m \in M, && \text{(LP-y.a)} \\ & \sum_{\bar{b} \in \bar{B}} y_l(\bar{b}) \leq 1, \quad \forall l \in L, && \text{(LP-y.b)} \\ & y_l(\bar{b}) \geq 0, \quad \forall \bar{b} \in \bar{B}, \quad \forall l \in L, && \text{(LP-y.c)} \end{aligned}$$

Lemma 5.2. *The linear program* (LPk^*) *has an integer optimal solution if and only if* (LP-y) *has an integer optimal solution. Moreover, if y^* is an integer optimal solution of* (LP-y)*, then $x^* = \chi(y^*)$ as in* (5.15) *is an optimal integer solution of* (LPk^*)*.*

This lemma shows that finding an optimal integer solution of (LPk^*) is equivalent to finding an optimal integer solution of (LP-y).

We finally show that the augmented trip organization problem is mathematically equivalent to an economy $\mathcal{G}$ with indivisible goods, and the existence of market equilibrium in our carpooling market is equivalent to the existence of Walrasian equilibrium of the economy. In $\mathcal{G}$, the set of indivisible "goods" is the rider set M and the set of agents is the route set L in the auxiliary network. Each agent l's value of any good bundle $\bar{b} \in \bar{B}$ is equivalent to the augmented trip value function $W_l(\bar{b})$. Moreover, each good m's price is equivalent to rider m's utility u^m. The vector of good allocation is y, where $y_l(\bar{b}) = 1$ if good bundle $\bar{b}$ is allocated to agent l. Given any y, for each $l \in L$, we denote the bundle of goods that is allocated to l as $\bar{b}_l$, i.e. $y_l(\bar{b}_l) = 1$. If no good is allocated to l (i.e. $\sum_{\bar{b}\in\bar{B}} y_l(\bar{b}) = 0$), then $\bar{b}_l = \emptyset$. The Walrasian equilibrium of economy $\mathcal{G}$ is defined as follows:

Definition 5.4 (Walrasian equilibrium Kelso Jr and Crawford [1982]). *A tuple (y^*, u^*) is a Walrasian equilibrium if*

(i) For any $l \in L$, $\bar{b}_l \in \arg\max_{\bar{b}\in\bar{B}} W_l(\bar{b}) - \sum_{m\in\bar{b}_l} u^m$, where $\bar{b}_l$ is the good bundle that is allocated to l given y^*

(ii) For any $m \in M$ that is not allocated to any agent, (i.e. $\sum_{l\in L}\sum_{\bar{b}\ni m} y_l^*(\bar{b}) = 0$), $u^{m*} = 0$.

In fact, we can show that (LP-y) has integer optimal solution if and only if Walrasian equilibrium exists in this equivalent economy:

Lemma 5.3. *The linear program* (LP-y) *has integer optimal solution if and only if a Walrasian equilibrium (y^*, u^*) exists in the equivalent economy. Furthermore, y^* is an integer optimal solution of* (LP-y)*, and $x^* = \chi(y^*)$ as in* (5.15) *is an optimal integer solution of* (LPk^*)*.*

In part 2, from Lemmas 5.2 – 5.3, we turn the problem of proving the existence of integer optimal solution in (LPk^*) to proving that the equivalent economy $\mathcal{G}$ has Walrasian equilibrium.

<u>Part 3.</u> In this final part, we show that if the carpool disutility parameter γ^m is homogeneous across all $m \in M$, then Walrasian equilibrium exists in the economy $\mathcal{G}$ constructed in Part 2.

To begin with, we introduce the following definition of gross substitutes condition on the augmented value function $\overline{V}$. In this definition, we utilize the notion of marginal value function $\overline{V}_r(\bar{b}'|\bar{b}) = \overline{V}_r(\bar{b} \cup \bar{b}') - \overline{V}_r(\bar{b})$ for all $r \in R$ and all $\bar{b}, \bar{b}' \subseteq M$.

Definition 5.5 (Gross Substitutes Reijnierse et al. [2002]). *For each $r \in R$, the augmented trip value function $\overline{V}_r$ is said to satisfy gross substitutes condition if*

(a) For any $\bar{b}, \bar{b}' \subseteq \bar{B}$ such that $\bar{b} \subseteq \bar{b}'$ and any $i \in M \setminus \bar{b}'$, $\overline{V}_r(i|\bar{b}') \leq \overline{V}_r(i|\bar{b})$.

(b) For all groups $\bar{b} \in \bar{B}$ and any $i, j, k \in M \setminus \bar{b}$,

$$\overline{V}_r(i, j|\bar{b}) + \overline{V}_r(k|\bar{b}) \leq \max\left\{\overline{V}_r(i|\bar{b}) + \overline{V}_r(j, k|\bar{b}),\ \overline{V}_r(j|\bar{b}) + \overline{V}_r(i, k|\bar{b})\right\}. \tag{5.17}$$

In Definition 5.5, *(a)* requires that the augmented value function $\overline{V}$ is submodular, i.e. the marginal valuation of $(\bar{b}, r)$ decreases in the size of group $\bar{b}$. Additionally, the gross substitutes condition also requires that the augmented value function satisfy *(b)*. This condition ensures that the sum of marginal values of $\{i, j\}$ and k is not strictly higher than that of both $i, \{j, k\}$ and $j, \{i, k\}$.

The following lemma shows that the augmented trip value function $\overline{V}$ satisfies gross substitutes condition if all riders have a homogeneous carpool disutility.

Lemma 5.4. *The augmented value function $\overline{V}_r$ satisfies gross substitutes for all $r \in R$ if all riders have homogeneous carpool disutility: $\gamma^m(d) \equiv \gamma(d)$ for all $d = 1, \ldots, A$ and all $m \in M$.*

In the economy $\mathcal{G}$, since each agent l's value function $W_l(\bar{b}) = \overline{V}_r(\bar{b})$ for all $\bar{b} \in \bar{B}$ and all $l \in L_r$, the agents' value functions W satisfy gross substitutes under the condition in Lemma 5.4. Moreover, the value functions W are also monotonic. From the following result, we know that a Walrasian equilibrium exists in economy with value functions that satisfy monotonicity and gross substitutes conditions.

Lemma 5.5 (Bikhchandani and Mamer [1997])**.** *If W_l satisfies the monotonicity and gross substitutes conditions for all $l \in L$, then Walrasian equilibrium (y^*, u^*) exists.*

Based on Lemmas 5.3, 5.4 and 5.5, we conclude the following:

Lemma 5.6. *The linear program* (LPk^*) *has an optimal integer solution if all riders have homogeneous carpool disutilities, i.e. $\gamma^m(d) \equiv \gamma(d)$ for all $m \in M$ and all $d = 1, \ldots, A$.*

Lemma 5.6 shows that (LPk^*) has an optimal integer solution. From Lemma 5.1, we know that this solution is also an optimal integer solution of (LP). Therefore, we can conclude Theorem 5.2 that market equilibrium exists when the network is series parallel and riders have homogeneous carpool disutilities. In Sec. 5.5 and 5.6, we assume that the sufficient conditions in Theorem 5.2 hold, and market equilibrium exists.

5.5 Computing Market Equilibrium

In this section, we present an algorithm for computing the market equilibrium (x^*, p^*, τ^*). The ideas behind the algorithm are based on Theorems 5.1, 5.2 and their proofs.

Computing optimal trip vector x^*. To begin with, one can obtain the optimal trip vector x^* following the proof of Theorem 5.2. In particular, we compute the route capacity vector k^* from Algorithm 1. From Lemma 5.1, we know that the optimal trip assignment vector x^* is an optimal integer solution of (LPk^*). Moreover, from Lemmas 5.2 – 5.6, we know that: (i) x^* can be derived from optimal solution y^* on the auxiliary network with the augmented trip value function W; and (ii) y^* is the same as the optimal good allocation in Walrasian equilibrium of the equivalent economy $\mathcal{G}$. We introduce the following well-known Kelso-Crawford algorithm (Algorithm 2) for computing Walrasian equilibrium y^*.

ALGORITHM 2: Kelso-Crawford Auction Kelso Jr and Crawford [1982]

Initialize: Set $u^m \leftarrow 0\ \forall m \in M$; $\bar{b}_l \leftarrow \emptyset,\ \forall l \in L$;
while *TRUE* **do**
  for $l \in L$ **do**
    $J_l \leftarrow \arg\max_{J \subseteq M \setminus \bar{b}_l} \phi_l(J|\bar{b}_l) \triangleq \left\{W_l(J \cup \bar{b}_l) - \sum_{m \in \bar{b}_l} u^m - \sum_{m \in J} (u^m + \epsilon)\right\}$
  if $J_l = \emptyset,\ \forall l \in L$ **then**
    break
  else
    Arbitrarily pick $\hat{l}$ with $J_{\hat{l}} \neq \emptyset$;
    $\bar{b}_{\hat{l}} \leftarrow \bar{b}_{\hat{l}} \cup J_{\hat{l}}$;
    $\bar{b}_{\hat{l}} \leftarrow \bar{b}_{\hat{l}} \setminus J_{\hat{l}},\ \forall l \neq \hat{l}$;
    $u^m \leftarrow u^m + \epsilon,\ \forall m \in J_{\hat{l}}$.
Return $\left(\bar{b}_l\right)_{l \in L}$

Algorithm 2 begins with all riders having zero utilities $u^m = 0$ and all routes in the auxiliary network being empty $\bar{b}_l = \emptyset$. In each iteration, we compute the set of riders J_l who are currently not assigned to route l and maximize the function $\phi_l(J_l|\bar{b}_l)$, which equals to the trip value minus the riders' utilities when the set J_l is added to $\bar{b}_l$. If there exists a route $\hat{l} \in L$ with $J_{\hat{l}} \neq \emptyset$, then we choose one of such route $\hat{l}$, and assign riders in $J_{\hat{l}}$ to $\hat{l}$. We increase the utilities of these riders in $J_{\hat{l}}$ by a small number ϵ.

Algorithm 2 terminates when $J_l = \emptyset$ for all $l \in L$. Given any $\epsilon < \frac{1}{2|M|}$, when the algorithm terminates, all routes are assigned with the rider set that maximizes its trip value minus riders' utilities. The trip vector based on $\left(\bar{b}_l\right)_{l \in L}$ is given by:

$$y_l^*(\bar{b}_l) = 1, \quad \text{and} \quad y_l^*(\bar{b}) = 0, \quad \forall \bar{b} \in \bar{B} \setminus \{\bar{b}_l\}, \quad \forall l \in L. \tag{5.18}$$

The following lemma shows that y^* is optimal under the conditions of monotonicity and gross substitutes.

Lemma 5.7 (Kelso Jr and Crawford [1982])**.** *For any $\epsilon < \frac{1}{2|M|}$, if the augmented value function W satisfies monotonicity and gross substitutes condition, then y^* as in* (5.18) *is an optimal integer solution of* (LP-y)*.*

Recall from Lemma 5.4, we know that when all riders have identical carpool disutility, i.e. $\gamma^m(d) = \gamma(d)$ for all $d = 1, \ldots, A$, then the augmented trip value function $\overline{V}$ satisfies gross substitutes condition. Since $W_l(\bar{b}) = \overline{V}_r(\bar{b})$ for all $l \in L_r$ and all $r \in R$, W also satisfies gross substitutes condition. Therefore, y^* is a Walrasian equilibrium good allocation vector in the equivalent economy $\mathcal{G}$, and from Lemmas 5.1 – 5.3, the vector $x^* = \chi(y^*)$ as in (5.15) is an optimal trip vector in market equilibrium.

In each iteration of Algorithm 2, we need to compute the set $J_l \in \arg\max_{J \subseteq M \setminus \bar{b}_l} \phi_l(J_l|\bar{b}_l)$ for each $l \in L$. Since the value function $W_l(\bar{b})$ is monotonic and satisfies gross substitutes condition, J_l can be computed by a greedy algorithm, in which riders are added to the set J_l one by one in decreasing order of the difference between the rider's marginal trip value $W_l(m|\bar{b}_l \cup J_l) = W_l(\{m\} \cup \bar{b}_l \cup J_l) - W_l(\bar{b}_l \cup J_l)$ and their utility u^m (Kelso Jr and Crawford [1982]). Since $W_l(\bar{b}) = \overline{V}_r(\bar{b})$ as in (5.14), and all riders have identical carpool disutility parameter, we can write $W_l(\bar{b})$ as follows:

$$\begin{aligned} W_l(\bar{b}) = \overline{V}_r(\bar{b}) &= \sum_{m \in h_r(\bar{b})} (\alpha^m - \beta^m t_r) - \sum_{m \in h_r(\bar{b})} \left(\pi(|h_r(\bar{b})|) + \gamma(|h_r(\bar{b})|) t_r\right) - (\sigma + \delta t_r)\, |h_r(\bar{b})| \\ &= \sum_{m \in h_r(\bar{b})} \eta_r^m - \theta(|h_r(\bar{b})|), \quad \forall l \in L_r, \quad \forall \bar{b} \in \bar{B}, \end{aligned}$$

where $\eta_r^m \triangleq \alpha^m - \beta^m t_r$, and $\theta(|h_r(\bar{b})|) = \left(\pi(|h_r(\bar{b})|) + \sigma\right) |h_r(\bar{b})| + \left(\gamma(|h_r(\bar{b})|) + \delta\right) |h_r(\bar{b})| t_r$. The representative rider group $h_r(\bar{b})$ for any trip $\left(\bar{b}, r\right) \in \bar{B} \times R$ can be constructed by selecting riders from $\bar{b}$ in decreasing order of η_r^m, and the last selected rider $\hat{m}$ (i.e. the rider in $h_r(\bar{b})$ with the minimum value of η_r^m) satisfies:

$$\eta_r^{\hat{m}} \geq \theta(|h_r(\bar{b})|) - \theta(|h_r(\bar{b})| - 1).$$

Thus, adding rider $\hat{m}$ to the set $h_r(\bar{b}) \setminus \{\hat{m}\}$ increases the trip value, but adding any other riders decrease the trip value, i.e.

$$\eta_r^m < \theta(|h_r(\bar{b})| + 1) - \theta(|h_r(\bar{b})|), \quad \forall m \in \bar{b} \setminus h_r(\bar{b}).$$

Therefore, the set $h_r(\bar{b})$ includes all riders with $\eta_r^m \geq \eta_r^{\hat{m}}$ in $\bar{b}$.

We can compute the set $J_l \leftarrow \arg\max_{J \subseteq M \setminus \bar{b}_l} \phi_l(J|\bar{b}_l)$ in each iteration of Algorithm 2 using Algorithm 3. In this algorithm, we first compute the size of the representative rider group $\tilde{h} = |h_r(\bar{b}_l)|$, then we add riders not in $\bar{b}_l$ into J_l greedily according to their marginal trip value minus utility. Note that for computing marginal trip value, we do not need to compute the augmented trip value function $W_l(\bar{b})$, but simply need to keep track of the representative rider group size $\tilde{h}$.

ALGORITHM 3: Computing J_l

Initialize: Set $J_l \leftarrow \emptyset$, $\tilde{h} \leftarrow 0$, $\tilde{b}_l \leftarrow \bar{b}_l$;
while *TRUE* **do**
    $\hat{m} \leftarrow \arg\max_{m \in \tilde{b}_l} \eta_l^m$;
    if $\eta_l^{\hat{m}} < \left(\theta(\tilde{h}+1) - \theta(\tilde{h})\right) t_l$ **then**
        break
    else
        $\tilde{h} \leftarrow \tilde{h} + 1$, $\tilde{b}_l \leftarrow \tilde{b}_l \setminus \{\hat{m}\}$
while *TRUE* **do**
    $\hat{j} \leftarrow \arg\max_{j \in S \setminus \left(\bar{b}_l \cup J_l\right)} \eta_l^j - u^j$;
    if $\eta_l^{\hat{j}} - u^{\hat{j}} < \left(\theta(\tilde{h}+1) - \theta(\tilde{h})\right) t_l$ **then**
        break
    else
        $\tilde{h} \leftarrow \tilde{h} + 1$, $J_l \leftarrow J_l \cup \{\hat{j}\}$
Return J_l

We next discuss the time complexity of Algorithm 2. The time complexity of computing

J_l as in Algorithm 3 is $O(|M|)$ for each $l \in L$ (each rider is counted at most once in Algorithm 3). Additionally, we know from Sec. 5.4 that the sum of route capacities given k^* equals to the maximum capacity of the network C. Thus $|L| = C$, and the time complexity of each iteration of Algorithm 2 is $O(|M|C)$. Moreover, riders' utilities are non-decreasing and at least one rider increases their utility by ϵ in each iteration. Besides, riders' utilities can not exceed the maximum trip value V_{max}, because otherwise $J_l = \emptyset$ for all $l \in L$ regardless of the assigned set $\bar{b}_l$; thus Algorithm 2 must terminate before the utility exceeds V_{max}. We can conclude that Algorithm 2 terminates in less than MV_{max}/ϵ iterations, and its time complexity is $O\left(\frac{V_{max}}{\epsilon}|M|^2 C\right)$.

We summarize that x^* is computed in the following two steps:

Step 1: Compute the optimal route capacity vector k^* from Algorithm 1. [9]

Step 2: Compute y^* from Algorithm 2. Derive the optimal trip organization vector $x^* = \chi(y^*)$.

Computing equilibrium payments p^* and toll prices τ^*. Given the optimal trip vector x^*, we compute the set of rider payments p^* and toll prices τ^* such that (x^*, p^*, τ^*) is a market equilibrium. Recall from Theorem 5.1, the riders' utilities and toll prices (u^*, τ^*) in any market equilibrium are optimal solutions of the dual program (D). Sec. 5.4 constructed the augmented trip value function $\overline{V}$, which satisfies monotonicity and gross substitutes conditions. Following the same proof ideas as in Theorem 5.1, we can show that the utility vector u^* and toll prices τ^* also can be solved from the following dual program with the augmented trip value function:

$$
\begin{aligned}
\min_{u,\tau} \quad & U(u,\tau) = \sum_{m \in M} u^m + \sum_{e \in E} q_e \tau_e \\
s.t. \quad & \sum_{m \in \bar{b}} u^m + \sum_{e \in r} \tau_e \geq \overline{V}_r(\bar{b}), \quad \forall \bar{b} \in \bar{B}, \quad \forall r \in R, && (\overline{\text{D}}.\text{a}) \\
& u^m \geq 0, \quad \tau_e \geq 0, \quad \forall m \in M, \quad \forall e \in E. && (\overline{\text{D}}.\text{b})
\end{aligned}
$$

The linear program $(\overline{\text{D}})$ has $|M|+|E|$ number of variables and $|R| \times |\bar{B}|$ number of constraints. This linear program can be solved by the ellipsoid method. In each iteration of this method,

[9]If the network is parallel, then this step can be omitted, and the vector $k^* = (q_r)_{r \in R}$.

we need to solve a separation problem to decide whether or not a solution (u, τ) is feasible, and if not find the constraint that it violates. Since the trip value function $\overline{V}$ is monotonic and satisfies the gross substitutes condition, we can solve the separation problem using Algorithm 3. For each route $r \in R$, we compute $\bar{b}_r \in \arg\max_{\bar{b} \in \bar{B}} \{\overline{V}_r(\bar{b}) - \sum_{m \in \bar{b}} u^m\} = \arg\max_{\bar{b} \in \bar{B}}$ using the same greedy algorithm as in Algorithm 3. Then, by checking whether or not $\sum_{m \in \bar{b}_r} u^m + \sum_{e \in r} \tau_e \geq \overline{V}_r(\bar{b}_r)$, we can determine if the constraint ($\overline{\text{D}}$.a) is satisfied for all route $r \in R$. In this way, we solve the separation problem in time polynomial in $|M|$ and $|R|$. Thus, the optimal solution of ($\overline{\text{D}}$) can also be solved by ellipsoid method in time polynomial in $|M|$ and $|R|$.

Finally, given any optimal solution (u^*, τ^*), the riders' payment vector p^* can be obtained from (5.4). Thus, we obtain (x^*, p^*, τ^*) as a market equilibrium.

Notice that the set of equilibrium utility and toll prices (u^*, τ^*) may not be singleton. From strong duality theory, we know that the sum of riders' equilibrium utilities and toll prices, must equal to the optimal social welfare given the organized trips in x^*, i.e. $\sum_{m \in M} u^{m*} + \sum_{e \in E} q_e \tau_e^* = S(x^*)$. Therefore, different market equilibria can result in different splits of social welfare between the riders' utilities and the collected toll prices. Next, we highlight a specific market equilibrium that provides the maximum share of social welfare to riders and collects the minimum tolls.

5.6 Strategyproofness and Maximum Rider Utilities

In this section, we consider the situation where the market is facilitated by a platform that implements a market equilibrium based on the reported preferences of each rider. Two questions arise in this situation: The first is whether or not riders truthfully report their preference parameters to the platform. The second is which market equilibrium is implemented and how it determines the splits between riders' utilities and collected tolls. We show that there exists a strategyproof market equilibrium under which riders truthfully report their preferences. Moreover, this market equilibrium also achieves the maximum utility for all riders and the total toll is the minimum.

We first introduce the definition of strategyproofness. To distinguish between the true preference parameters and the reported preference parameters, we denote the reported parameters as $\alpha^{'}$ and $\beta^{'}$.[10] The corresponding market equilibrium is denoted $\left(x^{*'}, p^{*'}, \tau^{*'}\right)$. The utility vector under market equilibrium with the true preference parameters (resp. reported preference parameters) u^* (resp. $u^{*'}$) can be computed as in (5.4). We say that a market equilibrium is strategyproof if no rider can gain higher utility by mis-reporting their preference parameters.

Definition 5.6 (Strategyproofness). *A market equilibrium (x^*, p^*, τ^*) is strategyproof if for any preference parameters $\alpha^{'} \neq \alpha$ and $\beta^{'} \neq \beta$, $u^{m*} \geq u^{m*'}$ for all $m \in M$.*

We next define the Vickery-Clark-Grove (VCG) Payment vector. For each $m \in M$, we denoted x^{-m*} as the optimal trip vector when rider m is not present. The social welfare for riders in $M \setminus \{m\}$ given the optimal trip vector x^{-m*} is denoted $S^{-m}(x^{-m*}) = \sum_{b \in B} \sum_{r \in R} V_r(b) x_r^{-m*}(b)$, and the social welfare for riders in $M \setminus \{m\}$ with x^* is $S_{-m}(x^*) = S(x^*) - \sum_{b \ni m} \sum_{r \in R} v_r^m(b) x_r^*(b)$.

Definition 5.7. *A VCG payment vector $p^\dagger = \left(p^{m\dagger}\right)_{m \in M}$ is given by:*

$$p^{m\dagger} = S_{-m}(x^{-m*}) - S_{-m}(x^*), \quad \forall m \in M. \tag{5.20}$$

In VCG payment vector (5.20), each rider m's payment is the difference of the total trip values for all other riders with and without rider m, i.e. $p^{m\dagger}$ is the externality of each rider m on all other riders. Under the optimal trip vector x^* and the VCG payment vector $p^\dagger$, the utility vector $u^\dagger = \left(u^{m\dagger}\right)_{m \in M}$ is given by:

$$u^{m\dagger} \overset{(5.4)}{=} \sum_{b \ni m} \sum_{r \in R} V_r(b) x_r^*(b) - p^{m\dagger} \overset{(5.20)}{=} S(x^*) - S_{-m}(x_{-m}^*), \quad \forall m \in M. \tag{5.21}$$

That is, the utility of each rider $m \in M$ is the difference of the optimal social welfare with and without rider m.

Lemma 5.8 (Vickrey [1961]). *A market equilibrium is strategyproof if the payment vector is $p^\dagger$.*

[10] We assume that riders have homogeneous carpool disutility that is known by the platform.

The next theorem shows that there exists a strategyproof market equilibrium $\left(x^*, p^\dagger, \tau^\dagger\right)$, in which the equilibrium payment vector is $p^\dagger$ and the riders' utility vector is $u^\dagger$. Moreover, all riders' utilities in this equilibrium are higher than that under any other market equilibrium, and the total collected tolls is the minimum.

Theorem 5.3. *A strategyproof market equilibrium $\left(x^*, p^\dagger, \tau^\dagger\right)$ exists, and the equilibrium utility vector is $u^\dagger$. Moreover, given any other market equilibrium (x^*, p^*, τ^*),*

$$u^{m\dagger} \geq u^{m*}, \quad \forall m \in M, \quad \textit{and} \quad \sum_{e \in E} q_e \tau_e^\dagger \leq \sum_{e \in E} q_e \tau_e^*.$$

We denote the set of u^* in the optimal solutions of the dual problem (D) as U^*. From Theorem 5.1, we know that the strategyproof equilibrium exists if and only if there exists a toll price vector $\tau^\dagger$ such that $\left(u^\dagger, \tau^\dagger\right)$ is an optimal solution of (D), i.e. $u^\dagger$ given by (5.21) is in U^*. Moreover, to show that $u^\dagger$ achieves the maximum equilibrium utility, we need to further prove that $u^\dagger$ is the maximum component in the set U^*.

We proceed in three steps: Firstly, Lemma 5.9 shows that the set U^* is equivalent to the set of utility vectors in the optimal solution set of the dual program of (LPk^*). Secondly, the set of optimal utility vectors in the dual program of (LPk^*) is the same as the set of prices in Walrasian equilibrium of the equivalent economy constructed in Sec. 5.4 (Lemma 5.10). Finally, the set of good prices in Walrasian equilibrium is a complete lattice, and the maximum component is $u^\dagger$ as in (5.21) (Lemma 5.11). Therefore, we can conclude that U^* is a complete lattice, and $u^\dagger$ is the maximum component in U^*.

We now present the formal statements of these lemmas and their proof ideas. The proofs are included in Appendix C.3.

Lemma 5.9. *A utility vector $u^* \in U^*$ if and only if there exists vector $\lambda^* = (\lambda_r^*)_{r \in R}$ such that (u^*, λ^*) is an optimal solution of the following linear program:*

$$\min_{u,\lambda} \quad \sum_{m \in M} u^m + \sum_{r \in R^*} k_r^* \lambda_r,$$

$$s.t. \quad \sum_{m \in b} u^m + \lambda_r \geq V_r(b) \quad \forall r \in R^*, \quad \forall b \in B, \tag{Dk^*.a}$$

$$u^m \geq 0, \lambda_r \geq 0, \quad \forall m \in M, \quad \forall r \in R^*, \tag{Dk^*.b}$$

where λ_r is the dual variable of constraint (LPk^*.b) *for each $r \in R$.*

In (Dk^*), the dual variable λ_r can be viewed as the toll price set on each route $r \in R^*$. We note that (Dk^*) is less restrictive than (D), which is the dual program on the original network, in two respects: Firstly, constraints (Dk^*) are only set for the set of routes R^* of the sub-network rather than on all routes in the whole network. Secondly, the toll prices λ in (Dk^*) are set on routes instead of on edges as in τ of (D). Any edge toll price vector can be equivalently represented as toll prices on routes by summing the tolls of all edges on any route. Therefore, given any feasible solution (u, τ) of (D), (u, λ) where $\lambda_r = \sum_{e \in r} \tau_e$ for each $r \in R^*$ is also feasible in (Dk^*).

We can check that for any optimal solution (u^*, τ^*) of (D), the vector (u^*, λ^*) – where $\lambda^*_r = \sum_{e \in r} \tau^*_e$ for each $r \in R^*$ – must also be optimal in (Dk^*). That is, the set U^* is a subset of the optimal utility vectors in (Dk^*). This result follows from strong duality theory and Lemma 5.1: From the strong duality theory, the optimal values of the objective function in (Dk^*) (resp. (LP)) equals to the optimal value of the primal problems (LP) (resp. (LPk^*)). From Lemma 5.1, we know that the optimal trip organization vector is the same in both (LP) and (LPk^*). Thus, the optimal value of (D) is the same as that of (LPk^*). Since the value of the objective function with (u^*, τ^*) equals to that with (u^*, λ^*), we know that (u^*, λ^*) must be an optimal solution of (Dk^*).

Furthermore, we can show that for any optimal solution u^* of (Dk^*), there must exist an edge toll vector τ^* such that (u^*, τ^*) is an optimal solution of ($\overline{\text{D}}$). That is, any equilibrium utility vector with route toll prices on the sub-network can also be induced by edge toll prices on the original network. This result relies on the fact that the network is series parallel, and it is proved by mathematical induction.

Lemma 5.9 enables us to characterize the riders' utility set U^* using the less restrictive dual program (Dk^*). Recall that in Sec. 5.4, we have shown that the trip organization problem on the constructed augmented network with the augmented value function is equivalent to an economy with indivisible goods (Lemma 5.3). The next lemma shows that the set U^* is the same as the set of Walrasian equilibrium prices in the equivalent economy.

Lemma 5.10. *A utility vector $u^* \in U^*$ if and only if there exists y^* such that (y^*, u^*) is a*

Walrasian equilibrium of the economy.

Moreover, since the augmented trip value function W is monotonic and satisfies gross substitutes condition, the set of Walrasian equilibrium price vectors is a lattice, and has a maximum component.

Lemma 5.11 (Gul and Stacchetti [1999]). *If the value function W satisfies the monotonicity and gross substitutes conditions, then the set of Walrasian equilibrium prices is a lattice and has a maximum component $u^\dagger = \left(u^{m\dagger}\right)_{m\in M}$ as in* (5.21).

From Lemmas 5.9 – 5.11, we know that $u^\dagger$ is the maximum component in the set U^*. That is, there exists a toll price vector $\tau^\dagger = \left(\tau_e^\dagger\right)_{e\in E}$ such that $\left(u^\dagger, \tau^\dagger\right)$ is an optimal solution of (D), and hence $\left(x^*, p^\dagger, \tau^\dagger\right)$ is a market equilibrium. From Lemma 5.8, we know that this market equilibrium is strategyproof. Moreover, all riders achieve the maximum equilibrium utilities in the equilibrium. Since $\sum_{m\in M} u^{m*} + \sum_{e\in E} q_e \tau_e^* = S(x^*)$ for any market equilibrium (x^*, u^*, τ^*), this also implies that the total amount of tolls $\sum_{e\in E} q_e \tau_e^\dagger$ that is collected in market equilibrium $\left(x^*, p^\dagger, \tau^\dagger\right)$ is the minimum. We thus conclude Theorem 5.3.

Finally, we discuss the computation of the market equilibrium $\left(x^*, p^\dagger, \tau^\dagger\right)$. In particular, the optimal trip assignment x^* can be computed in two steps described in Sec. 5.5 using Algorithms 1 – 2. Then, we re-run Algorithm 2 given k^* and rider set $M \setminus \{m\}$ to compute x^{-m*} for each $m \in M$. We compute the utility vector $u^\dagger$ (resp. payment vector $p^\dagger$) as in (5.21) (resp. (5.20)).

For any $e \in E$, we set $\tau_e^\dagger = 0$ if $\sum_{b\in B} \sum_{r\ni e} x_r^*(b) < q_e$. From $(\overline{\mathrm{D}})$, we know that $\tau^\dagger$ is any vector that satisfies the following constraints:

$$\begin{aligned}
\sum_{e\in r} \tau_e^\dagger &= \max_{\bar{b}\in\bar{B}} \overline{V}_r(\bar{b}) - \sum_{m\in\bar{b}} u^{m\dagger}, \quad \forall r \in R^*, \\
\sum_{e\in r} \tau_e^\dagger &\geq \max_{\bar{b}\in\bar{B}} \overline{V}_r(\bar{b}) - \sum_{m\in\bar{b}} u^{m\dagger}, \quad \forall r \in R \setminus R^*.
\end{aligned} \tag{5.23}$$

Finding a vector $\tau^\dagger$ that satisfies constraints in (5.23) is equivalent to solving a linear program with a constant objective function and feasibility constraints (5.23). This linear program can be computed by the ellipsoid method, in which the separation problem in each iteration is to check whether or not the toll price vector $\tau^\dagger$ satisfies the feasible constraints in (5.23). Since the augmented trip value function $\overline{V}$ satisfies monotonicity and gross substitutes condition, we can compute the right-hand-side value of the constraint in (5.23) using Algorithm 3 in time $O(|M|)$ for each $r \in R$. That is, the separation problem in each iteration can be computed in polynomial time of $|M|$ and $|R|$. Therefore, a toll vector $\tau^\dagger$ that satisfies (5.23) can be computed in polynomial time of $|M|$ and $|R|$.

5.7 Discussion

In this chapter, we studied the existence and computation of market equilibrium for organizing socially efficient carpooled trips over a transportation network using autonomous cars. We also identified a market equilibrium that is strategyproof and maximizes riders' utilities. Our approach can be used to analyze incentive mechanisms for sharing limited resources in networked environment.

One interesting direction for future work is to characterize equilibrium in a transportation market when riders belong to different classes that are differentiated by their carpool disutility levels. In this situation, riders with different carpool disutilies may be grouped into trips that are organized using different vehicle sizes to reflect the riders' car sharing preferences.

A more general problem is to design market with both autonomous and human-driven carpooled trips, wherein riders may have different preferences of over these service types. A pre-requisite to the design of such a market is quantitative evaluation of how autonomous and human-driven vehicles differ in terms of their utilization of road capacity and the incurred route travel times Jin et al. [2020]. Analysis of differentiated pricing and tolling schemes corresponding to trip assignments between the two service types is an interesting and relevant problem for future work.

Finally, the study of equilibrium outcomes and welfare implications when the autonomous carpooling market is not perfectly competitive is important from both market design and implementation viewpoints. When a few major entities (or platforms) control the provision of autonomous mobility services, it is important to study how the competition among these platforms affects the usage of road capacity, efficiency of carpooled trips, and toll prices needed to relieve network congestion. The results presented here can serve as a benchmark for evaluating the impacts of imperfect market competition in transportation markets enabled by autonomous cars.

Chapter 6

Security Analysis of Transportation Systems

6.1 Introduction

Transportation systems are undergoing a paradigm shift, thanks to the advances in computing and networking technologies that have enabled a range of functionalities for both infrastructure operators and travelers. The operations of information technologies and platform-based services featured in Chapters 2 – 5 rely on ubiquitous sensing and actuation capabilities, mobile and embedded computing with smartphones, and deep penetration of wireless communications networks. However, a significant drawback of information modernization is lowered security of transportation systems, caused by the exposure to cyber insecurities.

In recent years, several hacking incidents have been reported on transportation systems. For example, in December 2011, hackers executed an attack on the Northwest railway system for two days according to an official agency memo (Zetter [2017]). Hacking incidents to subway systems (both real and staged) have been also reported on the Toronto transit message system (Rosencrance [2006]), the ticketing systems on the Moscow subway (Owana [2012]) and the Bay Area Transit Systems (Hackett [2016]). More recently, successful cyber attacks have been demonstrated on traffic monitoring sensors (Zetter [2017], Reilly et al. [2015]), dynamic message signs (Mettler [2016]), traveler information systems (Bonaventure [2016]), and signal controllers (Jacobs [2014], Ghena et al. [2014]). Additionally, the emergence of network-based, (semi-)autonomous vehicle applications is expected to bring such threats to the forefront (Lowy [2015], Petit and Shladover [2015]), especially since most automobile manufacturers currently lack the capability to fully protect vehicle control systems against hacking.

Two of the well-recognized security concerns faced by infrastructure operators are: (i) *How to prioritize investments among facilities that are heterogeneous in terms of the impact that their compromise can have on the overall efficiency (or usage cost) of the system?* and (ii) *Whether or not an attacker can be fully deterred from launching an attack by proactively securing some of the facilities?*

In this chapter, we address these questions by focusing on the most basic form of *strategic interaction* between the system operator (defender) and an attacker. In particular, the attacker incurs a cost in compromising a facility and chooses among a set of critical facilities, possibly in a randomized manner. On the other hand, the defender faces the choice of which facilities to secure and with what level of security investment cost. The costs of attack/defense reflect the attacker/defender's *technological capabilities* in launching an attack/defense. If an undefended facility is targeted by the attacker, its functionality is compromised, and the outcome is evaluated as a reduction in the overall user welfare of the *physical infrastructure system*. Naturally, the defender aims to maintain a low usage cost, while the attacker wishes to increase the usage cost.

We model the attack and defense as a normal form (simultaneous) or a sequential (Stackelberg) game. The normal form game is relevant to situations in which the attacker cannot directly observe the chosen security plan, whereas the sequential game applies to situations where the defender proactively secures some facilities, and the attacker can observe the defense strategy. In both games, we provide a complete characterization of the equilibrium structure in terms of the relative vulnerability of different facilities and the costs of defense/attack. These results add value to the study on the allocation of defense resources on facilities against strategic adversaries (Powell [2007], Bier et al. [2007], Bell et al. [2008], Bier and Hausken [2013], Alderson et al. [2011], and Brown et al. [2006]).

Furthermore, we find that the defender's utility in the sequential game is no less than that in the normal form game. This phenomena has been long identified as the first mover advantage in two player games with mixed strategies (Basar and Olsder [1998] (pp. 126), Von Stengel and Zamir [2004]). We characterize the precise conditions on the attack and defense costs for which the first mover advantage for the defender is strictly positive. That is, we identify situations in which the defender can secure better system performance against the attacker or even completely deter the attack by proactively allocating security investments according to the equilibrium strategy.

The chapter is structured as follows: In Sec. 6.2, we introduce the model of both games, and discuss the modeling assumptions. We provide preliminary results to facilitate our analysis in Sec. 6.3. Sec. 6.4 characterizes NE, and Sec. 6.5 characterizes SPE. Sec. 6.6 compares both games. All proofs are included in Appendix D.

6.2 Model

6.2.1 Attacker-Defender Interaction: Normal Form versus Sequential Games

Consider an infrastructure system modeled as a set of components (facilities) E. To defend the system against an external malicious attack, the system operator (defender) can secure one or more facilities in E by investing in appropriate security technology. The set of facilities in question can include cyber or physical elements that are crucial to the functioning of the system. These facilities are potential targets for a malicious adversary whose goal is to compromise the overall functionality of the system by gaining unauthorized access to certain cyber-physical elements. The security technology can be a combination of proactive mechanisms (authentication and access control) or reactive ones (attack detection and response). Since our focus is on modeling the strategic interaction between the attacker and defender at a system level, we do not consider the specific functionalities of individual facilities or the protection mechanisms offered by various technologies.

We now introduce our game theoretic model. Let us denote a pure strategy of the defender as $s_d \subseteq E$, with $s_d \in S_d = 2^E$. The cost of securing any facility is given by the parameter $p_d \in \mathbb{R}_{>0}$. Thus, the total defense cost incurred in choosing a pure strategy s_d is

$|s_d| \cdot p_d$, where $|s_d|$ is the cardinality of s_d (i.e., the number of secured facilities). The attacker chooses to target a single facility $e \in E$ or not to attack. We denote a pure strategy of the attacker as $s_a \in S_a = E \cup \{\emptyset\}$. The cost of an attack is given by the parameter $p_a \in \mathbb{R}_{>0}$, and it reflects the effort that attacker needs to spend in order to successfully targets a single facility and compromise its operation.

We assume that prior to the attack, the usage cost of the system is $C_\emptyset$. This cost represents the level of efficiency with which the defender is able to operate the system for its users. A higher usage cost reflects lower efficiency. If a facility e is targeted by the attacker but not secured by the defender, we consider that e is compromised and the usage cost of the system changes to C_e. Therefore, given any pure strategy profile (s_d, s_a), the usage cost after the attacker-defender interaction, denoted $C(s_d, s_a)$, can be expressed as follows:

$$C(s_d, s_a) = \begin{cases} C_e, & \text{if } s_a = e, \text{ and } s_d \not\ni e, \\ C_\emptyset, & \text{otherwise.} \end{cases} \tag{6.1}$$

To study the effect of timing of the attacker-defender interaction, prior literature on security games has studied both normal form game and sequential games (Alpcan and Baysar [2010]). We study both models in our setting. In the normal form game, denoted Γ, the defender and the attacker move simultaneously. On the other hand, in the sequential game, denoted $\widetilde{\Gamma}$, the defender moves in the first stage and the attacker moves in the second stage after observing the defender's strategy. We allow both players to use mixed strategies. In Γ, we denote the defender's mixed strategy as $\sigma_d \triangleq (\sigma_d(s_d))_{s_d \in S_d} \in \Delta(S_d)$, where $\sigma_d(s_d)$ is the probability that the set of secured facilities is s_d. Similarly, a mixed strategy of the attacker is $\sigma_a \triangleq (\sigma_a(s_a))_{s_a \in S_a} \in \Delta(S_a)$, where $\sigma_a(s_a)$ is the probability that the realized action is s_a. Let $\sigma = (\sigma_d, \sigma_a)$ denote a mixed strategy profile. In $\widetilde{\Gamma}$, the defender's mixed strategy $\widetilde{\sigma}_d \triangleq (\widetilde{\sigma}_d(s_d))_{s_d \in S_d} \in \Delta(S_d)$ is defined analogously to that in Γ. The attacker's strategy is a map from $\Delta(S_d)$ to $\Delta(S_a)$, denoted by $\widetilde{\sigma}_a(\widetilde{\sigma}_d) \triangleq (\widetilde{\sigma}_a(s_a, \widetilde{\sigma}_d))_{s_a \in S_a} \in \Delta(S_a)$, where $\widetilde{\sigma}_a(s_a, \widetilde{\sigma}_d)$ is the probability that the realized action is s_a when the defender's strategy is $\widetilde{\sigma}_d$. A strategy profile in this case is denoted as $\widetilde{\sigma} = (\widetilde{\sigma}_d, \widetilde{\sigma}_a(\widetilde{\sigma}_d))$.

The defender's utility is comprised of two parts: the negative of the usage cost as given in (6.1) and the defense cost incurred in securing the system. Similarly, the attacker's utility is the usage cost net the attack cost. For a pure strategy profile (s_d, s_a), the utilities of defender and attacker can be respectively expressed as follows:

$$u_d(s_d, s_a) = -C(s_d, s_a) - p_d \cdot |s_d|, \quad u_a(s_d, s_a) = C(s_d, s_a) - p_a \cdot \mathbb{1}\{s_a \neq \emptyset\}.$$

For a mixed strategy profile (σ_d, σ_a), the expected utilities can be written as:

$$U_d(\sigma_d, \sigma_a) = \sum_{s_d \in S_d} \sum_{s_a \in S_a} u_d(s_d, s_a) \cdot \sigma_a(s_a) \cdot \sigma_d(s_d) = -\mathbb{E}_\sigma[C] - p_d \cdot \mathbb{E}_{\sigma_d}[|s_d|], \tag{6.2a}$$

$$U_a(\sigma_d, \sigma_a) = \sum_{s_d \in S_d} \sum_{s_a \in S_a} u_a(s_d, s_a) \cdot \sigma_a(s_a) \cdot \sigma_d(s_d) = \mathbb{E}_\sigma[C] - p_a \cdot \mathbb{E}_{\sigma_a}[|s_a|], \tag{6.2b}$$

where $\mathbb{E}_\sigma[C]$ is the expected usage cost, and $\mathbb{E}_{\sigma_d}[|s_d|]$ (resp. $\mathbb{E}_{\sigma_a}[|s_a|]$) is the expected number of defended (resp. targeted) facilities, i.e.:

$$\mathbb{E}_\sigma[C] = \sum_{s_a \in S_a} \sum_{s_d \in S_d} C(s_d, s_a) \cdot \sigma_a(s_a) \cdot \sigma_d(s_d),$$

$$\mathbb{E}_{\sigma_d}[|s_d|] = \sum_{s_d \in S_d} |s_d| \sigma_d(s_d), \quad \mathbb{E}_{\sigma_a}[|s_a|] = \sum_{e \in E} \sigma_a(e).$$

An equilibrium outcome of the game Γ is defined in the sense of Nash Equilibrium (NE). A strategy profile $\sigma^* = (\sigma_d^*, \sigma_a^*)$ is a NE if:

$$U_d(\sigma_d^*, \sigma_a^*) \geq U_d(\sigma_d, \sigma_a^*), \quad \forall \sigma_d \in \Delta(S_d),$$

$$U_a(\sigma_d^*, \sigma_a^*) \geq U_a(\sigma_d^*, \sigma_a), \quad \forall \sigma_a \in \Delta(S_a).$$

In the sequential game $\widetilde{\Gamma}$, the solution concept is that of a Subgame Perfect Equilibrium (SPE), which is also known as Stackelberg equilibrium. A strategy profile $\widetilde{\sigma}^* = (\widetilde{\sigma}_d^*, \widetilde{\sigma}_a^*(\widetilde{\sigma}_d))$ is a SPE if:

$$U_d(\widetilde{\sigma}_d^*, \widetilde{\sigma}_a^*(\widetilde{\sigma}_d^*)) \geq U_d(\widetilde{\sigma}_d, \widetilde{\sigma}_a^*(\widetilde{\sigma}_d)), \quad \forall \widetilde{\sigma}_d \in \Delta(S_d), \tag{6.3a}$$

$$U_a(\widetilde{\sigma}_d, \widetilde{\sigma}_a^*(\widetilde{\sigma}_d)) \geq U_a(\widetilde{\sigma}_d, \widetilde{\sigma}_a(\widetilde{\sigma}_d)), \quad \forall \widetilde{\sigma}_d \in \Delta(S_d), \quad \forall \widetilde{\sigma}_a(\widetilde{\sigma}_d) \in \Delta(S_a). \tag{6.3b}$$

Since both S_d and S_a are finite sets, and we consider mixed strategies, both NE and SPE exist.

6.2.2 Model Discussion

One of our main assumptions is that the attacker's capability is limited to targeting at most one facility, while the defender can invest in securing multiple facilities. Although this assumption appears to be somewhat restrictive, it enables us to derive analytical results on the equilibrium structure for a system with multiple facilities. The assumption can be justified in situations where the attacker can only target system components in a localized manner. Thus, a facility can be viewed as a set of collocated components that can be simultaneously targeted by the attacker. For example, in a transportation system, a facility can be a vulnerable link (edge), or a set of links that are connected by a vulnerable node (an intersection or a hub). In Sec. 6.7, we briefly discuss the issues in solving a more general game where multiple facilities can be simultaneously targeted by the attacker.

Secondly, our model assumes that the costs of attack and defense are identical across all facilities. We make this assumption largely to avoid the notational burden of analyzing the effect of facility-dependent attack/defense cost parameters on the equilibrium structures. In fact, as argued in Sec. 6.7, the qualitative properties of equilibria still hold when cost parameters are facility-dependent. However, characterizing the equilibrium regimes in this case can be quite tedious, and may not necessarily provide new insights on strategic defense investments.

Thirdly, we allow both players to choose mixed strategies. Indeed, mixed strategies are commonly considered in security games as a pure NE may not always exists. A mixed strategy entails a player's decision to introduce randomness in her behavior, i.e. the manner in which a facility is targeted (resp. secured) by the attacker (resp. defender). Consider for example, the problem of inspecting a transportation network facing risk of a malicious attack. In this problem, a mixed strategy can be viewed as randomized allocation of inspection effort on subsets of facilities. Mixed strategy of the attacker can be similarly interpreted.

Fourthly, we assume that the defender has the technological means to perfectly secure a facility. In other words, an attack on a secured facility cannot impact its operation. As we will see in Sec. 6.3, the defender's mixed strategy can be viewed as the level of security effort on each facility, where the effort level 1 (maximum) means perfect security, and 0 (minumum) means no security. Under this interpretation, the defense cost p_d is the cost of perfectly securing a unit facility (i.e., with maximum level of effort), and the expected defense cost is p_d scaled by the security effort defined by the defender's mixed strategy.

Fifthly, we do not consider a specific functional form for modeling the usage cost. In our model, for any facility $e \in E$, the difference between the post-attack usage cost C_e and the pre-attack cost $C_\emptyset$ represents the change of the usage cost of the system when e is compromised. This change can be evaluated based on the type of attacker-defender interaction one is interested in studying. For example, in situations when attack on a facility results in its complete disruption, one can use a connectivity-based metric such as the number of active source-destination paths or the number of connected components to evaluate the usage cost (Dziubiński and Goyal [2013] and Dziubiński and Goyal [2017]). On the other hand, in situations when facilities are congestible resources and an attack on a facility increases the users' cost of accessing it, the system's usage cost can be defined as the average cost for accessing (or routing through) the system. This cost can be naturally evaluated as the user cost in a Wardrop equilibrium (Bier and Hausken [2013]), although socially optimal cost has also been considered in the literature (Alderson et al. [2017]).

Finally, we note that for the purpose of our analysis, the usage cost as given in (6.1) fully captures the impact of player' actions on the system. For any two facilities $e, e^{'} \in E$, the ordering of C_e and $C_{e'}$ determines the relative scale of impact of the two facilities. As we show in Sec. 6.4–6.5, the order of cost functions in the set $\{C_e\}_{e \in \mathcal{E}}$ plays a key role in our analysis approach. Indeed the usage cost is intimately linked with the network structure and way of operation (for example, how individual users are routed through the network and how their costs are affected by a compromised facility). Barring a simple (yet illustrative) example, we do not elaborate further on how the network structure and/or the functional form of usage cost changes the interpretations of equilibrium outcome. We also do not discuss the computational aspects of arriving at the ordering of usage costs.

6.3 Rationalizable Strategies and Aggregate Defense Effort

We introduce two preliminary results that are useful in our subsequent analysis. Firstly, we show that the defender's strategy can be equivalently represented by a vector of facility-specific security effort levels. Secondly, we identify the set of rationalizable strategies of both players.

For any defender's mixed strategy $\sigma_d \in \Delta(S_d)$, the corresponding *security effort vector* is $\rho(\sigma_d) = (\rho_e(\sigma_d))_{e \in E}$, where $\rho_e(\sigma_d)$ is the probability that facility e is secured:

$$\rho_e(\sigma_d) = \sum_{s_d \ni e} \sigma_d(s_d). \tag{6.4}$$

In other words, $\rho_e(\sigma_d)$ is the level of security effort exerted by the defender on facility e under the security plan σ_d. Since $\sigma_d(s_d) \geq 0$ for any $s_d \in S_d$, we obtain that $0 \leq \rho_e(\sigma_d) = \sum_{s_d \ni e} \sigma_d(s_d) \leq \sum_{s_d \in S_d} \sigma_d(s_d) = 1$. Hence, any σ_d induces a valid probability vector $\rho \in [0,1]^{|E|}$. In fact, any vector $\rho \in [0,1]^{|E|}$ can be induced by at least one feasible σ_d. The following lemma provides a way to explicitly construct one such feasible strategy.

Lemma 6.1. *Consider any feasible security effort vector $\rho \in [0,1]^{|E|}$. Let m be the number of distinct positive values in ρ, and define $\rho_{(i)}$ as the i-th largest distinct value in ρ, i.e. $\rho_{(1)} > \cdots > \rho_{(m)}$. The following defender's strategy is feasible and induces ρ:*

$$\sigma_d(\{e \in E | \rho_e \geq \rho_{(i)}\}) = \rho_{(i)} - \rho_{(i+1)}, \quad \forall i = 1, \ldots, m-1 \tag{6.5a}$$

$$\sigma_d(\{e \in E | \rho_e \geq \rho_{(m)}\}) = \rho_{(m)}, \tag{6.5b}$$

$$\sigma_d(\emptyset) = 1 - \rho_{(1)}. \tag{6.5c}$$

For any remaining $s_d \in S_d$, $\sigma_d(s_d) = 0$.

We now re-express the player utilities in (6.2) in terms of $(\rho(\sigma_d), \sigma_a)$ as follows:

$$U_d(\sigma_d, \sigma_a) = -\sum_{s_a \in S_a} \left(\sum_{s_d \in S_d} \sigma_d(s_d) C(s_d, s_a) \right) \sigma_a(s_a) - \left(\sum_{s_d \in S_d} |s_d| \sigma_d(s_d) \right) p_d$$

$$= -\sum_{e\in E}\left(\sum_{s_d\in S_d}\sigma_d(s_d)C(s_d,e)\right)\sigma_a(e) - C_{\emptyset}\sigma_a(\emptyset) - \left(\sum_{e\in E}\rho_e(\sigma_d)\right)p_d$$

$$\overset{(6.1)}{=} -\sum_{e\in E}\left(\left(\sum_{s_d\ni e}\sigma_d(s_d)\right)C_{\emptyset} + \left(1-\sum_{s_d\ni e}\sigma_d(s_d)\right)C_e\right)\sigma_a(e) - C_{\emptyset}\sigma_a(\emptyset) - \left(\sum_{e\in E}\rho_e(\sigma_d)\right)p_d$$

$$= -\sum_{e\in E}\left(\rho_e(\sigma_d)\left(\left(C_{\emptyset}-C_e\right)\sigma_a(e)+p_d\right)+C_e\sigma_a(e)\right) - C_{\emptyset}\sigma_a(\emptyset), \tag{6.6a}$$

$$U_a(\sigma_d,\sigma_a) = \sum_{e\in E}\left(\rho_e(\sigma_d)\left(C_{\emptyset}-C_e\right)\sigma_a(e)+C_e\sigma_a(e)\right) + C_{\emptyset}\sigma_a(\emptyset) - \left(\sum_{e\in E}\sigma_a(e)\right)p_a. \tag{6.6b}$$

Thus, for any given attack strategy and any two defense strategies, if the induced security effort vectors are identical, then the corresponding utility for each player is also identical. Henceforth, we denote the player utilities as $U_d(\rho,\sigma_a)$ and $U_a(\rho,\sigma_a)$, and use σ_d and $\rho_e(\sigma_d)$ interchangeably in representing the defender's strategy. For the sequential game $\widetilde{\Gamma}$, we analogously denote the security effort vector given the strategy $\widetilde{\sigma}_d$ as $\tilde{\rho}(\widetilde{\sigma}_d)$, and the defender's utility (resp. attacker's utility) as $\widetilde{U}_d(\tilde{\rho},\widetilde{\sigma}_a)$ (resp. $\widetilde{U}_a(\tilde{\rho},\widetilde{\sigma}_a)$).

We next characterize the set of rationalizable strategies. Note that the post-attack usage cost C_e can increase or remain the same or even decrease, in comparison to the pre-attack cost $C_{\emptyset}$. Let the facilities whose damage result in an increased usage cost be grouped in the set $\bar{E}$. Similarly, let $\widehat{E}$ denote the set of facilities such that a damage to any one of them has no effect on the usage cost. Finally, the set of remaining facilities is denoted as $\breve{E}$. Thus:

$$\bar{E} \triangleq \left\{e\in E | C_e > C_{\emptyset}\right\}, \tag{6.7a}$$

$$\widehat{E} \triangleq \left\{e\in E | C_e = C_{\emptyset}\right\}, \tag{6.7b}$$

$$\breve{E} \triangleq \left\{e\in E | C_e < C_{\emptyset}\right\}. \tag{6.7c}$$

Clearly, $\bar{E}\cup\widehat{E}\cup\breve{E} = E$. The following proposition shows that in a rationalizable strategy profile, the defender does not secure facilities that are not in $\bar{E}$, and the attacker only considers targeting the facilities that are in $\bar{E}$.

Proposition 6.1. *The rationalizable action sets for the defender and attacker are given by* $2^{\bar{E}}$ *and* $\bar{E}\cup\{\emptyset\}$*, respectively. Hence, any equilibrium strategy profile* (ρ^*,σ_a^*) *in* Γ *(resp.*

$(\tilde{\rho}^*, \widetilde{\sigma}_a^*)$ *in* $\widetilde{\Gamma}$*) satisfies:*

$$\rho_e^* = \sigma_a^*(e) = 0, \qquad \forall e \in E \setminus \bar{E},$$
$$\tilde{\rho}_e^* = \widetilde{\sigma}_a^*(e, \tilde{\rho}) = 0, \quad \forall e \in E \setminus \bar{E}, \quad \forall \tilde{\rho} \in [0,1]^E.$$

If $\bar{E} = \emptyset$, then the attacker/defender does not attack/secure any facility in equilibrium. Henceforth, to avoid triviality, we assume $\bar{E} \neq \emptyset$. Additionally, we define a partition of facilities in $\bar{E}$ such that all facilities with identical C_e are grouped in the same set. Let the number of distinct values in $\{C_e\}_{e \in \bar{E}}$ be K, and $C_{(k)}$ denote the k-th highest distinct value in the set $\{C_e\}_{e \in \bar{E}}$. Then, we can order the usage costs as follows:

$$C_{(1)} > C_{(2)} > \cdots > C_{(K)} > C_{\emptyset}. \tag{6.8}$$

We denote $\bar{E}_{(k)}$ as the set of facilities such that if any $e \in \bar{E}_{(k)}$ is damaged, the usage cost $C_e = C_{(k)}$, i.e. $\bar{E}_{(k)} \triangleq \left\{e \in \bar{E} | C_e = C_{(k)}\right\}$. We also define $E_{(k)} \triangleq |\bar{E}_{(k)}|$. Clearly, $\cup_{k=1}^{K} \bar{E}_{(k)} = \bar{E}$, and $\sum_{k=1}^{K} E_{(k)} = |\bar{E}|$. Facilities in the same group have identical impact on the infrastructure system when compromised.

6.4 Normal Form Game Γ

In this section, we provide complete characterization of the set of NE for any given attack and defense cost parameters in game Γ. In Sec. 6.4.1, we show that Γ is strategically equivalent to a zero-sum game, and hence the set of attacker's equilibrium strategies can be solved by a linear program. In Sec. 6.4.2, we show that the space of cost parameters $(p_a, p_d) \in \mathbb{R}^2_{>0}$ can be partitioned into qualitatively distinct equilibrium regimes.

6.4.1 Strategic Equivalence to Zero-Sum Game

Our notion of strategic equivalence is the same as the best-response equivalence defined in Rosenthal [1974]. If Γ and another game Γ^0 are strategically equivalent, then given any strategy of the defender (resp. attacker), the set of attacker's (resp. defender's) best responses is identical in the two games. This result forms the basis of characterizing the set of NE.

We define the utility functions of the game Γ^0 as follows:

$$U_d^0(\sigma_d, \sigma_a) = -\mathbb{E}_\sigma[C] - \mathbb{E}_{\sigma_d}[|s_d|] \cdot p_d + p_a \cdot \mathbb{E}_{\sigma_a}[|s_a|], \tag{6.9a}$$

$$U_a^0(\sigma_d, \sigma_a) = \mathbb{E}_\sigma[C] + \mathbb{E}_{\sigma_d}[|s_d|] \cdot p_d - p_a \cdot \mathbb{E}_{\sigma_a}[|s_a|]. \tag{6.9b}$$

Thus, Γ^0 is a zero-sum game. We denote the set of defender's (resp. attacker's) equilibrium strategies in Γ^0 as Σ_d^0 (resp. Σ_a^0).

Lemma 6.2. *The normal form game Γ is strategically equivalent to the zero sum game Γ^0. The set of defender's (resp. attacker's) equilibrium strategies in Γ is $\Sigma_d^* \equiv \Sigma_d^0$ (resp. $\Sigma_a^* \equiv \Sigma_a^0$). Furthermore, for any $\sigma_d^* \in \Sigma_d^*$ and any $\sigma_a^* \in \Sigma_a^*$, (σ_d^*, σ_a^*) is an equilibrium strategy profile of Γ.*

Based on Lemma 6.2, the set of attacker's equilibrium strategies Σ_a^* can be expressed as the optimal solution set of a linear program.

Proposition 6.2. *The set Σ_a^* is the optimal solution set of the following optimization problem:*

$$\begin{aligned}
\max_{\sigma_a} \quad & V(\sigma_a) \\
s.t. \quad & V(\sigma_a) = \sum_{e \in \bar{E}} \min\{\sigma_a(e) \cdot (C_\emptyset - p_a) + p_d,\ \sigma_a(e) \cdot (C_e - p_a)\} + \sigma_a(\emptyset) \cdot C_\emptyset, && (6.10a)\\
& \sum_{e \in \bar{E}} \sigma_a(e) + \sigma_a(\emptyset) = 1, && (6.10b)\\
& \sigma_a(\emptyset) \geq 0, \quad \sigma_a(e) \geq 0, \quad \forall e \in \bar{E}. && (6.10c)
\end{aligned}$$

Furthermore, (6.10) *is equivalent to the following linear optimization program:*

$$\begin{aligned}
\max_{\sigma_a, v} \quad & \sum_{e \in \bar{E}} v_e + \sigma_a(\emptyset) \cdot C_\emptyset \\
s.t. \quad & \sigma_a(e) \cdot (C_\emptyset - p_a) + p_d - v_e \geq 0, \quad \forall e \in \bar{E}, && (6.11a)
\end{aligned}$$

$$\sigma_a(e) \cdot (C_e - p_a) - v_e \geq 0, \quad \forall e \in \bar{E}, \tag{6.11b}$$

$$\sum_{e \in \bar{E}} \sigma_a(e) + \sigma_a(\emptyset) = 1, \tag{6.11c}$$

$$\sigma_a(\emptyset) \geq 0, \quad \sigma_a(e) \geq 0, \quad \forall e \in \bar{E}. \tag{6.11d}$$

where $v = (v_e)_{e \in \bar{E}}$ is an $|\bar{E}|$-dimensional variable.

In Proposition 6.2, the objective function $V(\sigma_a)$ is a piecewise linear function in σ_a. Furthermore, given any σ_a and any $e \in \bar{E}$, we can write:

$$\begin{aligned} &\min \left\{\sigma_a(e) \cdot (C_\emptyset - p_a) + p_d,\ \sigma_a(e) \cdot (C_e - p_a)\right\} \\ =& \begin{cases} \sigma_a(e) \cdot (C_\emptyset - p_a) + p_d & \text{if } \sigma_a(e) > \frac{p_d}{C_e - C_\emptyset}, \\ \sigma_a(e) \cdot (C_e - p_a) & \text{if } \sigma_a(e) \leq \frac{p_d}{C_e - C_\emptyset}. \end{cases} \end{aligned} \tag{6.12}$$

Thus, we can observe that if $\sigma_a(e)$ equals to $p_d/\left(C_e - C_\emptyset\right)$, then $-\sigma_a(e) \cdot C_\emptyset - p_d = -\sigma_a(e) \cdot C_e$, i.e. if a facility e is targeted with the threshold attack probability $p_d/(C_e - C_\emptyset)$, the defender is indifferent between securing e versus not. The following lemma analyzes the defender's best response to the attacker's strategy, and shows that no facility is targeted with probability higher than the threshold probability in equilibrium.

Lemma 6.3. *Given any strategy of the attacker $\sigma_a \in \Delta(S_a)$, for any defender's security effort ρ that is a best response to σ_a, denoted $\rho \in BR(\sigma_a)$, the security effort ρ_e on each facility $e \in E$ satisfies:*

$$\rho_e \begin{cases} = 0, & \forall e \in \left\{\bar{E} | \sigma_a(e) < \frac{p_d}{C_e - C_\emptyset}\right\} \cup \widehat{E} \cup \check{E}, \\ \in [0,1], & \forall e \in \left\{\bar{E} | \sigma_a(e) = \frac{p_d}{C_e - C_\emptyset}\right\}, \\ = 1, & \forall e \in \left\{\bar{E} | \sigma_a(e) > \frac{p_d}{C_e - C_\emptyset}\right\}. \end{cases} \tag{6.13}$$

Furthermore, in equilibrium, the attacker's strategy σ_a^* satisfies:

$$\sigma_a^*(e) \leq \frac{p_d}{C_e - C_\emptyset}, \quad \forall e \in \bar{E}, \tag{6.14a}$$

$$\sigma_a^*(e) = 0, \qquad \forall e \in E \setminus \bar{E}. \tag{6.14b}$$

Lemma 6.3 highlights a key property of NE: The attacker does not target at any facility $e \in \bar{E}$ with probability higher than the threshold $p_d/(C_e - C_\emptyset)$, and the defender allocates a non-zero security effort only on the facilities that are targeted with the threshold probability.

Intuitively, if a facility e were to be targeted with a probability higher than the threshold $p_d/(C_e - C_\emptyset)$, then the defender's best response would be to secure that facility with probability 1, and the attacker's expected utility will be $-C_\emptyset - p_a\sigma_a(e)$, which is smaller than $-C_\emptyset$ (utility of no attack). Hence, the attacker would be better off by choosing the no attack action.

Now, we can re-write $V(\sigma_a)$ as defined in (6.10) as follows:

$$V(\sigma_a) \overset{(6.14)}{=} \sum_{e \in \{\bar{E} | \sigma_a(e) \leq \frac{p_d}{C_e - C_\emptyset}\}} \sigma_a(e) \left(C_e - p_a\right) + C_\emptyset \cdot \sigma_a(\emptyset), \tag{6.15}$$

and the set of attacker's equilibrium strategies maximizes this function.

6.4.2 Equilibrium Characterization

We are now in the position to introduce the equilibrium regimes. Each regime corresponds to a range of cost parameters such that the qualitative properties of equilibrium (i.e. the set of facilities that are targeted and secured) do not change in the interior of each regime.

We say that a facility e is *vulnerable* if $C_e - p_a > C_\emptyset$. Therefore, given any attack cost p_a, the set of vulnerable facilities is given by $\{\bar{E} | C_e - p_a > C_\emptyset\}$. Clearly, only vulnerable facilities are likely targets of the attacker. If $p_a > C_{(1)} - C_\emptyset$, then there are no vulnerable facilities. In contrast, if $p_a < C_{(1)} - C_\emptyset$, we define the following threshold for the per-facility defense cost:

$$\bar{p}_d(p_a) \triangleq \frac{1}{\sum_{e \in \{\bar{E} | C_e - p_a > C_\emptyset\}} \frac{1}{C_e - C_\emptyset}}. \tag{6.16}$$

We can check that for any $i = 1, \ldots, K-1$ (resp. $i = K$), if $C_{(i+1)} - C_\emptyset \leq p_a < C_{(i)} - C_\emptyset$

(resp. $0 < p_a < C_{(K)} - C_\emptyset$), then

$$\bar{p}_d(p_a) = \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1}. \tag{6.17}$$

Recall from Lemma 6.3 that $\sigma_a^*(e)$ is upper bounded by the threshold attack probability $p_d/(C_e - C_\emptyset)$. If the defense cost $p_d < \bar{p}_d(p_a)$, then $\sum_{k=1}^{i} \frac{E_{(k)} p_d}{C_{(k)} - C_\emptyset} < 1$, which implies that even when the attacker targets each vulnerable facility with the threshold attack probability, the total probability of attack is still smaller than 1. Thus, the attacker must necessarily choose not to attack with a positive probability. On the other hand, if $p_d > \bar{p}_d(p_a)$, then the no attack action is not chosen by the attacker in equilibrium.

Following the above discussion, we introduce two types of regimes depending on whether or not p_d is higher than the threshold $\bar{p}_d(p_a)$. In type I regimes, denoted as $\{\Lambda^i | i = 0, \ldots, K\}$, the defense cost $p_d < \bar{p}_d(p_a)$, whereas in type II regimes, denoted as $\{\Lambda_j | j = 1, \ldots, K\}$, the defense cost $p_d > \bar{p}_d(p_a)$. Hence, we say that p_d is "relatively low" (resp. "relatively high") in comparison to p_a in type I regimes (resp. type II regimes). We formally define these $2K + 1$ regimes as follows:

1. Type I regimes Λ^i, $i = 0, \ldots, K$:

   - If $i = 0$:

$$p_a > C_{(1)} - C_\emptyset, \text{ and } p_d > 0 \tag{6.18}$$

   - If $i = 1, \ldots, K - 1$:

$$C_{(i+1)} - C_\emptyset < p_a < C_{(i)} - C_\emptyset, \text{ and } 0 < p_d < \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1} \tag{6.19}$$

   - If $i = K$:

$$0 < p_a < C_{(K)} - C_\emptyset, \text{ and } 0 < p_d < \left(\sum_{k=1}^{K} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1} \tag{6.20}$$

2. Type II regimes, Λ_j, $j = 1, \dots, K$:

- If $j = 1$:

$$0 < p_a < C_{(1)} - C_{\emptyset}, \text{ and } p_d > \left(\frac{E_{(1)}}{C_{(1)} - C_{\emptyset}} \right)^{-1} \tag{6.21}$$

- If $j = 2, \dots, K$:

$$0 < p_a < C_{(j)} - C_{\emptyset}, \text{ and } \left(\sum_{k=1}^{j} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}} \right)^{-1} < p_d < \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}} \right)^{-1} \tag{6.22}$$

We now characterize equilibrium strategy sets Σ_d^* and Σ_a^* in the interior of each regime.[1]

Theorem 6.1. *The set of NE in each regime is as follows:*

1. Type I regimes Λ^i:

- If $i = 0$,

$$\rho_e^* = 0, \quad \forall e \in E \tag{6.23a}$$

$$\sigma_a^*(\emptyset) = 1. \tag{6.23b}$$

- If $i = 1, \dots, K$,

$$\rho_e^* = \frac{C_{(k)} - p_a - C_{\emptyset}}{C_{(k)} - C_{\emptyset}}, \qquad \forall e \in \bar{E}_{(k)}, \quad \forall k = 1, \dots, i \tag{6.24a}$$

$$\rho_e^* = 0, \qquad \forall e \in E \setminus \left(\cup_{k=1}^{i} \bar{E}_{(k)} \right) \tag{6.24b}$$

$$\sigma_a^*(e) = \frac{p_d}{C_{(k)} - C_{\emptyset}}, \qquad \forall e \in \bar{E}_{(k)}, \quad \forall k = 1, \dots, i \tag{6.24c}$$

$$\sigma_a^*(\emptyset) = 1 - \sum_{e \in \cup_{k=1}^{i} \bar{E}_{(k)}} \sigma_a^*(e). \tag{6.24d}$$

2. Type II regimes Λ_j:

[1] For the sake of brevity, we omit the discussion of equilibrium strategies when cost parameters lie exactly on the regime boundary, although this case can be addressed using the approach developed in this article.

- $j = 1$:

$$\rho_e^* = 0, \qquad \forall e \in E \tag{6.25a}$$

$$0 \leq \sigma_a^*(e) \leq \frac{p_d}{C_{(1)} - C_\emptyset} \quad \forall e \in \bar{E}_{(1)}, \tag{6.25b}$$

$$\sum_{e \in \bar{E}_{(1)}} \sigma_a^*(e) = 1. \tag{6.25c}$$

- $j = 2, \ldots, K$:

$$\rho_e^* = \frac{C_{(k)} - C_{(j)}}{C_{(k)} - C_\emptyset}, \quad \forall e \in \bar{E}_{(k)}, \quad \forall k = 1, \ldots, j-1 \tag{6.26a}$$

$$\rho_e^* = 0, \qquad \forall e \in E \setminus \left(\cup_{k=1}^{j-1} \bar{E}_{(k)}\right) \tag{6.26b}$$

$$\sigma_a^*(e) = \frac{p_d}{C_{(k)} - C_\emptyset}, \qquad \forall e \in \bar{E}_{(k)}, \quad \forall k = 1, \ldots, j-1 \tag{6.26c}$$

$$0 \leq \sigma_a^*(e) \leq \frac{p_d}{C_{(j)} - C_\emptyset}, \qquad \forall e \in \bar{E}_{(j)} \tag{6.26d}$$

$$\sum_{e \in \bar{E}_{(j)}} \sigma_a^*(e) = 1 - \sum_{k=1}^{j-1} \frac{p_d \cdot E_{(k)}}{C_{(k)} - C_\emptyset}. \tag{6.26e}$$

Let us discuss the intuition behind the proof of Theorem 6.1.

Recall from Proposition 6.2 and Lemma 6.3 that the set of attacker's equilibrium strategies Σ_a^* is the set of feasible mixed strategies that maximizes $V(\sigma_a)$ in (6.15), and the attacker never targets at any facility $e \in E$ with probability higher than the threshold $p_d/(C_e - C_\emptyset)$. Also recall that the costs $\{C_{(k)}\}_{k=1}^K$ are ordered according to (6.8). Thus, in equilibrium, the attacker targets the facilities in $\bar{E}_{(k)}$ with the threshold attack probability starting from $k = 1$ and proceeding to $k = 2, 3, \ldots K$ until either all the vulnerable facilities are targeted with the threshold attack probability (and no attack is chosen with remaining probability), or the total attack probability reaches 1.

Again, from Lemma 6.3, we know that the defender secures the set of facilities that are targeted with the threshold attack probability with positive effort. The equilibrium level of security effort ensures that the attacker gets an identical utility in choosing any pure strategy in the support of σ_a^*, and this utility is higher or equal to that of choosing any other pure strategy.

The distinctions between the two regime types are summarized as follows:

1. In type I regimes, the defense cost $p_d < \bar{p}_d(p_a)$. The defender secures all vulnerable facilities with a positive level of effort. The attacker targets at each vulnerable facility with the threshold attack probability, and the total probability of attack is less than 1.

2. In type II regimes, the defense cost $p_d > \bar{p}_d(p_a)$. The defender only secures a subset of targeted facilities with positive level of security effort. The attacker chooses the facilities in decreasing order of $C_e - C_\emptyset$, and targets each of them with the threshold probability until the attack resource is exhausted, i.e. the total probability of attack is 1.

6.5 Sequential game $\widetilde{\Gamma}$

In this section, we characterize the set of SPE in the game $\widetilde{\Gamma}$ for any given attack and defense cost parameters. The sequential game $\widetilde{\Gamma}$ is no longer strategically equivalent to a zero-sum game. Hence, the proof technique we used for equilibrium characterization in game Γ does not work for the game $\widetilde{\Gamma}$. In Sec. 6.5.1, we analyze the attacker's best response to the defender's security effort vector. We also identify a threshold level of security effort which determines whether or not the defender achieves full attack deterrence in equilibrium. In Sec. 6.5.2, we present the equilibrium regimes which govern the qualitative properties of SPE.

6.5.1 Properties of $\widetilde{\Gamma}$

By definition of SPE, for any security effort vector $\tilde{\rho} \in [0,1]^{|E|}$ chosen by the defender in the first stage, the attacker's equilibrium strategy in the second stage is a best response to $\tilde{\rho}$, i.e. $\widetilde{\sigma}_a^*(\tilde{\rho})$ satisfies (6.3b). As we describe next, the properties of SPE crucially depend on a threshold security effort level defined as follows:

$$\widehat{\rho}_e \triangleq \frac{C_e - p_a - C_\emptyset}{C_e - C_\emptyset}, \quad \forall e \in \bar{E}. \tag{6.27}$$

The following lemma presents the best response correspondence $BR(\tilde{\rho})$ of the attacker:

Lemma 6.4. *Given any $\tilde{\rho} \in [0,1]^{|E|}$, if $\tilde{\rho}$ satisfies $\tilde{\rho}_e \geq \widehat{\rho}_e$, for all $e \in \{\bar{E}|C_e - p_a > C_\emptyset\}$, then $BR(\tilde{\rho}) = \Delta(\bar{E}^* \cup \{\emptyset\})$, where:*

$$\bar{E}^* \triangleq \left\{\bar{E}\,|C_e - p_a > C_\emptyset, \quad \tilde{\rho}_e = \widehat{\rho}_e\right\}. \tag{6.28}$$

Otherwise, $BR(\tilde{\rho}) = \Delta(\bar{E}^\diamond)$, where:

$$\bar{E}^\diamond \triangleq \underset{e \in \{\bar{E}|C_e - p_a > C_\emptyset\}}{\operatorname{argmax}} \left\{\tilde{\rho}_e C_\emptyset + (1-\tilde{\rho}_e) C_e\right\}. \tag{6.29}$$

In words, if each vulnerable facility e is secured with an effort higher or equal to the threshold effort $\widehat{\rho}_e$ in (6.27), then the attacker's best response is to choose a mixed strategy with support comprised of all vulnerable facilities that are secured with the threshold level of effort (i.e., $\bar{E}^*$ as defined in (6.28)) and the no attack action. Otherwise, the support of attacker's strategy is comprised of all vulnerable facilities (pure actions) that maximize the expected usage cost (see (6.29)). In particular, no attack action is not chosen in attacker's best response.

Now recall that any SPE $(\tilde{\rho}^*, \widetilde{\sigma}_a^*(\tilde{\rho}^*))$ must satisfy both (6.3a) and (6.3b). Thus, for an equilibrium security effort $\tilde{\rho}^*$, an attacker's best response $\widetilde{\sigma}_a(\tilde{\rho}^*) \in BR(\tilde{\rho}^*)$ is an equilibrium strategy only if both these constraints are satisfied. The next lemma shows that depending on whether the defender secures each vulnerable facility e with the threshold effort $\widehat{\rho}_e$ or not, the total attack probability in equilibrium is either 0 or 1. Thus, the defender being the first mover determines whether the attacker is fully deterred from conducting an attack or not. Additionally, in SPE, the security effort on each vulnerable facility e is no higher than the threshold effort $\widehat{\rho}_e$, and the security effort on any other edge is 0.

Lemma 6.5. *Any SPE $(\tilde{\rho}^*, \widetilde{\sigma}_a^*(\tilde{\rho}^*))$ of the game $\widetilde{\Gamma}$ satisfies the following property:*

$$\sum_{e\in\bar{E}} \widetilde{\sigma}_a^*(e, \tilde{\rho}^*) = \begin{cases} 0, & \text{if } \tilde{\rho}_e^* \geq \widehat{\rho}_e, \quad \forall e \in \{\bar{E} | C_e - p_a > C_\emptyset\}, \\ 1, & \text{otherwise.} \end{cases}$$

Additionally, for any $e \in \{\bar{E}|C_e - p_a > C_\emptyset\}$, $\tilde{\rho}_e^* \leq \widehat{\rho}_e$. For any $e \in E \setminus \{\bar{E}|C_e - p_a > C_\emptyset\}$, $\tilde{\rho}_e^* = 0$.

The proof of this result is based on the analysis of following three cases:

<u>Case 1</u>: There exists at least one facility $e \in \{\bar{E}|C_e - p_a > C_\emptyset\}$ such that $\tilde{\rho}_e^* < \widehat{\rho}_e$. In this case, by applying Lemma 6.4, we know that $\widetilde{\sigma}_a^*(\tilde{\rho}^*) \in BR(\tilde{\rho}^*) = \Delta(\bar{E}^\diamond)$, where $\bar{E}^\diamond$ is defined in (6.29). Hence, the total attack probability is 1.

<u>Case 2</u>: For any $e \in \{\bar{E}|C_e - p_a > C_\emptyset\}$, $\tilde{\rho}_e^* > \widehat{\rho}_e$. In this case, the set $\bar{E}^*$ defined in (6.28) is empty. Hence, Lemma 6.4 shows that the total attack probability is 0.

<u>Case 3</u>: For any $e \in \{\bar{E}|C_e - p_a > C_\emptyset\}$, $\tilde{\rho}_e^* \geq \widehat{\rho}_e$, and the set $\bar{E}^*$ in (6.28) is non-empty. Again from Lemma 6.4, we know that $\widetilde{\sigma}_a^*(\tilde{\rho}^*) \in BR(\tilde{\rho}^*) = \Delta(\bar{E}^* \cup \{\emptyset\})$. Now assume that the attacker chooses to target at least one facility $e \in \bar{E}^*$ with a positive probability in equilibrium. Then, the defender can deviate by slightly increasing the security effort on each facility in $\bar{E}^*$. By introducing such a deviation, the defender's security effort satisfies the condition of Case 2, where the total attack probability is 0. Hence, this results in a higher utility for the defender. Therefore, in any SPE $(\tilde{\rho}^*, \widetilde{\sigma}_a^*(\tilde{\rho}^*))$, one cannot have a second stage outcome in which the attacker targets facilities in $\bar{E}^*$. We can thus conclude that the total attack probability must be 0 in this case.

In both Cases 2 and 3, we say that the attacker is *fully deterred*.

Clearly, these three cases are exhaustive in that they cover all feasible security effort vectors, and hence we can conclude that the total attack probability in equilibrium is either 0 or 1. Additionally, since the attacker is fully deterred when each vulnerable facility is secured with the threshold effort, the defender will not further increase the security effort beyond the threshold effort on any vulnerable facility. That is, only Cases 1 and 3 are possible in equilibrium.

6.5.2 Equilibrium Characterization

Recall that in Sec. 6.4, type I and type II regimes for the game Γ can be distinguished based on a threshold defense cost $\bar{p}_d(p_a)$. It turns out that in $\widetilde{\Gamma}$, there are still $2K+1$ regimes. Again, each regime denotes distinct ranges of cost parameters, and can be categorized either as type $\widetilde{\mathrm{I}}$ or type $\widetilde{\mathrm{II}}$. However, in contrast to Γ, the regime boundaries in this case are more complicated; in particular, they are non-linear in the cost parameters p_a and p_d.

To introduce the boundary $\widetilde{p}_d(p_a)$, we need to define the function $p_d^{ij}(p_a)$ for each $i = 1, \dots, K$ and $j = 1, \dots, i$ as follows:

$$p_d^{ij}(p_a) = \begin{cases} \frac{C_{(1)} - C_\emptyset}{\sum_{k=1}^{i} E_{(k)} - \sum_{k=1}^{i} \frac{p_a E_{(k)}}{C_{(k)} - C_\emptyset}}, & \text{if } j = 1, \\ \\ \frac{C_{(j)} - C_\emptyset}{(C_{(j)} - C_\emptyset) \cdot \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right) + \sum_{k=j}^{i} E_{(k)} - \sum_{k=1}^{i} \frac{p_a E_{(k)}}{C_{(k)} - C_\emptyset}}, & \text{if } j = 2, \dots, i. \end{cases} \tag{6.30}$$

For any $i = 1, \dots, K$, and any attack cost $C_{(i+1)} - C_\emptyset \leq p_a < C_{(i)} - C_\emptyset$, but $0 < p_a < C_{(K)} - C_\emptyset$ if $i = K$, the threshold $\widetilde{p}_d(p_a)$ is defined as follows:

$$\widetilde{p}_d(p_a) = \begin{cases} p_d^{ij}(p_a), & \text{if } \frac{\sum_{k=j+1}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}} \leq p_a < \frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}}, \text{ and } j = 1, \dots, i-1, \\ p_d^{ii}(p_a), & \text{if } 0 \leq p_a < \frac{E_{(i)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}}. \end{cases} \tag{6.31}$$

Lemma 6.6. *Given any attack cost $0 \leq p_a < C_{(1)} - C_\emptyset$, the threshold $\widetilde{p}_d(p_a)$ is a strictly increasing and continuous function of p_a.*

Furthermore, for any $0 < p_a < C_{(1)} - C_\emptyset$, $\widetilde{p}_d(p_a) > \bar{p}_d(p_a)$. If $p_a = 0$, $\widetilde{p}_d(0) = \bar{p}_d(0)$. If $p_a \to C_{(1)} - C_\emptyset$, $\widetilde{p}_d(p_a) \to +\infty$.

Since $\widetilde{p}_d(p_a)$ is a strictly increasing and continuous function function of p_a, the inverse function $\widetilde{p}_d^{-1}(p_d)$ is well-defined. Now we are ready to formally define the regimes for the game $\widetilde{\Gamma}$:

1. Type $\widetilde{\mathrm{I}}$ regimes $\widetilde{\Lambda}^i$, $i = 0, \dots, K$:

- If $i = 0$:

$$p_a > C_{(1)} - C_\emptyset, \text{ and } \quad p_d > 0. \tag{6.32}$$

- If $i = 1, \dots, K-1$:

$$C_{(i+1)} - C_\emptyset < p_a < C_{(i)} - C_\emptyset, \text{ and } \quad 0 < p_d < \widetilde{p}_d(p_a). \tag{6.33}$$

- If $i = K$:

$$0 < p_a < C_{(K)} - C_\emptyset, \text{ and } \quad 0 < p_d < \widetilde{p}_d(p_a). \tag{6.34}$$

2. Type $\widetilde{\text{II}}$ regimes $\widetilde{\Lambda}_j$, $j = 1, \dots, K$:

- If $j = 1$:

$$0 < p_a < \widetilde{p}_d^{-1}(p_d), \text{ and } \quad p_d > \left(\frac{E_{(1)}}{C_{(1)} - C_\emptyset}\right)^{-1} \tag{6.35}$$

- If $j = 2, \dots, K$:

$$0 < p_a < \widetilde{p}_d^{-1}(p_d), \text{ and } \quad \left(\sum_{k=1}^{j} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}\right)^{-1} < p_d < \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}\right)^{-1} \tag{6.36}$$

Analogous to the discussion in Section 6.4.2, we say p_d is "relatively low" in type $\widetilde{\text{I}}$ regimes, and "relatively high" in type $\widetilde{\text{II}}$ regimes. We now provide full characterization of SPE in each regime.

Theorem 6.2. *The defender's equilibrium security effort vector $\tilde{\rho}^* = (\tilde{\rho}_e^*)_{e \in E}$ is unique in each regime. Specifically, SPE in each regime is as follows:*

1. Type $\widetilde{\text{I}}$ regimes $\widetilde{\Lambda}^i$:

- *If* $i = 0$,

$$\tilde{\rho}_e^* = 0, \quad \forall e \in E, \tag{6.37a}$$

$$\widetilde{\sigma}_a^*(\emptyset, \tilde{\rho}) = 1, \quad \forall \tilde{\rho} \in [0,1]^{|E|}. \tag{6.37b}$$

- *If* $i = 1, \ldots, K$,

$$\tilde{\rho}_e^* = \frac{C_{(k)} - p_a - C_\emptyset}{C_{(k)} - C_\emptyset}, \quad \forall e \in \bar{E}_{(k)}, \quad \forall k = 1, \ldots, i, \tag{6.38a}$$

$$\tilde{\rho}_e^* = 0, \quad \forall e \in E \setminus \left(\cup_{k=1}^{i} \bar{E}_{(k)}\right), \tag{6.38b}$$

$$\widetilde{\sigma}_a^*(\emptyset, \tilde{\rho}^*) = 1, \tag{6.38c}$$

$$\widetilde{\sigma}_a^*(\tilde{\rho}) \in BR(\tilde{\rho}), \quad \forall \tilde{\rho} \in [0,1]^{|E|} \setminus \tilde{\rho}^*. \tag{6.38d}$$

2. *Type* $\widetilde{\text{II}}$ *regimes* $\widetilde{\Lambda}_j$*:*

- *If* $j = 1$,

$$\tilde{\rho}_e^* = 0, \quad \forall e \in E, \tag{6.39a}$$

$$\widetilde{\sigma}_a^*(\tilde{\rho}^*) \in \Delta(\bar{E}_{(1)}), \tag{6.39b}$$

$$\widetilde{\sigma}_a^*(\tilde{\rho}) \in BR(\tilde{\rho}), \quad \forall \tilde{\rho} \in [0,1]^{|E|} \setminus \tilde{\rho}^*. \tag{6.39c}$$

- *If* $j = 2, \ldots, K$,

$$\tilde{\rho}_e^* = \frac{C_{(k)} - C_{(j)}}{C_{(k)} - C_\emptyset}, \quad \forall e \in \bar{E}_{(k)}, \quad \forall k = 1, \ldots, j-1, \tag{6.40a}$$

$$\tilde{\rho}_e^* = 0, \quad \forall e \in E \setminus \left(\cup_{k=1}^{j-1} \bar{E}_{(k)}\right), \tag{6.40b}$$

$$\widetilde{\sigma}_a^*(\tilde{\rho}^*) \in \Delta\left(\cup_{k=1}^{j} \bar{E}_{(k)}\right), \tag{6.40c}$$

$$\widetilde{\sigma}_a^*(\tilde{\rho}) \in BR(\tilde{\rho}), \quad \forall \tilde{\rho} \in [0,1]^{|E|} \setminus \tilde{\rho}^*. \tag{6.40d}$$

In our proof of Theorem 6.2 (see Appendix D.3), we take the approach by first constructing a partition of the space $(p_a, p_d) \in \mathbb{R}^2_{>0}$ defined in (D.7), and then characterizing the SPE for cost parameters in each set in the partition (Lemmas D.1–D.2). Theorem 6.2 follows directly by regrouping/combining the elements of this partition such that each of the new partition has qualitatively identical equilibrium strategies.

From the discussion of Lemma 6.5, we know that only Cases 1 and 3 are possible in equilibrium, and that in any SPE, the security effort on each vulnerable facility e is no higher than the threshold effort $\widehat{\rho}_e$. It turns out that for any attack cost, depending on whether the defense cost is lower or higher than the threshold cost $\widetilde{p}_d(p_a)$, the defender either secures each vulnerable facility with the threshold effort given by (6.31) (type $\widetilde{\mathrm{I}}$ regime), or there is at least one vulnerable facility that is secured with effort strictly less than the threshold (type $\widetilde{\mathrm{II}}$ regimes):

1. In type $\widetilde{\mathrm{I}}$ regimes, the defense cost $p_d < \widetilde{p}_d(p_a)$. The defender secures each vulnerable facility with the threshold effort $\widehat{\rho}_e$. The attacker is fully deterred.

2. In type $\widetilde{\mathrm{II}}$ regimes, the defense cost $p_d > \widetilde{p}_d(p_a)$. The defender's equilibrium security effort is identical to that in NE of the normal form game Γ. The total attack probability is 1.

6.6 Comparison of Γ and $\widetilde{\Gamma}$

Sec. 6.6.1 deals with the comparison of players' equilibrium utilities in the two games. In Sec. 6.6.2, we compare the equilibrium regimes and discuss the distinctions in equilibrium properties of the two games. This leads us to an understanding of the effect of timing of play, i.e. we can identify situations in which the defender gains by proactively investing in securing all of the vulnerable facilities at an appropriate level of effort.

6.6.1 Comparison of Equilibrium Utilities

The equilibrium utilities in both games are unique, and can be directly derived using Theorems 6.1 and 6.2. We denote the equilibrium utilities of the defender and attacker in regime Λ^i (resp. Λ_j) as $U_d^{\Lambda^i}$ and $U_a^{\Lambda^i}$ (resp. $U_d^{\Lambda_j}$ and $U_a^{\Lambda_j}$) in Γ, and $\widetilde{U}_d^{\widetilde{\Lambda}^i}$ and $\widetilde{U}_a^{\widetilde{\Lambda}^i}$ (resp. $\widetilde{U}_d^{\widetilde{\Lambda}_j}$ and $\widetilde{U}_a^{\widetilde{\Lambda}_j}$) in regime $\widetilde{\Lambda}^i$ (resp. $\widetilde{\Lambda}_j$) in $\widetilde{\Gamma}$.

Proposition 6.3. *In both Γ and $\widetilde{\Gamma}$, the equilibrium utilities are unique in each regime. Specifically,*

1. *Type I ($\widetilde{\mathrm{I}}$) regimes Λ^i ($\widetilde{\Lambda}^i$):*

   - *If $i = 0$:*

$$U_d^{\Lambda_0} = \widetilde{U}_d^{\widetilde{\Lambda}^0} = -C_\emptyset, \text{ and } \quad U_a^{\Lambda_0} = \widetilde{U}_a^{\widetilde{\Lambda}^0} = C_\emptyset.$$

   - *If $i = 1, \ldots, K$:*

$$\begin{aligned} U_d^{\Lambda^i} &= -C_\emptyset - \left(\sum_{k=1}^{i} E_{(k)} \right) p_d, & \text{and} \quad U_a^{\Lambda^i} = C_\emptyset, \\ \widetilde{U}_d^{\widetilde{\Lambda}^i} &= -C_\emptyset - \left(\sum_{k=1}^{i} \frac{(C_e - p_a - C_\emptyset)\, E_{(k)}}{C_e - C_\emptyset} \right) p_d, & \text{and} \quad \widetilde{U}_a^{\widetilde{\Lambda}^i} = C_\emptyset. \end{aligned}$$

2. *Type II ($\widetilde{\mathrm{II}}$) regimes Λ_j ($\widetilde{\Lambda}_j$):*

   - *If $j = 1$:*

$$U_d^{\Lambda_1} = \widetilde{U}_d^{\widetilde{\Lambda}_1} = -C_{(1)}, \text{ and } \quad U_a^{\Lambda_1} = \widetilde{U}_a^{\widetilde{\Lambda}_1} = C_{(1)} - p_a.$$

   - *If $j = 2, \ldots, K$:*

$$U_d^{\Lambda_j} = \widetilde{U}_d^{\widetilde{\Lambda}_j} = -C_{(j)} - \sum_{k=1}^{j-1} \frac{\left(C_{(k)} - C_{(j)}\right) p_d E_{(k)}}{C_{(k)} - C_\emptyset}, \text{ and } \quad U_a^{\Lambda_j} = \widetilde{U}_a^{\widetilde{\Lambda}_j} = C_{(j)} - p_a.$$

From our results so far, we can summarize the similarities between the equilibrium outcomes in Γ and $\widetilde{\Gamma}$. While most of these conclusions are fairly intuitive, the fact that they are common to both game-theoretic models suggests that the timing of defense investments do not play a role as far as these insights are concerned. Firstly, the support of both players equilibrium strategies tends to contain the facilities, whose compromise results in a high usage cost. The defender secures these facilities with a high level of effort in order to reduce the probability with which they are targeted by the attacker. Secondly, the attack and defense costs jointly determine the set of facilities that are targeted or secured in equilibrium. On one hand, the set of vulnerable facilities increases as the cost of attack decreases. On the other hand, when the cost of defense is sufficiently high, the attacker tends to conduct an attack with probability 1. However, as the defense cost decreases, the attacker randomizes the attack on a larger set of facilities. Consequently, the defender secures a larger set of facilities with positive effort, and when the cost of defense is sufficiently small, all vulnerable facilities are secured by the defender. Thirdly, each player's equilibrium payoff is non-decreasing in the opponent's cost, and non-increasing in her own cost. Therefore, to increase her equilibrium payoff, each player is better off as her own cost decreases and the opponent's cost increases.

6.6.2 First Mover Advantage

We now focus on identifying parameter ranges in which the defender has the first mover advantage, i.e., the defender in SPE has a strictly higher payoff than in NE. To identify the first mover advantage, let us recall the expressions of type I regimes for Γ in (6.18)–(6.20) and type $\widetilde{\mathrm{I}}$ regimes for $\widetilde{\Gamma}$ in (6.32)–(6.34). Also recall that, for any given cost parameters p_a and p_d, the threshold $\bar{p}_d(p_a)$ (resp. $\widetilde{p}_d(p_a)$) determines whether the equilibrium outcome is of type I or type II regime (resp. type $\widetilde{\mathrm{I}}$ or $\widetilde{\mathrm{II}}$ regime) in the game Γ (resp. $\widetilde{\Gamma}$). Furthermore, from Lemma 6.6, we know that the cost threshold $\bar{p}_d(p_a)$ in Γ is smaller than the threshold $\widetilde{p}_d(p_a)$ in $\widetilde{\Gamma}$. Thus, for all $i = 1, \ldots, K$, the type I regime Λ^i in Γ is a proper subset of the type $\widetilde{\mathrm{I}}$ regime $\widetilde{\Lambda}^i$ in $\widetilde{\Gamma}$. Consequently, for any $(p_a, p_d) \in \mathbb{R}^2_{>0}$, we can have one of the following three cases:

(a) $0 < p_d < \bar{p}_d(p_a)$: The defense cost is relatively low in both Γ and $\widetilde{\Gamma}$. We denote the set of (p_a, p_d) that satisfy this condition as L (*low* cost). That is,

$$L \triangleq \{(p_a, p_d) \,|0 < p_d < \bar{p}_d(p_a)\} = \cup_{i=0}^{K} \Lambda^i. \tag{6.41}$$

(b) $\bar{p}_d(p_a) < p_d < \widetilde{p}_d(p_a)$: The defense cost is relatively high in Γ, but relatively low in $\widetilde{\Gamma}$. We denote the set of (p_a, p_d) that satisfy this condition as M (*medium* cost). That is,

$$M \triangleq \{(p_a, p_d)\,|\bar{p}_d(p_a) < p_d < \widetilde{p}_d(p_a)\} = \cup_{i=1}^{K} \left(\widetilde{\Lambda}^i \setminus \Lambda^i\right). \tag{6.42}$$

(c) $p_d > \widetilde{p}_d(p_a)$: The defense cost is relatively high in both Γ and $\widetilde{\Gamma}$. We denote the set of (p_a, p_d) that satisfy this condition as H (*high* cost). That is,

$$H \triangleq \{(p_a, p_d)\,|p_d > \widetilde{p}_d(p_a)\} = \cup_{j=1}^{K} \widetilde{\Lambda}_j.$$

We next compare the properties of NE and SPE for cost parameters in each set based on Theorems 6.1 and 6.2, and Propositions 6.3.

- Set L:

  Attacker: In Γ, the total attack probability is nonzero but smaller than 1, whereas in $\widetilde{\Gamma}$, the attacker is fully deterred. The attacker's equilibrium utility is identical in both games, i.e., $U_a = \widetilde{U}_a$.

  Defender: The defender chooses identical equilibrium security effort in both games, i.e. $\rho^* = \tilde{\rho}^*$, but obtains a higher utility in $\widetilde{\Gamma}$ in comparison to that in Γ, i.e., $U_d < \widetilde{U}_d$.

- Set M:

  Attacker: In Γ, the attacker conducts an attack with probability 1, whereas in $\widetilde{\Gamma}$ the attacker is fully deterred. The attacker's equilibrium utility is lower in $\widetilde{\Gamma}$ in comparison to that in Γ, i.e., $U_a > \widetilde{U}_a$.

  Defender: The defender secures each vulnerable facility with a strictly higher level of effort in $\widetilde{\Gamma}$ than in Γ, i.e. $\tilde{\rho}_e^* > \rho_e^*$ for each vulnerable facility $e \in \{E|C_e - p_a > C_\emptyset\}$. The defender's equilibrium utility is higher in $\widetilde{\Gamma}$ in comparison to that in Γ, i.e., $U_d < \widetilde{U}_d$.

- Set H:

  Attacker: In both games, the attacker conducts an attack with probability 1, and obtains identical utilities, i.e. $U_a = \widetilde{U}_a$.

Defender: The defender chooses identical equilibrium security effort in both games, i.e., $\rho^* = \tilde{\rho}^*$, and obtains identical utilities, i.e. $U_d = \widetilde{U}_d$.

Importantly, the key difference between NE and SPE comes from the fact that in $\widetilde{\Gamma}$, the defender as the leading player is able to influence the attacker's strategy in her favor. Hence, when the defense cost is relatively medium or low (both sets M and L), the defender can proactively secure all vulnerable facilities with the threshold effort to fully deter the attack, which results in a higher defender utility in $\widetilde{\Gamma}$ than in Γ. Thus, we say the defender has the first-mover advantage when the cost parameters lie in the set M or L. However, the reason behind the first-mover advantage differs in each set:

- In set M, the defender needs to proactively secure all vulnerable facilities with strictly higher effort in $\widetilde{\Gamma}$ than that in Γ to fully deter the attacker.

- In set L, the defender secures facilities in $\widetilde{\Gamma}$ with the same level of effort as that in Γ, and the attacker is still deterred with probability 1.

On the other hand, in set H, the defense cost is so high that the defender is not able to secure all targeted facilities with an adequately high level of security effort. Thus, the attacker conducts an attack with probability 1 in both games, and the defender no longer has first-mover advantage.

Finally, for the sake of illustration, we compute the parameter sets L, M, and H for transportation network with three facilities (edges); see Fig. 6-1. If an edge $e \in E$ is not damaged, then the cost function is $\ell_e(w_e)$, which increases in the edge load w_e. If edge e is successfully compromised by the attacker, then the cost function changes to $\ell_e^{\otimes}(w_e)$, which is higher than $\ell_e(w_e)$ for any edge load $w_e > 0$. The network faces a set of non-atomic travelers with total demand $D = 10$. We define the usage cost in this case as the average cost of travelers in Wardrop equilibrium Correa and Stier-Moses [2011]. Therefore, the usage costs corresponding to attacks to different edges are $C_1 = 20$, $C_2 = 19$, $C_3 = 18$ and the pre-attack usage cost is $C_\emptyset = 17$. From (6.8), $K = 3$, and $\bar{E}_{(1)} = \{e_1\}$, $\bar{E}_{(2)} = \{e_2\}$ and $\bar{E}_{(3)} = \{e_3\}$. In Fig. 6-2, we illustrate the regimes of both Γ and $\widetilde{\Gamma}$, and the three sets H, M, and L distinguished by the thresholds $\bar{p}_d(p_a)$ and $\widetilde{p}_d(p_a)$.

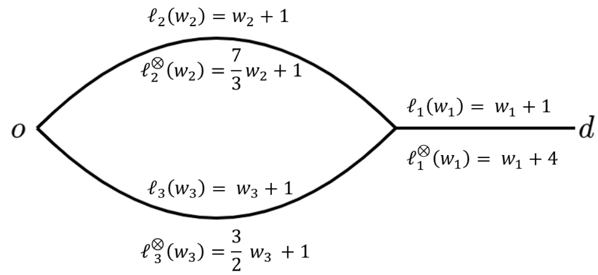


Figure 6-1: Three edge network

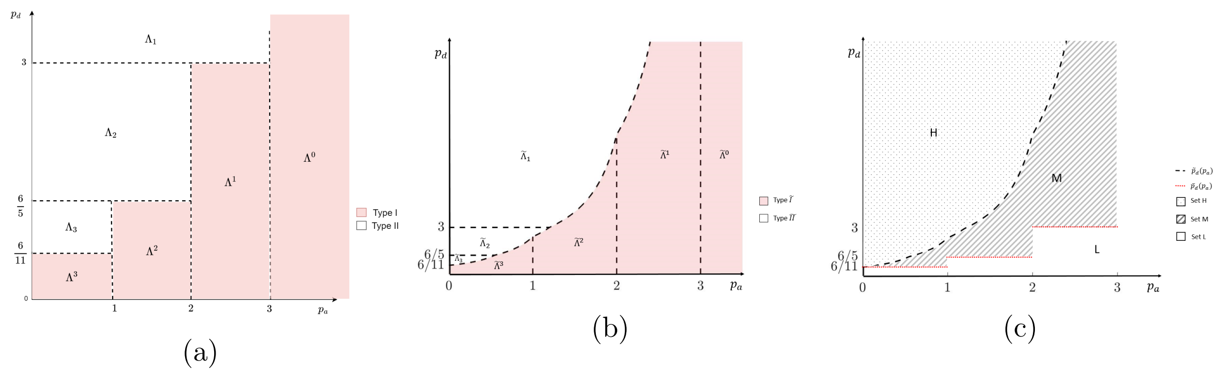


Figure 6-2: (a) Regimes of NE in Γ, (b) Regimes of SPE in $\widetilde{\Gamma}$, (c) Comparison of NE and SPE.

6.7 Model Extensions and Discussion

In this section, we discuss how relaxing our modeling assumptions influence our main results. Our discussion centers around extending our results when the following modeling aspects are included: facility-dependent cost parameters, less than perfect defense, and attacker's ability to target multiple facilities.

Facility-dependent attack and defense costs. Our techniques for equilibrium characterization of games Γ and $\widetilde{\Gamma}$ — as presented in Sections 6.4 and 6.5 respectively — can be generalized to the case when attack/defense costs are non-homogeneous across facilities. We denote the attack (resp. defense) cost for facility $e \in E$ as $p_{a,e}$ (resp. $p_{d,e}$). However, an explicit characterization of equilibrium regimes in each game can be quite complicated due to the multidimensional nature of cost parameters.

In normal form game Γ, it is easy to show that the attacker's best response correspondence in Lemma 6.3 holds except that the threshold attack probability for any facility $e \in \bar{E}$ now becomes $p_{d,e}/(C_e - C_\emptyset)$. The set of vulnerable facilities is given by $\{E|C_e - p_{a,e} > C_\emptyset\}$. The attacker's equilibrium strategy is to order the facilities in decreasing order of $C_e - p_{a,e}$, and target the facilities in this order each with the threshold probability until either all vulnerable facilities are targeted or the total probability of attack reaches 1. As in Theorem 6.1, the former case happens when the cost parameters lie in a type I regime, and the latter case happens for type II regimes, although the regime boundaries are more complicated to describe. In equilibrium, the defender chooses the security effort vector to ensure that the attacker is indifferent among choosing any of the pure actions that are in the support of equilibrium attack strategy.

In the sequential game $\widetilde{\Gamma}$, Lemmas 6.4 and 6.5 can be extended in a straightforward manner except that the threshold security effort for any vulnerable facility $e \in \{E|C_e - C_\emptyset > p_{a,e}\}$ is given by $\widehat{\rho}_e = (C_e - p_{a,e} - C_\emptyset)/(C_e - C_\emptyset)$. The SPE for this general case can be obtained analogously to Theorem 6.2, i.e. comparing the defender's utility of either securing all vulnerable facilities with the threshold effort to fully deter the attack, or choosing a strategy that is identical to that in Γ. These cases happen when the cost parameters lie in (suitably defined) Type $\widetilde{\mathrm{I}}$ and Type $\widetilde{\mathrm{II}}$ regimes, respectively. The main conclusion of our analysis also holds: the defender obtains a higher utility by proactively defending all vulnerable facilities when the facility-dependent cost parameters lie in type $\widetilde{\mathrm{I}}$ regimes.

Less than perfect defense in addition to facility-dependent cost parameters. Now consider that the defense on each facility is only successful with probability $\gamma \in (0, 1)$, which is an exogenous technological parameter. For any security effort vector ρ, the actual probability that a facility e is not compromised when targeted by the attacker is $\gamma\rho_e$. Again our results on NE and SPE in Sec. 6.4 – Sec. 6.5 can be readily extended to this case. However, the expressions for thresholds for attack probability and security effort level need to be modified. In particular, for Γ, in Lemma 6.3, the threshold attack probability on any facility $e \in \bar{E}$ is $p_{d,e}/\gamma(C_e - C_\emptyset)$. For $\widetilde{\Gamma}$, the threshold security effort $\widehat{\rho}_e$ for any vulnerable facility $e \in \{E|C_e - C_\emptyset > p_{d,e}\}$ is $(C_e - p_{a,e} - C_\emptyset)/\gamma(C_e - C_\emptyset)$. If this threshold is higher than 1 for a particular facility, then the defender is not able to deter the attack from targeting it.

Attacker's ability to target multiple facilities. If the attacker is not constrained to targeting a single facility, his pure strategy set would be $S_a = 2^E$. Then for a pure strategy profile (s_d, s_a), the set of compromised facilities is given by $s_a \setminus s_d$, and the usage cost $C_{s_a \setminus s_d}$. Unfortunately, our approach cannot be straightforwardly applied to this case. This is because the mixed strategies cannot be equivalently represented as probability vectors with elements representing the probability of each facility being targeted or secured. In fact, for a given attacker's strategy, one can find two feasible defender's mixed strategies that induce an identical security effort vector, but result in different players utilities. Hence, the problem of characterizing defender's equilibrium strategies cannot be reduced to characterizing the equilibrium security effort on each facility. Instead, one would need to account for the attack/defense probabilities on all the subsets of facilities in E. This problem is beyond the scope of our paper, although a related work Dahan and Amin [2015] has made some progress in this regard.

Finally, we briefly comment on the model where all the three aspects are included. So long as players' strategy sets are comprised of mixed strategies, the defender's equilibrium utility in $\widetilde{\Gamma}$ must be higher or equal to that in Γ. This is because in $\widetilde{\Gamma}$, the defender can always choose the same strategy as that in NE to achieve a utility that is no less than that in Γ. Moreover, one can show the existence of cost parameters such that the defender has strictly higher equilibrium utility in SPE than in NE. In particular, consider that the attacker's cost parameters $(p_{a,e})_{e \in E}$ in this game are such that there is only one vulnerable facility $\bar{e} \in E$ such that $C_{\bar{e}} - C_{\emptyset} > p_{a,\bar{e}}$, and the threshold effort on that facility $\widehat{\rho}_{\bar{e}} = (C_{\bar{e}} - p_{a,\bar{e}} - C_{\emptyset}) / \gamma(C_{\bar{e}} - C_{\emptyset}) < 1$. In this case, if the defense cost $p_{d,\bar{e}}$ is sufficiently low, then by proactively securing the facility $\bar{e}$ with the threshold effort $\widehat{\rho}_{\bar{e}}$, the defender can deter the attack completely and obtain a strictly higher utility in $\widetilde{\Gamma}$ than that in Γ. Thus, for such cost parameters, the defender gets the first mover advantage in equilibrium.

Chapter 7

Aggregate Demand Prediction in Transportation Networks

7.1 Introduction and Problem Formulation

Chapters 2 – 6 have focused on building analytical models and tools that study the incentives and strategic interactions of humans in transportation networks. In this chapter, we develop empirical tools for predicting the aggregate demand in multi-model transportation networks.

Consider the setting, where travelers commute from one region, denoted as A, to another region, denoted as B, either by driving or taking public transit such as bus or subway (see Fig. 7-1). The two regions can be two city centers or one city center and one suburban area, where travelers regularly commute between them. Each region is covered by a highway network and a public transit network, and the two regions are connected by one or several roads and transit lines.

The goal of our study is to predict the aggregate fraction of driving demand compared to the demand of taking public transit from region A to B. In particular, we predict how the demand fraction changes with the travel time of the multimodal transportation networks, which often fluctuates due to recurrent and non-recurrent disruptions. This demand fraction is useful for the transportation authority to anticipate the aggregate change of demand in response to the travel time fluctuation, and to efficiently manage traffic flows and transit schedules.

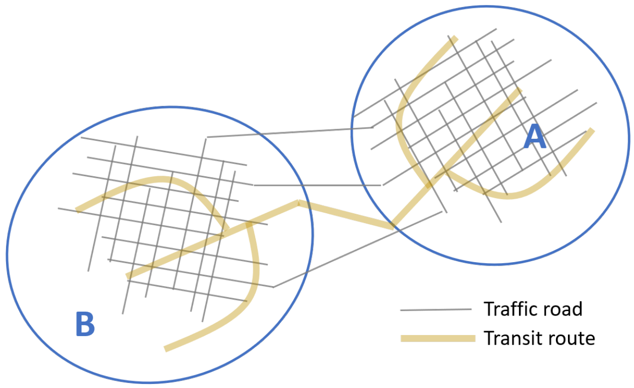


Figure 7-1: Regions A and B connected by highways and transit networks.

The challenge of predicting the aggregate demand fraction is two fold: First, we do not have individual-level data including the origins and destinations of trips, choices between driving and taking transit, and choices of routes in each mode. Instead, our prediction only relies on the aggregate traffic flow measurement that are collected from embedded sensors at specific locations of the network. Second, the prediction of demand fraction relies on the high-dimensional data of travel time costs on all segments in both the highway network and the transit network. Our prediction needs to account for the heterogeneous impact of travel time on these segments on the aggregate demand fraction.

Our machine learning method tackles these two challenges, and achieve high accuracy in an empirical study of driving and transit demand prediction in San Francisco Bay area. We present the prediction model and machine learning methods in Sec. 7.2, and demonstrate the empirical result in Sec. 7.3. We also provide a behavioral justification of our prediction model in Appendix E.

7.2 Prediction Model

We consider a set of days I. For each day $i \in I$, we divide the time period of our study into T intervals, where each $t \in T$ refers to the time interval $[\tau_t, \tau_{t+1}]$. The interval length $\tau_{t+1} - \tau_t$ is identical for all $t \in T$. For each day $i \in I$ and each time interval $t \in T$, the aggregate demand of transit ridership from region A to B is q^b_{ti}, and the aggregate demand of driving is q_{ti}^d. The demand fraction of driving in time interval $t \in T$ on day $i \in I$ is:

$$y_{ti} = \frac{q_{ti}^d}{q_{ti}^d + q_{ti}^b}, \tag{7.1}$$

Given y_{ti}, we directly obtain the demand fraction of transit as $1 - y_{ti}$.

We predict the driving fraction y_{ti} using the travel time of segments in the traffic network and the public transit network. In the traffic network, each segment is defined as the range from an on-ramp to the next off-ramp. In the transit network, each segment is defined as a single stop. We denote the set of all segments as $N = N^d \cup N^b$, where N^d (resp. N^b) is the set of segments in the traffic (resp. transit) network. For each $i \in I$ and each $t \in T$, the average travel time of segment $n \in N$ is x_{ti}^n. Then, the vector of average travel time of all segments for i and t is $x_{ti} = (x_{ti}^n)_{n \in N}$.

Our prediction model is trained separately for each time interval $t \in T$ because the sets of travelers are different for different time intervals, and their demand patterns are also different. For each $t \in T$, we predict the driving demand ratio y_{ti} as in (7.1) using the average travel time vector in time intervals $t, t-\delta, \ldots, t-k\delta$, where $\delta > 0$ is a positive time lag, time interval $t - j\delta$ corresponds $[\tau_t - j\delta, \tau_{t+1} - j\delta]$ for any $j = 0, \ldots, k$, and k is the maximum number of time lags that are incorporated in the model. We denote the travel time vector for time interval $t - j\delta$ in day i as $x_{(t-j\delta)i}$.

We use a kernel function $\phi_t(\cdot) : \mathbb{R}_{\geq 0}^{|N|(k+1)} \to \mathbb{R}$ that transforms the travel time vector $\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$ into a real-valued number. We predict the driving fraction y_{ti} as a logistic function of the kernel value $\phi_t\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$ plus a noise term ψ_{ti} with identical and independent distribution.

$$y_{ti} = \frac{1}{1 + \exp\left\{\phi_t\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)\right\}} + \psi_{ti}. \tag{7.2}$$

Our prediction model is motivated by the behavioral foundation of binary choice model for individual travelers (Ben-Akiva and Bierlaire [1999]), see Appendix E for behavioral justification of our model. The difference between our prediction and choice modeling is that we focus on predicting the aggregate demand fraction instead of individual mode choices. Additionally, we only use data on aggregate demand of driving and transit. We do not observe the mode choices at individual level.

We can re-write (7.2) equivalently as follows:

$$\log\left(\frac{q_{ti}^b}{q_{ti}^d}\right) \overset{(7.1)}{=} \log\left(\frac{1}{y_{ti}} - 1\right) = \phi_t\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right) + \epsilon_{ti}, \tag{7.3}$$

where ϵ_{ti} is also identically and independently distributed for all $i \in I$.

The input vector of the kernel function $\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$ has in total $|N|(k+1)$ variables, where $|N|$ is the number of segments in driving and transit networks, and k is the maximum number of time lags. To avoid over-fitting and improve the prediction accuracy, we employ a class of dimension reduction methods to train the kernel function. In particular, we consider *(i)* naive subset variable regression; *(ii)* ridge regression; *(iii)* LASSO regression; *(iv)* principal components regression; *(v)* random forest method.

In methods *(i)* – *(iv)*, we adopt a linear kernal function:

$$\phi_t\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right) = \gamma_t + \sum_{j=0}^{k} (\beta_t^j)' \cdot x_{(t-j\delta)i}. \tag{7.4}$$

(i) Naive subset variable regression: For each $t \in T$, we select a subset of regressors $\tilde{x}_{(t-\tilde{j}\delta)i}$, where $\tilde{j}$ is a single selected time lag in $\{0, 1, \ldots, k\}$, and $\tilde{x}_{(t-\tilde{j}\delta)i}$ is a sub-vector of $x_{t-\tilde{j}\Delta}$ that only includes segments within r distance to the middle point between region A and B. In (7.4), we estimate γ_t and the coefficients that correspond to the selected variables $\tilde{x}_{(t-\tilde{j}\delta)i}$ using ordinary least square. We set the rest of the parameters as zero. In naive subset variable regression, the hyper-parameter is (α, r), where α governs the chosen time lag of the model and r governs the number of segments included in the regression.

(ii) Ridge regression: For each $t \in T$, the ridge regression coefficients minimizes the sum of least squares error and an L-2 norm regularization of the coefficients $\beta_t = \left(\beta_t^j\right)_{j=0}^k$:

$$\min_{\gamma_t, \beta_t} \sum_{i \in I} \left(\log\left(\frac{q_{ti}^b}{q_{ti}^d}\right) - \gamma_t - \sum_{j=0}^{k} \left(\beta_t^j\right)' x_{(t-j\delta)i}\right)^2 + \lambda \|\beta_t\|^2,$$

The hyper parameter $\lambda \geq 0$ governs the weight of the regularization term – the sum of squares of the coefficients β_t.

(iii) LASSO regression: For each $t \in T$, the lasso coefficients minimizes the sum of least squares error and an L-1 norm regularization of the coefficients β_t

$$\min_{\gamma_t,\beta_t} \sum_{i\in I}\left(\log\left(\frac{q_{ti}^b}{q_{ti}^d}\right) - \gamma_t - \sum_{j=0}^{k}\left(\beta_t^j\right)' x_{(t-j\Delta)i}\right)^2 + \lambda\|\beta_t\|_1,$$

and λ is the hyper parameter that governs the weight of the regularization term.

(iv) Principal components regression (PCR). We consider the following transformed linear regression:

$$\log\left(\frac{q_{ti}^b}{q_{ti}^d}\right) = \gamma_t + \sum_{\ell\in L}\eta_{t\ell}z_{t\ell}, \tag{7.5}$$

where $(z_{t\ell})_{\ell=1}^{L}$ is the first L principal component vectors constructed from the original input $\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$.The principal component vectors are constructed sequentially: the first principal vector $z_{t1} = \left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)' \cdot v_{t1}$ is the projection of the original vector $\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$ onto the direction v_{t1} such that the variance of z_{t1} is maximized. For each $\ell = 2, \ldots, L$, after constructing the first $\ell-1$ principal vectors, we construct the ℓ-th vector by finding a $v_{t\ell}$ that is orthogonal to $v_{t1}, \ldots, v_{t\ell-1}$, and $z_{t\ell} = \left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)' v_{t\ell}$ achieves the maximum variance.

The total number of the principal vectors included in the regression L is the hyper parameter, and L is smaller than the total number of original regressors $|N|(k+1)$ in (7.2). Moreover, the coefficient γ_t in the transformed regression (7.5) is the same as that in (7.3), and the PCR coefficients in the original regression $\beta_t = \left(\beta_t^0, \beta_t^1, \ldots, \beta_t^k\right)$ can be computed from the transformed coefficients $\eta_t = (\eta_{t1}, \ldots, \eta_{tL})$ as follows:

$$\beta_t = \eta_t^{'} \cdot v_t.$$

and $v_t = (v_{t1}, \ldots, v_{tL})$.

(v) Random forest. For each hour t, given any hyper parameters M and L, we compute the random forest regression as follows:

1. Select M data points $\left(\log\left(\frac{q_{ti}^b}{q_{ti}^d}\right), x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)_{i \in M}$ randomly out of the training set to build a regression tree

2. Repeat step 1 for L times to build L trees

3. For a new data point, predict the dependent variable use each one of the L trees, and compute the average of the L outcomes.

Cross validation. We select the optimal hyper parameter in each one of the four methods using 10-fold cross validation. For each time interval $t \in T$, we partition the data set $\{q_{ti}^b, q_{ti}^d, x_{ti}\}_{i \in I}$ into 10 equal-size folds at random. We train our model using 9 out of 10 folds of data, and test the trained model on the remaining one fold. For any given hyper-parameter, we iterate this process over all folds, and compute the root-mean-square error (RMSE) as the square root of the squared test error:

$$RMSE = \sqrt{\frac{1}{10}\sum_{f=1}^{10}\frac{1}{|I_f|}\sum_{i \in I_f}(y_{ti} - \frac{1}{1+\exp\{\hat{\phi}_t^{-f}\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)\}})} \tag{7.6}$$

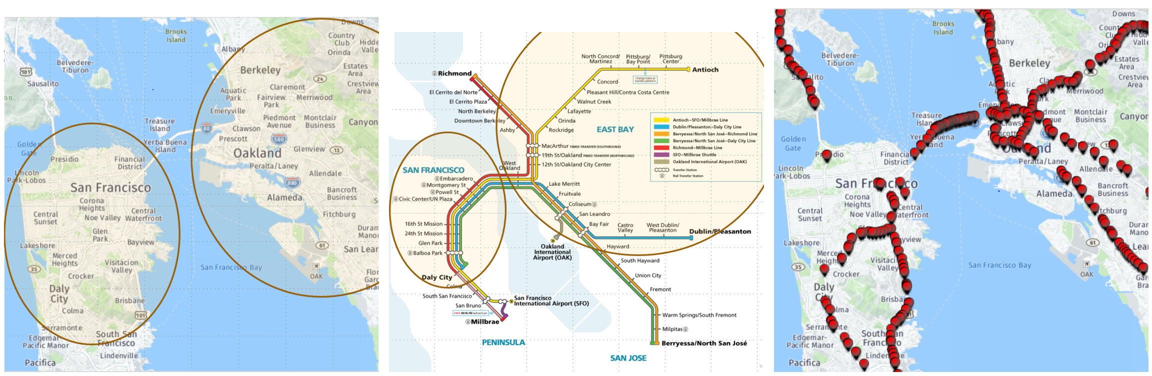
where $\{I_f\}_{f=1}^{10}$ is the 10-fold partition of the set I and $\hat{\phi}_t^{-f}\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$ is the kernel function trained using all but the f-th fold $I \setminus I_f$ as the training data set. For each regression method, we compute the optimal hyper parameter that minimizes the RMSE.

7.3 Empirical Study

7.3.1 Highways and Transit System in San Francisco Bay Area

In this section, we apply our model and method in Sec. 7.2 to predict the driving demand fraction in San Francisco Bay area. Our analysis focuses on travelers whose origins are in the the East-Bay area (region A), and the destinations are in region B that includes City of San Francisco and Daly City. The two regions are connected by highways and the Bay Area Rapid Transit (BART) system. We demonstrate the locations of the two regions in Fig. 7-2a, and the BART stations in Fig. 7-2b.

We use three data sources: (1) The Caltrans Performance Measurement System (PeMS) dataset of 5 minute aggregate traffic flow (total number of cars) and average speed measured by loop detectors embedded on the main highways in California, see Fig 7-2c for the locations of the loop detectors in the two regions. This dataset also provides performance metric of each detector – the percentage of traffic counts that is not imputed; (2) The BART origin-destination pair dataset that reports hourly origin-destination demand taken from user tap-in/tap-out information for each pair of BART stations; (3) The California Highway Patrol (CHP) incidents detail dataset that documents the duration, location, type (collision, hazard, advisory, etc.) of traffic incidents, and the measures taken after incidents such as lane closure, road cleaning, etc.



(a) (b) (c)

Figure 7-2: (a) Two regions; (b) BART stations; (c) Loop detectors

We study the hourly prediction of aggregate driving fraction between the two regions. Each t is a one-hour time interval, and T is the set of 18 one-hour intervals from 4:00 to 22:00 – the BART operating hours. Our analysis includes all *incident-free workdays* from Jan. 1st 2019 to Dec. 31st 2019. We do not include holidays and weekends, because the travel pattern and mode choices during these days are different from that in weekdays.[1] We also filter out data points of hours during which there is a major incidents that cause lane closure according to the reports from CHP. This is because travelers may not have complete information of the incidents, and their mode choices during incident hours may not fully

[1]Travelers mainly take trips for work during workdays, and take trips for leisure during weekends and holidays. Additionally, on weekends, the San Francisco Bay bridge has a different toll price, and the BART has a different schedule. These factors also affect travelers' mode choices on weekends.

account for the non-recurrent delays caused by these incidents. Furthermore, we delete data points collected by loop detectors that have less than 80% non-imputed traffic counts.

We measure the total driving demand of each hour t by the total flow recorded by the loop detector on the Bay bridge in hour t.[2,3] The total BART demand of each hour t is measured by aggregating the ridership over all pairs of BART stations, where the origin is in A and the destination is in B.

We consider the part of the traffic network that covers the range of the main highways in areas A, B and the Bay bridge. Each highway segment of this network is a stretch of highway bounded by ramps (i.e. entrances, exits, splits, etc), and the average travel time of the segment is taken to be the length of the segment (from ramp to ramp) divided by the average speed (averaged over readings across all intermediate detectors).[4] There are 217 segments in our network.

We do not include the BART travel time vector in our analysis due to the lack of station-to-station BART travel time data. This is equivalent to assuming that BART travel time is approximately constant, and thus does not affect the aggregate driving fraction. This assumption is mostly consistent with the "Customer On Time Performance" record provided by BART authority, which shows that the over 90% of the trips are made on time for most days in the fiscal year of 2018.[5] As we will show in the next section, our prediction is fairly accurate even though we only account for the driving time on highways.

7.3.2 Results and Discussion

In Fig. 7-3, we show the 10-fold cross validation root mean squared error for each method and each hour. The boxes extend from the first (Q1) to the third quartile (Q3) while the whiskers extend beyond the box by 1.5 times the interquartile range (Q3-Q1). Observations beyond the whiskers are considered outliers. We show that all five methods achieve fairly accurate prediction of the driving demand ratio with the mean RMSE less than 0.04 in all

[2]We aggregate the 5-minute flow data into hourly-flow data.

[3]Loop detectors on the Bay bridge provide consistent flow estimate. We choose the flow recorded by detector 404416, because this detector has 100% non-imputed traffic counts for the period of our study.

[4]For segments without intermediate detectors, the travel time is taken to be the average of the immediate upstream and downstream detectors on the same highway.

[5]The data is not available for 2019.

hours except for the hour 5:00 - 6:00.[6] In particular, all methods achieve RMSE less than 0.02 during morning rush hours from 7:00 - 10:00. In fact, the fluctuation of driving demand fraction during this period of time is the highest of all times (the variance of driving fraction is higher than 0.12), which implies that a significant proportion of travelers adjust their mode choices based on the travel time during morning commute. Our methods accurately predict travelers' response to morning hour driving time. This prediction is useful for the traffic authority to adjust traffic and transit management plans according to the predicted demand change.

Moreover, all five methods achieve comparable accuracy levels in terms of mean RMSE. The RMSE of the random forest method is slightly lower than the other four methods, and the RMSE of the naive subset variable regression is slightly higher. This observation is intuitive because the random forest method has the flexibility to capture the non-linear relationship between $\log\left(\frac{q^b_{ti}}{q^d_{ti}}\right)$ and the costs of segments, while all other four methods assume that $\log\left(\frac{q^b_{ti}}{q^d_{ti}}\right)$ is linear in the travel time costs. On the other hand, the naive subset variable regression has slightly higher RMSE than other methods since only the subsections that are nearest to the Bay bridge are selected instead of segments in the entire network.

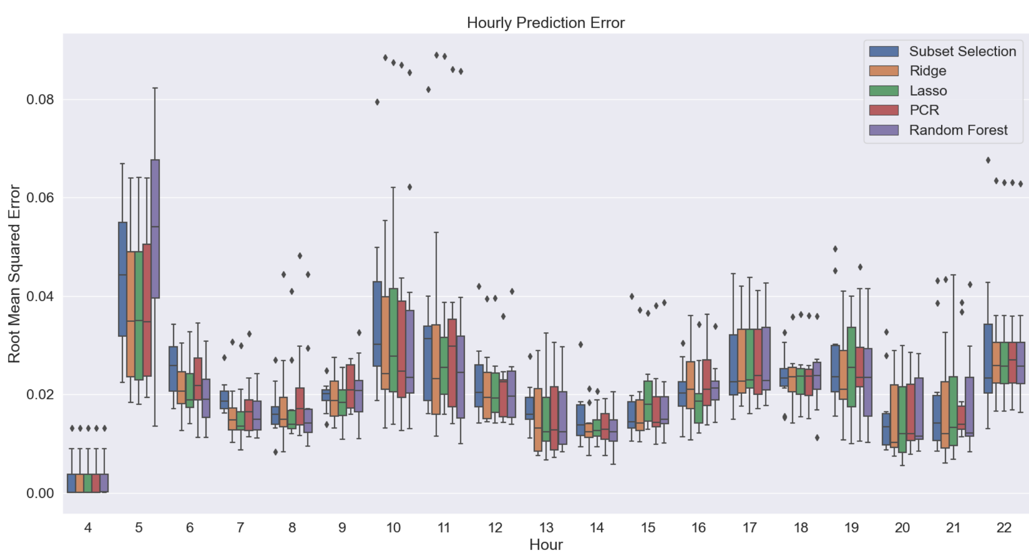


Figure 7-3: Root mean squared error of predictions

[6]During 5:00 - 6:00, the mean RMSE of ridge, Lasso and PCR are below 0.04, and that of subset selection and random forest regression are between 0.04 and 0.06.

Next, we visualize the geospatial distribution of the model coefficients in all methods as heatmaps. Specifically, in naive subset variable regression, ridge regression, LASSO regression, and principal component regression, we sum the segment coefficients β_t^j over all lag values $j = 0, \ldots, k$ for each segment in the network, and colored on a spectrum from red to green, which correspond to lower (more negative) and higher (more positive) coefficients respectively. For random forest, we color segments on a spectrum of green (least important) to red (most important) as determined by an impurity-based measure of importance. We color segments with no influence on the prediction as gray. In the case of LASSO, which rewards sparse coefficient estimates, gray segments have coefficient 0. For the random forest, gray segments suggest no splitting criteria are employed. Fig. 7-4 – 7-8 demonstrate the heatmaps of each method in the hours 7:00 - 8:00, 9:00 - 10:00, 11:00 - 12:00, 13:00 - 14:00, 15:00 - 16:00 and 17:00 - 18:00, respectively.

The model coefficients demonstrate the influence of the travel time of each segment on the aggregate driving demand ratio. We expect segments whose driving times have a greater influence on reducing driving ratio to be more negatively correlated with the dependent variable, and thus have a more negative coefficient (red). The heatmaps of the coefficients provide useful information for the traffic authority to identify the critical segments in the road network, where the congestion delay results in travelers shifting from driving to taking public transit.

From Fig. 7-4 – 7-8, we find that all prediction methods demonstrate similar patterns in relative criticality of segments in shaping the driving fraction, and the pattern changes over time. Particularly, we find that in the morning, congestion on segments of I80-W has a negative influence on the driving fraction, while in the afternoon, congestion on I580-W has a higher impact. One likely explanation of this observation is a shift in demands from origins along I80-W in the morning to demands farther east along I580-W in the afternoon.

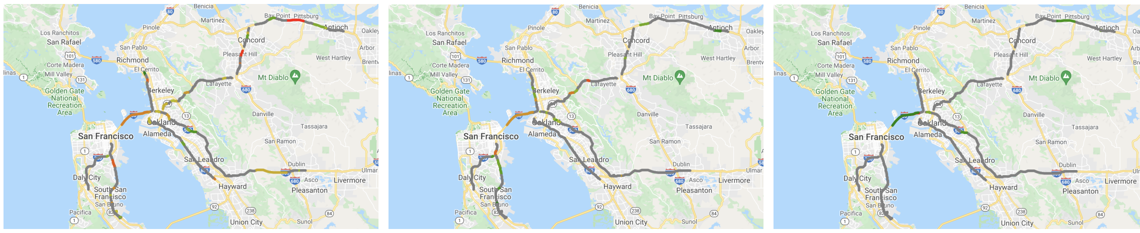
We also note that congestion on segments that are closer to the Bay bridge have a higher impact on reducing driving fraction. This is intuitive since congestion on these segments affect a larger proportion of trips that cross the bridge. In addition, the heap-maps identify a few segments that are further away from the bridge, but congestion on these segments have relatively higher impact on reducing the driving fraction. We find that these are the segments close to BART stations. This may be due to the fact that travelers whose origins or destinations are closer to BART stations are more like to shift from driving to taking BART when the driving time cost is high.

Finally, we observe that in methods that select a subset of segments instead of assigning non-zero weights to all segments, i.e. naive subset variable regression, LASSO regression, and random forests, more segments are assigned with non-zero weights during morning rush hours compared to rest of the day. This implies that more segments experience high variability in driving time during morning rush hours than in other hours, and more travelers are more responsive to the driving time fluctuation in these hours. This leads to the high variability of driving demand fraction in the morning.

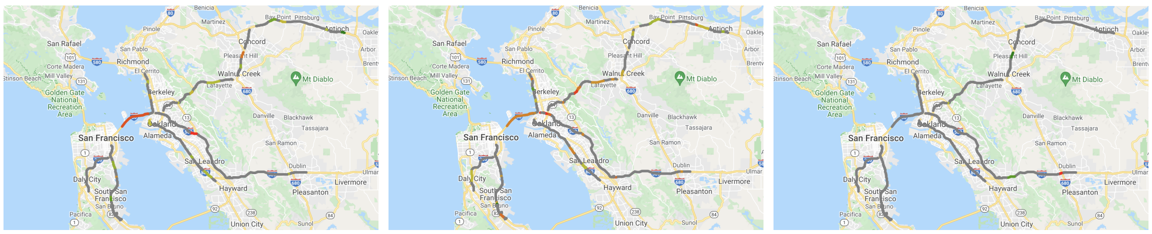
(a) 7:00 - 8:00 (b) 9:00 - 10:00 (c) 11:00 - 12:00

(d) 13:00 - 14:00 (e) 15:00 - 16:00 (f) 17:00 - 18:00

Figure 7-4: Naive subset variable regression: Weights on road segments

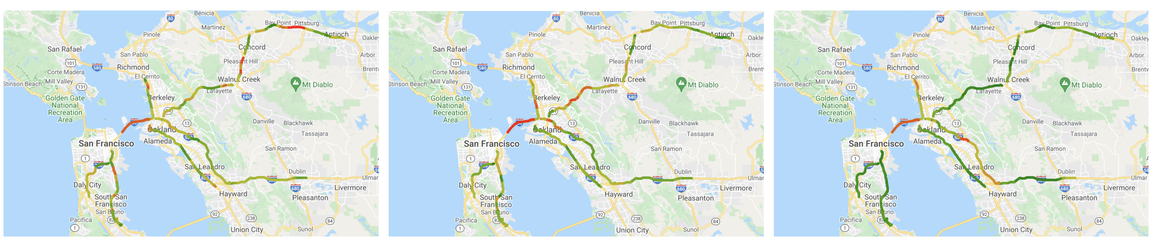


(a) 7:00 - 8:00 (b) 9:00 - 10:00 (c) 11:00 - 12:00

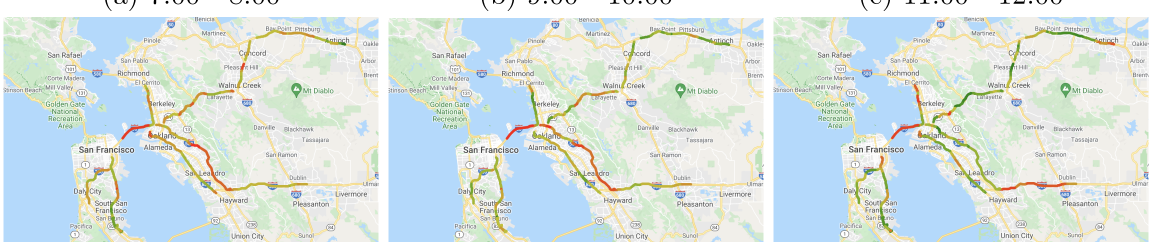


(d) 13:00 - 14:00 (e) 15:00 - 16:00 (f) 17:00 - 18:00

Figure 7-5: LASSO regression: Weights on road segments

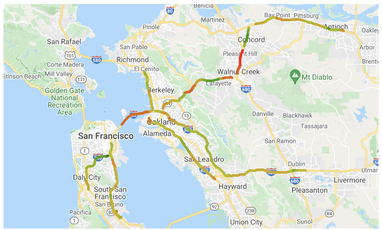


(a) 7:00 - 8:00 (b) 9:00 - 10:00 (c) 11:00 - 12:00

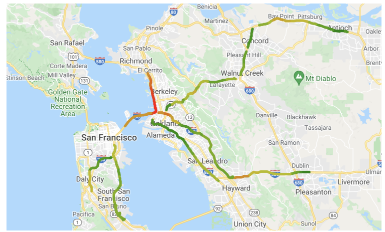


(d) 13:00 - 14:00 (e) 15:00 - 16:00 (f) 17:00 - 18:00

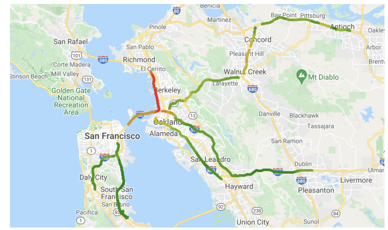
Figure 7-6: Ridge regression: Weights on road segments



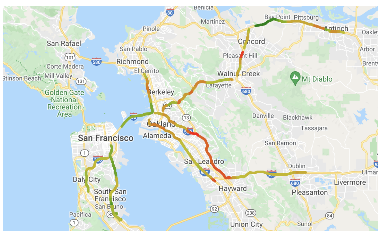
(a) 7:00 - 8:00



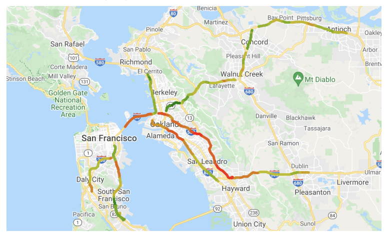
(b) 9:00 - 10:00



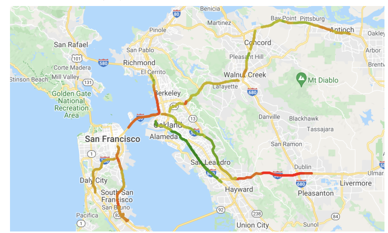
(c) 11:00 - 12:00



(d) 13:00 - 14:00

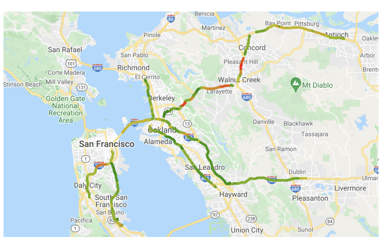


(e) 15:00 - 16:00

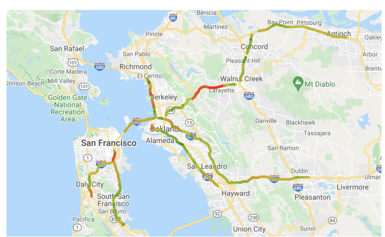


(f) 17:00 - 18:00

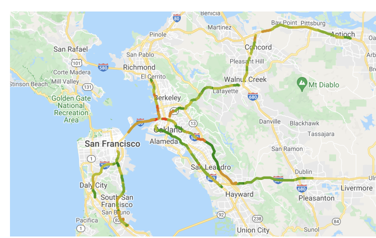
Figure 7-7: Principal component regression: Weights on road segments



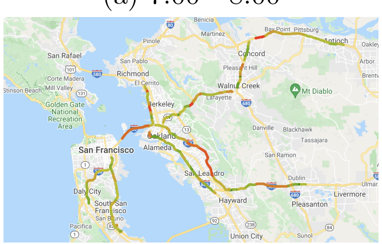
(a) 7:00 - 8:00



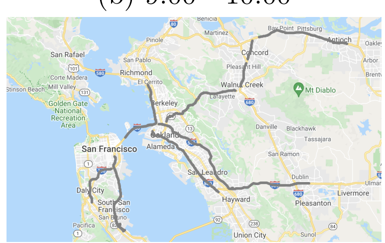
(b) 9:00 - 10:00



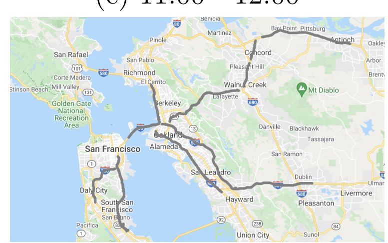
(c) 11:00 - 12:00



(d) 13:00 - 14:00



(e) 15:00 - 16:00



(f) 17:00 - 18:00

Figure 7-8: Random forest: Weights on road segments

Chapter 8

Conclusion and Future Work

This thesis focuses on analyzing the role of platforms and autonomous systems in today's transportation networks. We propose a modeling framework that addresses the strategic nature of human-platform interactions and the physical constraints of the infrastructure networks. We develop game-theoretic tools that study the value of information and optimal information design for routing games in uncertain networks, and multi-agent strategic learning via information platforms. We designed a welfare-improving market mechanism for autonomous carpooling services and developed game-theoretic models for security of cyber-physical systems. We also present a machine learning method for predicting aggregate demand in multimodal transportation networks. Our results provide guidelines to engineer both efficiency and resiliency in the design of the complex transportation systems.

Beyond this thesis, the following questions are interesting directions for future research:

1. How to achieve stronger learning guarantees via strategic experimentation on information platforms?

In Chapter 4, we have shown that learning induced by strategic agents may not recover complete information environment unless certain conditions are satisfied. The main reason that hinders complete information learning is the fact that data is generated endogenously from agents' utility maximizing decisions in games. Additionally, since the realized payoff information of each agent is aggregated and shared to all agents by the public information platform, no agent has an incentive to explore off-equilibrium strategies. An important direction of future research is to study the design of platforms that incentives agents to explore off-equilibrium strategies so that learning eventually aggregates complete information of the unknown environment. The design of such platform requires providing monetary incentives for agents who explore off-equilibrium strategies, or to retain some payoff information as private only for the agents who explore.

2. How to design incentive mechanisms under uncertain road capacities?

In Chapter 5, we demonstrate the design of carpooling market mechanism with fixed and known road capacity. In practice, the capacity of road segments is uncertain due to recurrent or non-recurrent disruptions. The design of incentive mechanism needs to account for the uncertain capacity, and assign trips in a manner such that the capacity constraint is satisfied with high probability. In this case, travelers' valuation of trips should also rely on the expected delay of the trip time cost that is computed based on the distribution of road capacity.

3. How to predict the dynamics of aggregate agent behavior during prolonged network disruptions?

In Chapter 7, our machine learning methods accurately predict the driving demand fraction in transportation networks with recurrent disturbances. When testing our prediction models using data collected during non-recurrent disruptions, we find that our prediction remains to be accurate when the duration of disruptions caused by the incident is short (less than 15 minutes). On the other hand, we find that the prediction error of driving demand fraction is high when incidents significantly reduce the road capacity (lane closures) and cause prolonged disruptions. This implies that during incident hours, new methods are needed to account for travelers' dynamic adjustment of travel decisions. Such methods need to reflect the fact that travelers' response to incident is typically delayed due to the lack of perfect information.

Appendix A

Supplementary Material for Chapter 2

A.1 Proofs of Section 2.4

Proof of Lemma 2.1. First note that $\Phi(q)$, as defined in (2.12), is a continuous and differentiable function of the strategy profile q. To show that $\Phi(q)$ is a weighted potential function of $\Gamma(\lambda)$, we write the first order derivative of $\Phi(q)$ with respect to $q_r^i(t^i)$:

$$\begin{aligned}\frac{\partial \Phi(q)}{\partial q_r^i(t^i)} &= \sum_{s\in S}\sum_{t^{-i}\in T^{-i}} \pi(s,t^i,t^{-i}) \sum_{e\in r} c_e^s\left(w_e(t^i,t^{-i})\right) \\ &\overset{(2.8)}{=} \Pr(t^i)\mathbb{E}[c_r(q)|t^i], \quad \forall r\in R, \quad \forall t^i\in T^i, \quad \forall i\in I. \end{aligned} \tag{A.1}$$

Thus, $\Phi(q)$ satisfies (2.11) with $\gamma(t^i)=\Pr(t^i)$ for any $t^i\in T^i$ and any $i\in I$. ☐

Proof of Lemma 2.2. Since each $c_e^s(w_e(t))$ is differentiable in $w_e(t)$, we know that $\breve{\Phi}(w)$ is twice differentiable with respect to w. The first order partial derivative of $\breve{\Phi}(w)$ with respect to $w_e(t)$ can be written as: $\frac{\partial \breve{\Phi}(w)}{\partial w_e(t)} = \sum_{s\in S}\pi(s,t)c_e^s\left(w_e\left(t\right)\right)$ for any $e\in E$, and any $t\in T$. Also, the second order derivative of $\breve{\Phi}(w)$ can be written as follows:

$$\frac{\partial^2\breve{\Phi}(w)}{\partial w_e(t)\partial w_{e'}(t')} = \begin{cases} \sum_{s\in S}\pi\left(s,t\right)\frac{dc_e^s(w_e(t))}{dw_e(t)}, & \text{if } e=e' \text{ and } t=t', \\ 0, & \text{otherwise,}\end{cases} \qquad \forall e,e'\in E, \quad \forall t,t'\in T.$$

Since for any $e\in E$ and any $s\in S$, $c_e^s(w_e)$ is increasing in w_e, $\sum_{s\in S}\pi\left(s,t\right)\frac{dc_e^s(w_e(t))}{dw_e(t)} > 0$. Thus, the Hessian matrix of $\breve{\Phi}(w)$ has positive elements on the diagonal and 0 in all other entries, i.e. it is positive definite. Therefore, $\check{\Phi}(w)$ is strictly convex in w. $\square$

Proof of Theorem 2.1. We first show that any minimum of (OPT-Q) is a Bayesian Wardrop equilibrium. The Lagrangian of (OPT-Q) is given by (2.15), where $\mu = (\mu^{t^i})_{t^i \in T^i, i \in I}$ and $\nu = (\nu_r^{t^i})_{r \in R, t^i \in T^i, i \in I}$ are Lagrange multipliers associated with the constraints (2.4a) and (2.4b), respectively. For any optimal solution q, there must exist μ and ν such that (q, μ, ν) satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_r^i(t^i)} = \frac{\partial \Phi}{\partial q_r^i(t^i)} - \mu^{t^i} - \nu_r^{t^i} &= 0, && \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I, && \text{(KKT.1)}\\
\nu_r^{t^i} q_r^i(t^i) &= 0, && \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I, && \text{(KKT.2)}\\
\nu_r^{t^i} &\geq 0, && \forall r \in R, \quad \forall t^i \in T^i, \quad \forall i \in I. && \text{(KKT.3)}
\end{aligned}
$$

Using (A.1) and (KKT.1), we have $\frac{\partial \Phi(q)}{\partial q_r^i(t^i)} = \Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu^{t^i} + \nu_r^{t^i}$ for any $r \in R$, and $t^i \in T^i$, $i \in I$. From (KKT.2), we see that for any $r \in R$, and $t^i \in T^i$, $i \in I$, if $q_r^i(t^i) > 0$, the corresponding Lagrange multiplier $\nu_r^{t^i} = 0$, and $\Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu^{t^i}$. However, if $q_r^i(t^i) = 0$, then $\Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu^{t^i} + \nu_r^{t^i} \geq \mu^{t^i}$. Thus, for any $r \in R$, and $t^i \in T^i$, $i \in I$:

$$
q_r^i(t^i) > 0 \quad \Rightarrow \quad \Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu^{t^i} \leq \mu^{t^i} + \nu_{r'}^{t^i} = \Pr(t^i)\mathbb{E}[c_{r'}(q)|t^i], \quad \forall r' \in R.
$$

From (2.9), we conclude that an optimal solution of (OPT-Q) is a Bayesian Wardrop equilibrium.

Next, we show that any equilibrium q^* of the game $\Gamma(\lambda)$ is an optimal solution of (OPT-Q). Consider a pair of Lagrange multipliers $\bar{\mu}$ (resp. $\bar{\nu}$)) corresponding to the constraints (4a) (resp. (4b)), where $\bar{\mu}^{t^i} = \min_{r \in R} \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i]$ and $\bar{\nu}_r^{t^i} = \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \bar{\mu}^{t^i}$. We can easily check that (KKT.1) and (KKT.3) are satisfied by $(q^*, \bar{\mu}, \bar{\nu})$. Since q^* is an equilibrium, we know from (2.9) that for a route $r \in R$, and $t^i \in T^i$, $i \in I$, if $q_r^{i*}(t^i) > 0$, then $\mathbb{E}[c_r(q^*)|t^i] = \min_{r \in R} \mathbb{E}[c_r(q^*)|t^i]$ and consequently $\bar{\nu}_r^{t^i} = \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \bar{\mu}^{t^i} = 0$. This implies that (KKT.2) is also satisfied by $(q^*, \bar{\mu}, \bar{\nu})$. Noting that $\Phi(q) \equiv \check{\Phi}(w)$, where the induced edge load w is linear in q (see (2.7)), and that $\check{\Phi}(w)$ is strictly convex in w (Lemma 2.2), we conclude that $\Phi(q)$ is a convex function of q. Furthermore, since $Q(\lambda)$ is a convex polytope, (OPT-Q) is a convex problem. Thus, KKT conditions are also sufficient for optimality, and any equilibrium q^* is an optimal solution of (OPT-Q).

Finally, for any λ, we can use equations (2.7) and (2.14) to re-express (OPT-Q) as an optimization problem whose solution gives an equilibrium edge load $w^*(\lambda)$:

$$\begin{aligned} \min_{w,q} \quad & \breve{\Phi}(w) \\ s.t. \quad & q \in Q(\lambda), \quad w_e(t) = \sum_{r \ni e} \sum_{i \in I} q_r^i(t^i), \quad \forall t \in T, \quad \forall e \in E. \end{aligned} \tag{A.2}$$

Clearly, the feasible set of the above problem is a convex polytope. From Lemma 2.2, $\breve{\Phi}(w)$ is strictly convex in w. Therefore, the equilibrium edge load $w^*(\lambda)$ is unique. $\square$

Lemma A.1. *(Theorem 2 in Wachsmuth [2013]) The Lagrange multiplies μ^* and ν^* associated with any $q^* \in Q^*(\lambda)$ at the optimum of* (OPT-Q) *are unique if and only if the LICQ condition is satisfied in that the gradients of the set of tight constraints in* (2.4a)-(2.4b) *at the optimum are linearly independent.*

Proof of Lemma 2.3. Let the set of constraints that are tight at optimum of (OPT-Q) in (2.4b) be denoted as $\mathcal{B}$. Assume for the sake of contradiction that LICQ does not hold, i.e. the set of equality constraints (2.4a) and the elements in the set $\mathcal{B}$ are linearly dependent. Now, note that the constraint sets (2.4a) and (2.4b) are each comprised of linearly independent affine functions. Hence, there must exist a type $\bar{t}^i$ such that the gradient of the corresponding equality constraint (i.e. $\sum_{r \in R} q_r^{i*}(\bar{t}^i) = \lambda^i D$) is linearly dependent with the elements in the set $\mathcal{B}$, which implies that $q_r^{i*}(\bar{t}^i) = 0, \forall r \in R$. However, this violates the equality constraint in (2.4a) as $\sum_{r \in R} q_r^{i*}(\bar{t}^i) = \lambda^i D \neq 0$, and we arrive at a contradiction.

Since LICQ holds, for any equilibrium strategy profile $q^* \in Q^*(\lambda)$, the corresponding μ^* and ν^* must be unique. Following the proof of Theorem 2.1, we conclude that for any $q^* \in Q^*(\lambda)$, (q^*, μ^*, ν^*) satisfies the KKT conditions, and $\mu^{t^i *}$ and $\nu_r^{t^i *}$ can be written as (2.16a) and (2.16b), respectively.

Finally, noting that the equilibrium edge load is unique (Theorem 2.1), μ^* and ν^* in (2.16a)-(2.16b) are thus unique in equilibrium. $\square$

Proof of Proposition 2.1. Step I: We show that any $q \in Q(\lambda)$ induces a route flow f that satisfies (2.17a)-(2.17d). From (2.6), we obtain that for any $t^i, \tilde{t}^i \in T^i$, any $t^{-i}, \tilde{t}^{-i} \in T^{-i}$, and any $i \in I$, f satisfies (2.17a):

$$f_r(t^i, t^{-i}) - f_r(\tilde{t}^i, t^{-i}) = q_r^i(t^i) + \sum_{j \in I \setminus \{i\}} q_r^j(t^j) - q_r^i(\tilde{t}^i) - \sum_{j \in I \setminus \{i\}} q_r^j(t^j)$$
$$= q_r^i(t^i) + \sum_{j \in I \setminus \{i\}} q_r^j(\tilde{t}^j) - q_r^i(\tilde{t}^i) - \sum_{j \in I \setminus \{i\}} q_r^j(\tilde{t}^j) = f_r(t^i, \tilde{t}^{-i}) - f_r(\tilde{t}^i, \tilde{t}^{-i}).$$

From (2.4a) and (2.4b), we can directly conclude that f must also satisfy (2.17b) and (2.17c). Additionally,

$$D - \sum_{r \in R} \min_{t^i \in T^i} f_r(t^i, t^{-i}) \overset{(2.6)}{=} D - \sum_{r \in R} \sum_{j \in I \setminus \{i\}} q_r^j(t^j) - \sum_{r \in R} \min_{t^i \in T^i} q_r^i(t^i)$$
$$\overset{(2.4a)}{=} D - \sum_{j \in I \setminus \{i\}} \lambda^j D - \sum_{r \in R} \min_{t^i \in T^i} q_r^i(t^i) = \lambda^i D - \sum_{r \in R} \min_{t^i \in T^i} q_r^i(t^i) \overset{(2.4b)}{\leq} \lambda^i D, \quad \forall t^{-i} \in T^{-i}, \ \forall i \in I.$$

Therefore, f satisfies (2.17d). Thus, any feasible route flow must satisfy (2.17a)-(2.17d).

Step II: Next, we show that for any route flow $f \in F(\lambda)$ (i.e. f that satisfies constraints (2.17a)-(2.17d)), the set of feasible strategies that induce f can be characterized by (2.22). For any route $r \in R$, the linear system of equations (2.6) has $\prod_{i \in I} |T^i|$ equations in $\sum_{i \in I} |T^i|$ variables. Note that for any given $\widehat{t} = \left(\widehat{t^i}\right)_{i \in I} \in T$, the following equations are linearly independent:

$$\begin{aligned} \sum_{i \in I} q_r^i(\widehat{t^i}) &= f_r(\widehat{t}), \\ q_r^i(t^i) + \sum_{j \in I \setminus \{i\}} q_r^j(\widehat{t^j}) &= f_r(t^i, \widehat{t}^{-i}), \quad \forall t^i \in T^i \setminus \{\widehat{t^i}\}, \quad \forall i \in I. \end{aligned} \tag{A.3}$$

We then show that given any $t \in T$, $\sum_{i \in I} q_r^i(t^i) = f_r(t)$ is a linear combination of the equations in (A.3). Following (2.17a), we can write:

$$\sum_{i \in I} f_r(t^i, \widehat{t}^{-i}) - (|I| - 1) f_r(\widehat{t}) = f_r(t^1, \widehat{t}^{-1}) + f_r(t^2, \widehat{t}^{-2}) + \sum_{i=3}^{|I|} f_r(t^i, \widehat{t}^{-i}) - (|I| - 1) f_r(\widehat{t})$$
$$\overset{(2.17a)}{=} f_r(t^1, t^2, \widehat{t}^{-1-2}) + f_r(\widehat{t}) + \sum_{i=3}^{|I|} f_r(t^i, \widehat{t}^{-i}) - (|I| - 1) f_r(\widehat{t})$$

$$=f_r(t^1,t^2,\widehat{t}^{-1-2})+\sum_{i=3}^{|I|} f_r(t^i,\widehat{t}^{-i})-(|I|-2)f_r(\widehat{t}),$$

where $\widehat{t}^{-1-2}=(\widehat{t}^3,\cdots,\widehat{t}^{|I|})$. We apply the same procedure iteratively for another $|I|-2$ times:

$$\sum_{i\in I} f_r(t^i,\widehat{t}^{-i})-(|I|-1)f_r(\widehat{t})=f_r(t),\quad \forall t\in T. \tag{A.4}$$

Now for any $r\in R$ and $t\in T$, we can write $\sum_{i\in I} q_r^i(t^i)=\sum_{i\in I}\left(q_r^i(t^i)+\sum_{j\in I\setminus\{i\}} q_r^j(\widehat{t^j})\right)-(|I|-1)\sum_{i\in I} q_r^i(\widehat{t^i}) \overset{(A.3)}{=} \sum_{i\in I} f_r(t^i,\widehat{t}^{-i})-(|I|-1)f_r(\widehat{t}) \overset{(A.4)}{=} f_r(t)$. Thus, for any $r\in R$, the linear system (2.6) is comprised of $\sum_{i\in I}|T^i|$ variables, and any constraint can indeed be expressed as a linear combination of $\sum_{i\in I}|T^i|-|I|+1$ independent equations in (A.3). From the rank-nullity theorem, we conclude that the dimension of null space of this linear map is $|I|-1$. Then, for any $r\in R$, any $i\in I$, setting $q_r^i(\widehat{t^i})=\chi_r^i$, any solution of (2.6) can be expressed as (2.22), where $\widehat{t}\in T$ is an arbitrary type profile. Additionally, $\sum_{i\in I}\chi_r^i=\sum_{i\in I} q_r^i(\widehat{t^i})=f_r(\widehat{t})$. Thus, χ satisfies (2.23b), i.e. for each $r\in R$, out of the $|I|$ variables in $\{\chi_r^i\}_{i\in I}$, $|I|-1$ are free, and the remaining one is obtained from (2.23b). We can conclude that the strategy profile q as defined in (2.22) induces the route flow f. It remains to be shown that if q is a feasible strategy profile, χ must satisfy (2.23a) and (2.23c) as well. Since q satisfies (2.4a), we obtain that $\lambda^i D \overset{(2.4a)}{=} \sum_{r\in R} q_r^i(t^i) \overset{(2.22)}{=} \sum_{r\in R}\left(f_r(t^i,\widehat{t}^{-i})-f_r(\widehat{t^i},\widehat{t}^{-i})+\chi_r^i\right) \overset{(2.17b)}{=} \sum_{r\in R}\chi_r^i$ for any $i\in I$, i.e. χ satisfies (2.23a). Additionally, from (2.4b), $0\le q_r^i(t^i) \overset{(2.22)}{=} f_r(t^i,\widehat{t}^{-i})-f_r(\widehat{t^i},\widehat{t}^{-i})+\chi_r^i$ for any $r\in R$ and any $t^i\in T^i$. Thus, $\chi_r^i\ge \max_{t^i\in T^i}\left\{f_r(\widehat{t^i},\widehat{t}^{-i})-f_r(t^i,\widehat{t}^{-i})\right\}$, i.e. χ satisfies (2.23c).

Step III: Finally, we show that the set of χ satisfying (2.23) is non-empty, i.e., any $f\in F(\lambda)$ can be induced by at least one feasible strategy profile q. Consider any $f\in F(\lambda)$, we explicitly construct the following χ, and show that such χ satisfies (2.23):

$$\chi_r^i=\gamma_r\cdot\left(\lambda^i D-\sum_{r\in R}\max_{t^i\in T^i}\left(f_r(\widehat{t})-f_r(t^i,\widehat{t}^{-i})\right)\right)+\max_{t^i\in T^i}\left(f_r(\widehat{t})-f_r(t^i,\widehat{t}^{-i})\right),\ \forall r\in R,\ \forall i\in I, \tag{A.5}$$

where $\widehat{t}$ is any arbitrary type profile, and

$$\gamma_r = \frac{f_r(\widehat{t}) - \sum_{i\in I} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)}{\sum_{r\in R} \left[f_r(\widehat{t}) - \sum_{i\in I} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)\right]},$$

if $\sum_{r\in R} \left[f_r(\widehat{t}) - \sum_{i\in I} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)\right] \neq 0$, and $\gamma_r = 0$ otherwise.

First, we check that the $(\chi_r^i)_{r\in R, i\in I}$ as defined in (A.5) satisfies (2.23c). Note that $\gamma_r \geq 0$. To see this, since for any $t \in T$, $\sum_{i\in I} f_r(t^i, \widehat{t}^{-i}) - (|I|-1)f_r(\widehat{t}) \overset{(A.4)}{=} f_r(t) \geq 0$, we know that $\min_{t\in T} \sum_{i\in I} f_r(t^i, \widehat{t}^{-i}) - (|I|-1)f_r(\widehat{t}) = \min_{t\in T} f_r(t) \geq 0$. Thus, for any $r \in R$, we obtain:

$$f_r(\widehat{t}) - \sum_{i\in I} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right) = \min_{t\in T} \sum_{i\in I} f_r(t^i, \widehat{t}^{-i}) - (|I|-1)f_r(\widehat{t}) = \min_{t\in T} f_r(t) \geq 0. \tag{A.6}$$

Hence, we can conclude that $\gamma_r \geq 0$. Next, $\lambda^i D - \sum_{r\in R} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right) \overset{(2.17b)}{=} \lambda^i D - \left(D - \sum_{r\in R} \min_{t^i\in T^i} f_r(t^i, \widehat{t}^{-i})\right) \overset{(2.17d)}{\geq} 0$. Using the above inequalities, we obtain that χ_r^i as considered in (A.5) satisfies (2.23c).

Second, we check χ_r^i satisfies (2.23a). If $\sum_{r\in R} \left[f_r(\widehat{t}) - \sum_{i\in I} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)\right] > 0$, then:

$$\begin{aligned}
\sum_{r\in R} \chi_r^i &= \sum_{r\in R} \gamma_r \cdot \left(\lambda^i D - \sum_{r\in R} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)\right) + \sum_{r\in R} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right) \\
&= \left(\lambda^i D - \sum_{r\in R} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)\right) + \sum_{r\in R} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right) = \lambda^i D.
\end{aligned}$$

On the other hand, if $\sum_{r\in R} \left[f_r(\widehat{t}) - \sum_{i\in I} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)\right] = 0$, we obtain that:

$$\begin{aligned}
0 &= \sum_{r\in R} \left[f_r(\widehat{t}) - \sum_{i\in I} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)\right] \\
&\overset{(2.17b)}{=} D - \sum_{i\in I} \left(\sum_{r\in R} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right)\right) \overset{(2.17d)}{\geq} D - \sum_{i\in I} \lambda^i D = 0,
\end{aligned}$$

which implies that for any $i \in I$, $\sum_{r\in R} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right) = \lambda^i D$. Since in this case, $\gamma_r = 0$, we can conclude that $\sum_{r\in R} \chi_r^i = \sum_{r\in R} \max_{t^i\in T^i} \left(f_r(\widehat{t}) - f_r(t^i, \widehat{t}^{-i})\right) = \lambda^i D$, i.e.

χ satisfies (2.23a).

Finally, χ_r^i also satisfies (2.23b). If $\sum_{r\in R}\left[f_r(\widehat{t}) - \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\right] > 0$, we have:

$$\begin{aligned}
\sum_{i\in I}\chi_r^i &= \gamma_r\cdot\sum_{i\in I}\left(\lambda^i D - \sum_{r\in R}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\right) + \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\\
&= \gamma_r\cdot\left(D - \sum_{i\in I}\sum_{r\in R}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\right) + \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\\
&\overset{(2.17b)}{=} \gamma_r\cdot\left(\sum_{r\in R}\left[f_r(\widehat{t}) - \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\right]\right) + \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\\
&= f_r(\widehat{t}) - \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right) + \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right) = f_r(\widehat{t}).
\end{aligned}$$

If $\sum_{r\in R}\left[f_r(\widehat{t}) - \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\right] = 0$, then we have

$$0 = \sum_{r\in R}\left[f_r(\widehat{t}) - \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right)\right] \overset{(A.6)}{=} \sum_{r\in R}\min_{t\in T} f_r(t) \geq 0,$$

which implies that for any $r\in R$, $\min_{t\in T} f_r(t) = 0$. In this case, $\gamma_r = 0$, and thus $\sum_{i\in I}\chi_r^i = \sum_{i\in I}\max_{t^i\in T^i}\left(f_r(\widehat{t}) - f_r(t^i,\widehat{t}^{-i})\right) \overset{(A.6)}{=} f_r(\widehat{t}) - \min_{t\in T} f_r(t) = f_r(\widehat{t})$, i.e. χ satisfies (2.23b).

Therefore, if f satisfies (2.17a)-(2.17d), we can conclude that the χ in (A.5) satisfies (2.23), i.e. the set of χ satisfying (2.23) is non-empty. We already showed in Step II that q as defined in (2.22) with parameter χ satisfying (2.23) is a feasible strategy profile, and q induces f. Therefore, if f satisfies (2.17a)-(2.17d), there exists a feasible q that induces f, i.e. any $f\in F(\lambda)$ is a feasible route flow.

In summary, we have shown that any feasible route flow satisfies (2.17) (Step I); For any f that satisfies (2.17), the set of feasible strategy profiles that induce f can be written in (2.22)-(2.23) (Step II); Such set is non-empty, and hence f is feasible (Step III). We can thus conclude that the set of feasible route flows is $F(\lambda)$, and the set of feasible strategies that induce f is as in (2.22)-(2.23). $\square$

Proof of Proposition 2.2. From Proposition 2.1, we know that the set of feasible route flows is the set $F(\lambda)$ characterized by (2.17a)-(2.17d). Additionally, the weighted potential function in (2.12) can be equivalently written as a function of f given by (2.13). Therefore, the minimum of (OPT-F) is equal to that in (OPT-Q), and the set of optimal solutions is the set of equilibrium route flows. $\square$

A.2 Proofs of Section 2.5

Lemma A.2. *The route flows $f^{ij,\dagger} \in F^{ij,\dagger}$ induce a unique edge load $w^{ij,\dagger}$.*

Proof of Lemma A.2. Following (2.7) and (OPT-F), any edge load $w^{ij,\dagger}$ induced by route flows in $F^{ij,\dagger}$ (which we defined as optimal solution set of (OPT-F^{ij})) is an optimal solution of the following optimization problem:

$$\begin{aligned}
\min_{w} \quad & \breve{\Phi}(w), \\
s.t. \quad & w_e(t) = \sum_{r \ni e} f_r(t), \quad \forall t \in T, \quad \forall e \in E, \\
& f \text{ satisfies (2.17a), (2.17b), (2.17c), (IIC)}\setminus\{i,j\}, \text{(IIC}_{ij}).
\end{aligned}$$

The constraints (2.17a), (2.17b), (2.17c) are linear constraints. Following from (2.21), constraints (IIC)$\setminus\{i,j\}$, (IIC$_{ij}$) are each equivalent to a set of linear constraints. Additionally, w is a linear function of f, thus the feasible set of w in this optimization problem must also be a convex polytope. Since $\breve{\Phi}(w)$ is a strictly convex function in w, the optimal solution $w^{ij,\dagger}$ is unique. $\square$

Proof of Lemma 2.4. First, we show that both thresholds $\underline{\lambda}^i$ and $\bar{\lambda}^i$ belong to the interval $[0, 1-|\lambda^{-ij}|]$. Since $\underline{\lambda}^i$ is attainable on the set $F^{ij,\dagger}$, there exists $\tilde{f}^{ij,\dagger} \in F^{ij,\dagger}$ such that:

$$\underline{\lambda}^i = \frac{1}{D}\widehat{J}^i(\tilde{f}^{ij,\dagger}) \overset{(2.19)}{=} \frac{1}{D}\left(D - \sum_{r\in R}\min_{t^i \in T^i} \tilde{f}^{ij,\dagger}_r\left(t^i, t^{-i}\right)\right) \geq \frac{1}{D}\left(D - \sum_{r\in R}\tilde{f}^{ij,\dagger}_r\left(\widehat{t}^i, t^{-i}\right)\right) \overset{(2.17b)}{=} 0.$$

Similarly, we can check that $\bar{\lambda}^i \leq 1 - |\lambda^{-ij}|$.

Additionally, we know for any $f^{ij,\dagger} \in F^{ij,\dagger}$:

$$\bar{\lambda}^i \overset{(2.26)}{\geq} \frac{1}{D}\left\{\left(1 - |\lambda^{-ij}|\right)D - \widehat{J}^j(f^{ij,\dagger})\right\} \overset{(\text{IIC}_{ij})}{\geq} \frac{1}{D}\widehat{J}^i(f^{ij,\dagger}) \overset{(2.26)}{\geq} \underline{\lambda}^i.$$

Therefore, $0 \leq \underline{\lambda}^i \leq \bar{\lambda}^i \leq 1 - |\lambda^{-ij}|$. $\square$

For any two populations $i, j \in I$, we can compute the threshold $\underline{\lambda}^i$ as follows:

$$
\begin{aligned}
\min \quad & y \\
s.t. \quad & D - \sum_{r \in R} f_r(t_r^i, \widehat{t}^{-i}) \leq y \cdot D, \quad \forall t_1^i \in T^i, \dots, \forall t_{|R|}^i \in T^i, \\
& f^{ij,\dagger} \in F^{ij,\dagger},
\end{aligned} \tag{A.7}
$$

where $F^{ij,\dagger}$ is the polytope defined in (2.25). Therefore, (A.7) is a linear programming. Analogously, the threshold $\bar{\lambda}^i$ is the optimal value of the following linear program:

$$
\begin{aligned}
\max \quad & y \\
s.t. \quad & -|\lambda^{-ij}|D + \sum_{r \in R} f_r(t_r^j, \widehat{t}^{-j}) \geq y \cdot D, \quad \forall t_1^j \in T^j, \dots \forall t_{|R|}^j \in T^j, \\
& f^{ij,\dagger} \in F^{ij,\dagger}.
\end{aligned} \tag{A.8}
$$

Proof of Theorem 2.2. $\left[\text{Regime } \Lambda_1^{ij}\right]$: First, we show by contradiction that the constraint (IIC_i) is tight for any equilibrium route flow. Assume that for a given $\lambda \in \Lambda_1^{ij}$, there exists an equilibrium route flow f^* such that (IIC_i) is not tight. From Proposition 2.2, we know that f^* is an optimal solution of (OPT-F). Since (OPT-F) is a convex optimization problem, f^* is still a minimizer of $\widehat{\Phi}(f)$ if we drop the constraint (IIC_i). Additionally, the constraints (IIC_i) and (IIC_j) implies that f^* must also satisfy (IIC_{ij}). Thus, such f^* is an optimal solution of the following problem:

$$
\min_f \quad \widehat{\Phi}(f), \quad s.t. \quad \text{(2.17a), (2.17b), (2.17c), (IIC}_{ij}\text{), and (IIC)}\backslash\{i\}. \tag{A.9}
$$

Moreover, the threshold $\bar{\lambda}^i$ defined in (2.26) is attained by a route flow, say $\tilde{f}^{ij,\dagger}$, in the set $F^{ij,\dagger}$. Thus, we can write: $1 - |\lambda^{-ij}| - \frac{1}{D}\widehat{J}^j(\tilde{f}^{ij,\dagger}) = \bar{\lambda}^i \overset{\text{(Lemma 2.4)}}{\geq} \underline{\lambda}^i \overset{\text{(2.27a)}}{>} \lambda^i$. Rearranging, we obtain: $\frac{1}{D}\widehat{J}^j(\tilde{f}^{ij,\dagger}) < 1 - |\lambda^{-ij}| - \lambda^i = \lambda^j$, and so such $\tilde{f}^{ij,\dagger}$ also satisfies (IIC_j). Since $\tilde{f}^{ij,\dagger}$ is an optimal solution of (2.25), which minimizes the same objective function as (A.9) but without the constraint (IIC_j), we thus know that $\tilde{f}^{ij,\dagger}$ is also an optimal solution in (A.9). Since the induced edge load is unique, we can conclude that the edge load induced by f^* must be identical to that induced by $\tilde{f}^{ij,\dagger}$, which is $w^{ij,\dagger}$. Then, from (2.25), we have $f^* \in F^{ij,\dagger}$. Therefore, from (2.26), we can write $\underline{\lambda}^i \leq \frac{1}{D}\widehat{J}^i(f^*)$. Since we assumed that (IIC$_i$) is not binding in equilibrium, we obtain: $\frac{1}{D}\widehat{J}^i(f^*) < \lambda^i < \underline{\lambda}^i \leq \frac{1}{D}\widehat{J}^i(f^*)$, which is a contradiction. Thus, (IIC$_i$) must be tight in equilibrium for any λ in regime Λ_1^{ij}.

Finally, following the tightness of (IIC$_i$) at optimum of (OPT-F), by rearranging the constraint (IIC$_{ij}$) in (2.28), we have $\widehat{J}^j(f^*) \leq (1 - |\lambda^{-ij}|)\, D - \widehat{J}^i(f^*) = \lambda^j D$. Thus, (IIC$_j$) is guaranteed to hold in Regime Λ_1^{ij} given the constraint (IIC$_{ij}$) and the fact that (IIC$_i$) is tight at the optimum of (OPT-F). Hence, (IIC$_j$) can be dropped in (OPT-F) without changing the optimal solution set.

$\left[\text{Regime } \Lambda_3^{ij}\right]$: Analogous to the proof given for regime Λ_1^{ij}, we can argue that constraint (IIC$_j$) is tight in any equilibrium for any λ in regime Λ_3^{ij}. By imposing constraint (IIC$_{ij}$), (IIC$_i$) can be dropped from the constraint set in (OPT-F) without changing the optimal solution set.

$\left[\text{Regime } \Lambda_2^{ij}\right]$: To study this regime, we need two additional thresholds

$$\underline{\underline{\lambda}}^i \triangleq \frac{1}{D} \max_{f^{ij,\dagger} \in F^{ij,\dagger}} \left\{ \widehat{J}^i(f^{ij,\dagger}) \right\}, \quad \bar{\bar{\lambda}}^i \triangleq \frac{1}{D} \min_{f^{ij,\dagger} \in F^{ij,\dagger}} \left\{ \left(1 - |\lambda^{-ij}|\right) D - \widehat{J}^j(f^{ij,\dagger}) \right\}.$$

From (2.26), we can check that $\underline{\lambda}^i \leq \underline{\underline{\lambda}}^i$, and $\bar{\bar{\lambda}}^i \leq \bar{\lambda}^i$.

For any $\lambda^i \in [\underline{\lambda}^i, \underline{\underline{\lambda}}^i]$, we argue that $F^*(\lambda) \subseteq F^{ij,\dagger}$. Since the set $F^{ij,\dagger}$ as defined by (2.25) is a bounded polytope, and $\underline{\lambda}^i$ (resp. $\underline{\underline{\lambda}}^i$) is the minimum (resp. maximum) value of the continuous function $\widehat{J}^i(f^{ij,\dagger})$ on $F^{ij,\dagger}$, we know from the mean value theorem that there exists a $\tilde{f}^{ij,\dagger} \in F^{ij,\dagger}$ satisfying: $\lambda^i = \frac{1}{D}\widehat{J}^i(\tilde{f}^{ij,\dagger})$. Such $\tilde{f}^{ij,\dagger}$ also satisfies constraint (IIC$_j$). Therefore, $\tilde{f}^{ij,\dagger}$ satisfies all the constraints in (2.17), and minimizes $\widehat{\Phi}(f)$. So $\tilde{f}^{ij,\dagger}$ is an equilibrium route flow, which implies that $F^*(\lambda) \cap F^{ij,\dagger} \neq \emptyset$. Since the equilibrium edge load vector is unique, and the edge load induced by $\tilde{f}^{ij,\dagger}$ is $w^{ij,\dagger}$, we must have $w^*(\lambda) = w^{ij,\dagger}$. Furthermore, from (2.25), $F^{ij,\dagger}$ is a superset of all feasible route flows that can induce $w^{ij,\dagger}$. Therefore, $F^*(\lambda) \subseteq F^{ij,\dagger}$ for any $\lambda^i \in [\underline{\lambda}^i, \underline{\underline{\lambda}}^i]$. Similarly, we can argue that for any $\lambda^i \in [\bar{\bar{\lambda}}^i, \bar{\lambda}^i]$, $F^*(\lambda) \subseteq F^{ij,\dagger}$.

To prove that $F^*(\lambda) \subseteq F^{ij,\dagger}$ for any λ in regime Λ_2^{ij}, we need to argue two cases $\underline{\underline{\lambda}}^i \geq \bar{\bar{\lambda}}^i$ and $\underline{\underline{\lambda}}^i < \bar{\bar{\lambda}}^i$ separately. If $\underline{\underline{\lambda}}^i \geq \bar{\bar{\lambda}}^i$, then $[\underline{\lambda}^i, \bar{\lambda}^i] \subseteq [\underline{\lambda}^i, \underline{\underline{\lambda}}^i] \cup [\bar{\bar{\lambda}}^i, \bar{\lambda}^i]$. Therefore, $F^*(\lambda) \subseteq F^{ij,\dagger}$ for any

λ in regime Λ_2^{ij}. If $\underline{\underline{\lambda}}^i < \bar{\bar{\lambda}}^i$, for any $\lambda^i \in (\underline{\underline{\lambda}}^i, \bar{\bar{\lambda}}^i)$, we can check that any $f^{ij,\dagger} \in F^{ij,\dagger}$ satisfies the constraint (IIC$_i$): $\frac{1}{D}\widehat{J}^i(f^{ij,\dagger}) \leq \underline{\underline{\lambda}}^i < \lambda^i$. Additionally, since $\lambda^i < \bar{\bar{\lambda}}^i \leq 1 - |\lambda^{-ij}| - \frac{1}{D}\widehat{J}^j(f^{ij,\dagger})$, we know that $\frac{1}{D}\widehat{J}^j(f^{ij,\dagger}) < 1 - |\lambda^{-ij}| - \lambda^i = \lambda^j$, i.e. $f^{ij,\dagger}$ also satisfies the constraint (IIC$_j$). Thus, any $f^{ij,\dagger} \in F^{ij,\dagger}$ is an equilibrium route flow, i.e. $F^*(\lambda) = F^{ij,\dagger}$ for any $\lambda^i \in (\underline{\underline{\lambda}}^i, \bar{\bar{\lambda}}^i)$. Combined with the fact that $F^*(\lambda) \subseteq F^{ij,\dagger}$ for any $\lambda \in [\underline{\lambda}^i, \underline{\underline{\lambda}}^i] \cup [\bar{\bar{\lambda}}^i, \bar{\lambda}^i]$, we know that $F^*(\lambda) \subseteq F^{ij,\dagger}$ for any λ in regime Λ_2^{ij}. □

Corollary A.1. *If the game $\Gamma(\lambda)$ has a parallel-route network, then the equilibrium route flow f^* is unique. Moreover, if there are two populations, then the equilibrium strategy profile is unique in regime Λ_1^{12} or Λ_3^{12}, and can be written as follows:*

$$\text{In regime } \Lambda_1^{ij}: \quad q_r^{1*}(t^1) = f_r^*(t^1, \widehat{t}^2) - \min_{\widehat{t}^1 \in T^1} f_r^*(\widehat{t}^1, \widehat{t}^2), \quad \forall r \in R, \quad \forall t^1 \in T^1, \tag{A.10a}$$

$$q_r^{2*}(t^2) = \min_{\widehat{t}^1 \in T^1} f_r^*(\widehat{t}^1, t^2), \quad \forall r \in R, \quad \forall t^2 \in T^2, \tag{A.10b}$$

$$\text{In regime } \Lambda_3^{ij}: \quad q_r^{1*}(t^1) = \min_{\widehat{t}^2 \in T^2} f_r^*(t^1, \widehat{t}^2), \quad \forall r \in R, \quad \forall t^1 \in T^1, \tag{A.10c}$$

$$q_r^{2*}(t^2) = f_r^*(\widehat{t}^1, t^2) - \min_{\widehat{t}^2 \in T^2} f_r^*(\widehat{t}^1, \widehat{t}^2), \quad \forall r \in R, \quad \forall t^2 \in T^2, \tag{A.10d}$$

where $\left(\widehat{t}^1, \widehat{t}^2\right)$ is any type profile.

Proof of Corollary A.1. Given a parallel route network, we immediately obtain the uniqueness of f^* from Theorem 2.1. Then from Proposition 2.1, any strategy profile that can induce f^* can be expressed as in (2.22). In regime Λ_1^{12}, we know from Theorem 2.2 that the constraint (IIC$_1$) is tight in equilibrium. Therefore, from (2.23a) and (2.23c), we obtain:

$$\lambda^1 D \overset{(2.23a)}{=} \sum_{r \in R} \chi_r^1 \overset{(2.23c)}{\geq} \sum_{r \in R} \max_{t^1 \in T^1} \left(f_r^*(\widehat{t}^1, \widehat{t}^2) - f_r^*(t^1, \widehat{t}^2)\right) \overset{(2.19)}{=} \widehat{J}^1(f^*) = \lambda^1 D.$$

Thus, (2.23c) is tight for any $r \in R$, i.e. $\chi_r^1 = \max_{t^1 \in T^1}\left(f_r^*(\widehat{t}^1, \widehat{t}^2) - f_r^*(t^1, \widehat{t}^2)\right)$. Additionally, from (2.23a), $\chi_r^2 = \min_{t^1 \in T^1} f_r^*(t^1, \widehat{t}^2)$. Thus, χ as defined in (2.23) is unique. Following (2.22), we can obtain the unique q^* as defined in (A.10a)-(A.10b). Analogously, we can argue that the equilibrium strategy profile is also unique in regime Λ_3^{12}, and is written as in (A.10c)-(A.10d). □

Proof of Proposition 2.3. $\left[\text{Regime } \Lambda_1^{ij}\right]$: Consider any population size vector $\lambda \in \Lambda_1^{ij}$, there exists a sufficiently small $\epsilon > 0$ such that $\lambda' = \lambda + \epsilon z^{ij} \in \Lambda_1^{ij}$, i.e. $\lambda^{i'} = \lambda^i + \epsilon > \lambda^i$, $\lambda^{j'} = \lambda^j - \epsilon < \lambda^j$, and the sizes of all other populations remain unchanged. Consider any equilibrium route flow $f^*(\lambda) \in F^*(\lambda)$ and any $f^*(\lambda') \in F^*(\lambda')$. We know from Theorem 2.2 that constraint (IIC_i) is tight in equilibrium, and thus $f^*(\lambda)$ and $f^*(\lambda')$ satisfy: $\frac{1}{D}\widehat{J}^i(f^*(\lambda)) = \lambda^i < \lambda^{i'} = \frac{1}{D}\widehat{J}^i(f^*(\lambda'))$. Consequently, any equilibrium route flow $f^*(\lambda)$ for size vector λ is in the feasible domain of (2.28) for size vector λ', but $f^*(\lambda) \notin F^*(\lambda')$, because $\widehat{J}^i\left(f^*(\lambda)\right) = \lambda^i < \lambda^{i'}$, i.e. the constraint ($\text{IIC}_i$) is satisfied, but not tight. Since $f^*(\lambda') \in F^*(\lambda')$, we must have $\Psi(\lambda') = \widehat{\Phi}(f^*(\lambda')) < \widehat{\Phi}(f^*(\lambda)) = \Psi(\lambda)$.

Additionally, from (2.24), we know that $\Psi(\lambda') = \breve{\Phi}(w^*(\lambda')) < \breve{\Phi}(w^*(\lambda)) = \Psi(\lambda)$. Thus, the unique equilibrium edge load $w^*(\lambda)$ necessarily changes in the direction z^{ij} in regime Λ_1^{ij}.

$\left[\text{Regime } \Lambda_2^{ij}\right]$: From Theorem 2.2, $F^*(\lambda) \subseteq F^{ij,\dagger}$ for any $\lambda \in \Lambda_2^{ij}$. Since the equilibrium edge load is unique, we know $w^*(\lambda) = w^{ij,\dagger}$. From (2.24) we can conclude that $\Psi(\lambda) = \breve{\Phi}(w^{ij,\dagger})$. Thus, $\Psi(\lambda)$ as well as $w^*(\lambda)$ remain fixed in regime Λ_2^{ij}.

$\left[\text{Regime } \Lambda_3^{ij}\right]$: Following similar argument in regime Λ_1^{ij}, we conclude that $\Psi(\lambda)$ monotonically increases in the direction z^{ij} in regime Λ_3^{ij}. As a result, $w^*(\lambda)$ changes when λ is perturbed in the direction z^{ij} in regime Λ_3^{ij}. $\square$

Lemma A.3. *(Fiacco and Kyparisis [1986], page 102) The value of the potential function in equilibrium, $\Psi(\lambda)$, is convex with respect to λ if in* (OPT-Q), *$\Phi(q)$ is convex in q, and the constraints are affine in q and λ.*

Lemma A.4. *(Fiacco [2009], page 3469) If in* (OPT-Q)*, the objective function $\Phi(q)$ is convex and continuously differentiable in q, and additionally the set of equilibria q^* and the set of Lagrange multipliers μ^*, ν^* are nonempty and bounded, then $\Psi(\lambda)$ is continuous and directionally differentiable in λ. Furthermore, for any given $i, j \in I$, the directional derivative of $\Psi(\lambda)$ in the direction z^{ij} is $\nabla_{z^{ij}}\Psi(\lambda) = \min_{q^* \in Q^*(\lambda)} \max_{\substack{(\mu^*,\nu^*) \\ \in (M(q^*),N(q^*))}} \nabla_\lambda L(q^*, \mu^*, \nu^*, \lambda) z^{ij}$, where $M(q^*)$ and $N(q^*)$ are the sets of Lagrange multipliers μ^* and ν^* in* (2.15) *associated with the equilibrium $q^* \in Q^*(\lambda)$.*

Proof of Lemma 2.5. Since in (OPT-Q), the weighted potential function $\Phi(q)$ is convex in q, and the constraints (2.4a)-(2.4b) are affine in q and λ, from Lemma A.3, we know that the optimal value of the potential function $\Psi(\lambda)$ is convex in λ.

Next, we can check that (OPT-Q) satisfies the following conditions: (1) The potential function $\Phi(q)$ is continuously differentiable in q, and constraints (2.4a)-(2.4b) are linear in q and λ; (2) The optimal solution set $Q^*(\lambda)$ is non-empty and bounded (Theorem 2.1). The Lagrange multipliers at the optimum of (OPT-Q) are unique, and bounded (Lemma 2.3). Therefore, from Lemma A.4, we know that $\Psi(\lambda)$ is differentiable in the direction z^{ij}, and $\nabla_{z^{ij}}\Psi(\lambda)$ can be expressed as:

$$\begin{aligned}\nabla_{z^{ij}}\Psi(\lambda) &= \min_{q^*\in Q^*(\lambda)} \max_{\substack{(\mu^*,\nu^*)\\ \in (M(q^*),N(q^*))}} \nabla_\lambda \mathcal{L}(q^*,\mu^*,\nu^*,\lambda)\cdot z^{ij} \\ &\overset{(2.15)}{=} \min_{q^*\in Q^*(\lambda)} \max_{\substack{(\mu^*,\nu^*)\\ \in (M(q^*),N(q^*))}} \left(\sum_{t^i\in T^i}\mu^{*t^i} - \sum_{t^j\in T^j}\mu^{*t^j}\right)\cdot D,\end{aligned}$$

where $M(q^*)$ (resp. $N(q^*)$) is the set of optimal Lagrange multipliers μ^* (resp. ν^*) associated with the equilibrium strategy q^*. From Lemma 2.3, since both μ^* and ν^* are unique in equilibrium, $\nabla_{z^{ij}}\Psi(\lambda)$ can be simplified:

$$\begin{aligned}\nabla_{z^{ij}}\Psi(\lambda) &= \left(\sum_{t^i\in T^i}\mu^{*t^i} - \sum_{t^j\in T^j}\mu^{*t^j}\right) D \\ &\overset{(2.16a)}{=} \left(\sum_{t^i\in T^i}\min_{r\in R}\Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \sum_{t^j\in T^j}\min_{r\in R}\Pr(t^j)\mathbb{E}[c_r(q^*)|t^j]\right) D \\ &\overset{(2.10)}{=} \left(C^{i*}(\lambda) - C^{j*}(\lambda)\right) D = -V^{ij*}(\lambda)\cdot D.\end{aligned}$$

□

Proof of Theorem 2.3. First, we know from Proposition 2.3 that in direction z^{ij}, $\Psi(\lambda)$ decreases in regime Λ_1^{ij}, does not change in regime Λ_2^{ij} and increases in regime Λ_3^{ij}. Following Lemma 2.5, we directly obtain that $V^{ij*}(\lambda) > 0$ in Λ_1^{ij}, $V^{ij*}(\lambda) = 0$ in Λ_2^{ij}, and $V^{ij*}(\lambda) < 0$ in Λ_3^{ij}.

Next, from Lemma 2.5, we know that $\Psi(\lambda)$ is convex in λ. Hence, for any $i, j \in I$, the directional derivative $\nabla_{z^{ij}}\Psi(\lambda)$ is non-decreasing in z^{ij}. From (2.29), $V^{ij*}(\lambda)$ is non-increasing in z^{ij}. □

Proof of Proposition 2.4. Since the interim belief of population j, $\beta^j(s, t^{-j}|t^j)$ in (2.30) is independent with type t^j, the equilibrium strategy of the uninformed population $q^{j*}(t^j)$ must be identical across all $t^j \in T^j$. Following (2.18) and (2.19), the impact of information metric $J^j(q^*) = \widehat{J}^j(f^*) = 0$ for any $q^* \in Q^*(\lambda)$, $f^* \in F^*(\lambda)$ and any λ. For the sake of contradiction, we assume that the regime Λ_3^{ij} is non-empty. From Theorem 2.2, we know that the constraint (IIC$_j$) must be tight in equilibrium when λ is in regime Λ_3^{ij}. However, since $\widehat{J}^j(f^*) = 0$ for any λ, the constraint (IIC$_j$) is tight only when $\lambda^j = 0$, i.e. $\lambda^i = 1 - |\lambda^{-ij}|$. This implies that the regime Λ_3^{ij} is indeed empty. Thus, there are at most two regimes Λ_1^{ij} and Λ_2^{ij}. Following Proposition 2.3, we can conclude that $C^{j*}(\lambda) \geq C^{i*}(\lambda)$.

Example A.1. We consider the game with two populations on two parallel routes (r_1 and r_2) with the following cost functions: $c_1^{\mathbf{n}}(f_1) = f_1 + 15$, $c_1^{\mathbf{a}}(f_1) = 3f_1 + 15$, $c_2(f_2) = 20f_2 + 30$. The prior distribution θ, the total demand D, and the information environment are the same as that in Example 2.1. Although both populations receive the accurate signal of the state with positive probability, we have $\bar{\lambda}^1 = 1$ as the impact of information on population 2 is zero. Since the free flow travel time on r_2 is much higher than that on r_1, population 2 travelers exclusively uses r_1 regardless of the received signal, see Fig. A-1.

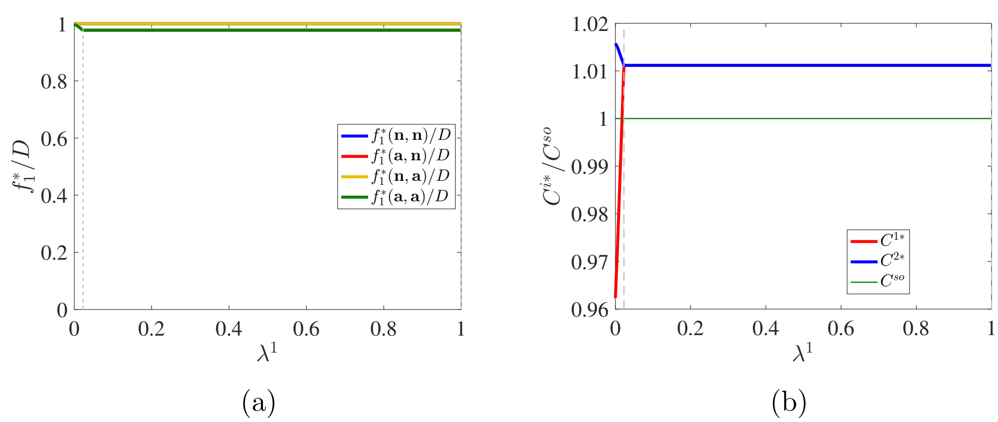
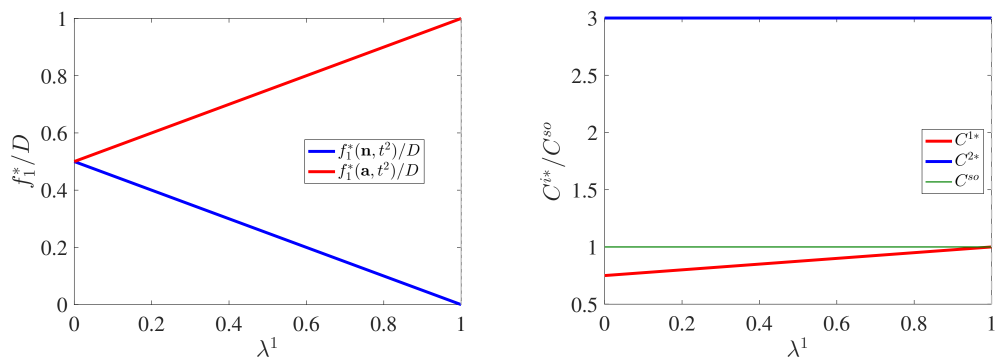


Figure A-1: Effects of varying population sizes for Example A.1: (a) Equilibrium route flows on r_1; (b) Equilibrium population costs.

Example A.2. Consider a game with two populations on two parallel routes (r_1 and r_2). There are two states: $\{s_1, s_2\}$, each state is realized with probability 0.5. The cost functions are: $c_1^{s_1}(f_1) = f_1 + 10$, $c_1^{s_2}(f_1) = f_1 + 1$, $c_2^{s_1}(f_2) = f_2 + 1$, $c_2^{s_2}(f_2) = f_2 + 10$. Population 1 is completely informed, and population 2 is uninformed. The total demand is $D = 1$. We now obtain that $\underline{\lambda}^1 = \bar{\lambda}^1 = 1$; thus, regimes Λ_2^{12} and Λ_3^{12} are empty sets, and population 1 has strictly lower expected cost than population 2 for any $\lambda^1 \in (0, 1)$, see Fig. A-2.



(a) (b)

Figure A-2: Effects of varying population sizes for Example A.2: (a) Equilibrium route flows on r_1; (b) Equilibrium population costs.

A.3 Proofs of Section 2.6

Proof of Proposition 2.5. Firstly, we prove that for any $\lambda \in \Lambda^\dagger$, $F^*(\lambda) \subseteq F^\dagger$. From the definition of $\Lambda^\dagger$ in (2.33), we know that for any $\lambda \in \Lambda^\dagger$, there exists at least one route flow $f^\dagger \in F^\dagger$ satisfying the constraints in (OPT-F), and hence such $f^\dagger$ is a feasible solution of the optimization problem (OPT-F); thus $\widehat{\Phi}(f^\dagger) \geq \Psi(\lambda)$. Additionally, since $f^\dagger$ is an optimal solution of (2.31), which has the same objective function as (OPT-F) but without the constraints (IIC), we conclude that $\widehat{\Phi}(f^\dagger) \leq \Psi(\lambda)$ for any feasible λ (including $\lambda \in \Lambda^\dagger$). Thus, $\Psi(\lambda) = \widehat{\Phi}(f^\dagger)$, and $f^\dagger$ is an equilibrium route flow. Analogous to the argument in proof of Theorem 2.1, the equilibrium edge load equals to $w^\dagger$. Since the set $F^\dagger$ in (2.32) contains all route flows such that the induced edge load is $w^\dagger$, we can conclude that the set of equilibrium route flow $F^*(\lambda) \subseteq F^\dagger$ for any $\lambda \in \Lambda^\dagger$.

Next, we prove that $\Lambda^\dagger = \arg\min_\lambda \Psi(\lambda)$. We have argued in the first part of the proof that for any $f^\dagger \in F^\dagger$, $\widehat{\Phi}(f^\dagger) \leq \Psi(\lambda)$ for any feasible λ; and since for any $\lambda \in \Lambda^\dagger$, $F^*(\lambda) \subseteq F^\dagger$, we have $\Psi(\lambda) = \widehat{\Phi}(f^\dagger)$. Therefore, $\widehat{\Phi}(f^\dagger) = \min_\lambda \Psi(\lambda)$, and $\Lambda^\dagger \subseteq \arg\min_\lambda \Psi(\lambda)$. Additionally, for any $\lambda \in \arg\min_\lambda \Psi(\lambda)$, we have $\Psi(\lambda) = \min_\lambda \Psi(\lambda) = \widehat{\Phi}(f^\dagger)$. Since $F^\dagger$ includes all route flows that satisfy (2.17a)-(2.17c) and attain the minimum value of $\Psi(\lambda)$, any equilibrium route flow $f^* \in F^*(\lambda)$ for $\lambda \in \arg\min_\lambda \Psi(\lambda)$ must be in $F^\dagger$. Hence, such λ must also be in $\Lambda^\dagger$ defined in (2.33), i.e. $\arg\min_\lambda \Psi(\lambda) \subseteq \Lambda^\dagger$. We can therefore conclude that $\Lambda^\dagger = \arg\min_\lambda \Psi(\lambda)$.

From Lemma 2.5, we know that the function $\Psi(\lambda)$ is convex in λ. Additionally, the set of feasible population size vector λ is a closed convex set. Consequently, the set $\Lambda^\dagger = \arg\min_\lambda \Psi(\lambda)$ is convex and non-empty.

Finally, we show that $w^*(\lambda) = w^\dagger$ if and only if $\lambda \in \Lambda^\dagger$. From the first part of the proof, we know that $F^*(\lambda) \subseteq F^\dagger$. Therefore, the unique equilibrium edge load is $w^\dagger$, which does not depend on λ. Additionally, for any feasible $\lambda \notin \Lambda^\dagger$, from the second part of the proof, we know that $\Psi(\lambda) > \check{\Phi}(w^\dagger)$. Thus, $w^*(\lambda) \neq w^\dagger$. $\square$

Proof of Theorem 2.4. Firstly, we show for any $\lambda \in \Lambda^\dagger$, all travelers have identical costs in equilibrium. Consider any $i, j \in I$ such that $\lambda^i > 0$ and $\lambda^j > 0$, the directional derivative of $\Psi(\lambda)$ in the direction z^{ij}, $\nabla_{z^{ij}} \Psi(\lambda)$, must be 0. Otherwise, $\Psi(\lambda)$ strictly decreases in the direction z^{ij} (resp. z^{ji}) if $\nabla_{z^{ij}} \Psi(\lambda) < 0$ (resp. if $\nabla_{z^{ij}} \Psi(\lambda) > 0$), which contradicts the fact that $\Lambda^\dagger = \arg\min_\lambda \Psi(\lambda)$ as in Proposition 2.5. From Lemma 2.5, we know that $C^{i*}(\lambda) = C^{j*}(\lambda)$. Therefore, any two populations with positive size have identical costs in equilibrium.

If any $\lambda \in \Lambda^\dagger$ satisfies $\lambda^i > 0$ for all $i \in I$, then the first step in our proof is sufficient to show that (2.34) is satisfied. Otherwise, for any $\lambda \in \Lambda^\dagger$, and any degenerate population $i \in \{I | \lambda^i = 0\}$, we need to show that $C^{i*}(\lambda) \geq C^{j*}(\lambda)$, where $\lambda^j > 0$. Since $\Lambda^\dagger = \arg\min_\lambda \Psi(\lambda)$, we know that $\Psi(\lambda)$ must be non-decreasing in the direction z^{ij}. Thus, we obtain: $\nabla_{z^{ij}} \Psi(\lambda) \stackrel{(2.29)}{=} \left(C^{i*}(\lambda) - C^{j*}(\lambda)\right) D \geq 0$. Thus, $C^{i*}(\lambda) \geq C^{j*}(\lambda)$. The first and the second steps together show that any $\lambda \in \Lambda^\dagger$ satisfies (2.34), and hence is a vector of equilibrium adoption rates.

Finally, we show that for any feasible $\lambda \notin \Lambda^\dagger$, (2.34) is not satisfied. Since $\Lambda^\dagger = \arg\min_\lambda \Psi(\lambda)$, for any $\lambda \notin \Lambda^\dagger$, we can claim that there must exist a direction z^{ij} such that $\Psi(\lambda)$ decreases in the direction z^{ij}, $\nabla_{z^{ij}}\Psi(\lambda) < 0$. Otherwise, λ is a local minimum of $\Psi(\lambda)$, and since $\Psi(\lambda)$ is convex in λ (Lemma 2.5), λ is a global minimum, which contradicts the fact that $\lambda \notin \Lambda^{\dagger}$. For such a direction z^{ij}, there are two possible cases: (1) $\lambda^i > 0$ and $\lambda^j > 0$. In this case, from (2.29), $C^{i*}(\lambda) \neq C^{j*}(\lambda)$, and thus travelers do not have identical costs in equilibrium. (2) $\lambda^i = 0$ and $\lambda^j > 0$. In this case, from (2.29), we must have $\nabla_{z^{ij}}\Psi(\lambda) = \left(C^{i*}(\lambda) - C^{j*}(\lambda)\right) D < 0$. Therefore, $C^{j*}(\lambda) > C^{i*}(\lambda)$, which implies that travelers in population j has incentive to change subscription to platform i. To sum up, in either case, $\lambda \notin \Lambda^{\dagger}$ cannot be a vector of equilibrium adoption rates. □

A.4 Extension to Networks with Multiple Origin-destination Pairs

In this section, we extend our model to networks with multiple origin-destination pairs, and show that all the results presented in the paper still hold. Consider a network with a set of origin-destination (o-d) pairs K. Each o-d pair $k \in K$ is connected by the set of routes R_k. The set of all routes is $R = \cup_{k \in K} R_k$. The demand of travelers between o-d pair $k \in K$ is $D_k > 0$. The information environment – state, platforms, signals and common prior –is the same as that introduced in Sec. 2.3.1. The fraction of travelers between o-d pair $k \in K$ who subscribe to platform $i \in I$ is λ^i_k. A feasible size vector $\lambda = (\lambda^i_k)_{k \in K, i \in I}$ satisfies $\lambda^i_k \geq 0$ for any $k \in K$ and any $i \in I$, and $\sum_{i \in I} \lambda^i_k = 1$ for any $k \in K$. We denote the strategy profile as $q = (q^i_{r,k}(t^i))_{r \in R_k, i \in I, t^i \in T^i, k \in K}$, where $q^i_{r,k}(t^i)$ is the amount of travelers in population i who take route r between o-d pair k when the signal is t^i. A strategy profile q is feasible if it satisfies:

$$\sum_{r \in R_k} q^i_{r,k}(t^i) = \lambda^i_k \cdot D_k, \quad \forall i \in I, \quad \forall t^i \in T^i, \quad \forall k \in K,$$

$$q^i_{r,k}(t^i) \geq 0, \quad \forall r \in R_k, \quad \forall i \in I, \quad \forall t^i \in T^i, \quad \forall k \in K.$$

For any feasible strategy profile q, the induced route flow vector is $f = (f_{r,k}(t))_{r \in R_k, k \in K, t \in T}$, where $f_{r,k}(t)$ is the flow on route r induced by travelers between o-d pair k when the type profile is t:

$$f_{r,k}(t) = \sum_{i \in I} q^i_{r,k}(t^i), \quad \forall r \in R_k, \quad \forall k \in K, \quad \forall t \in T. \tag{A.11}$$

The aggregate edge load on edge $e \in E$ when the type profile is t can be written as follows:

$$w_e(t) = \sum_{i \in I} \sum_{k \in K} \sum_{r \in \{R_k | r \ni e\}} q^i_{r,k}(t^i) \overset{\text{(A.11)}}{=} \sum_{k \in K} \sum_{r \in \{R_k | r \ni e\}} f_{r,k}(t). \tag{A.12}$$

The expected cost of each route $r \in R$ given type $t^i \in T^i$, denoted $\mathbb{E}[c_r(q)|t^i]$, can be written as in (2.8). A feasible strategy profile q^* is a Bayesian Wardrop equilibrium if for any $k \in K$, any $i \in I$, and any $t^i \in T^i$:

$$\forall r \in R_k, \quad q^{i*}_{r,k}(t^i) > 0 \quad \Rightarrow \quad \mathbb{E}[c_r(q^*)|t^i] \leq \mathbb{E}[c_{r'}(q^*)|t^i], \quad \forall r' \in R_k.$$

We now state the extensions of our results to the network with K o-d pairs. Firstly, we can check that the following function of q is a weighted potential function of the Bayesian congestion game with K o-d pairs:

$$\Phi(q) = \sum_{e \in E} \sum_{s \in S} \sum_{t \in T} \pi(s,t) \int_0^{\sum_{i \in I} \sum_{k \in K} \sum_{r \in \{R_k | r \ni e\}} q^i_{r,k}(t^i)} c^s_e(z) dz,$$

Therefore, given any size vector λ, the set of equilibrium strategy profiles $Q^*(\lambda)$ can be solved by the same convex optimization problem (OPT-Q) in Theorem 2.1. The equilibrium edge load $w^*(\lambda)$ is unique.

Secondly, with simple modifications, we characterize the set of feasible route flow vectors F as follows:

$$F \triangleq \{f \mid \forall k \in K, \ (f_{r,k}(t))_{r \in R_k, t \in T} \text{ satisfies (2.17a) – (2.17d)}\},$$

and the set of equilibrium route flows $F^*(\lambda)$ can be solved by the optimization problem (OPT-F) in Proposition 2.2. Particularly, the information impact constraint (2.17d) for o-d pair $k \in K$ and population $i \in I$ now becomes:

$$D_k - \sum_{r \in R_k} \min_{t^i \in T^i} f_{r,k}(t^i, t^{-i}) \leq \lambda_k^i D_k, \quad \forall t^{-i} \in T^{-i}, \quad \forall i \in I, \quad \forall k \in K. \tag{A.13}$$

That is, the impact of information sent by platform i on the route flows between o-d pair k is bounded by the amount of travelers who subscribe to platform i and travel between o-d pair k.

Thirdly, for any o-d pair k and any pair of platforms i and j, we can analogously analyze how the equilibrium outcomes change with the sizes λ_k^i and λ_k^j, while the sizes of all other populations remain unchanged. We denote this direction of perturbing λ as z_k^{ij}. We can show that Theorem (2.2) holds: three regimes (one or two may be empty) can be distinguished by whether or not the information impact all the travelers between o-d pair k who subscribe to platform i (resp. j), i.e. whether or not (A.13) is tight at the optimum of (OPT-F).

Fourthly, the relative value of information, denoted $V_k^{ij*}(\lambda)$, is the travel cost saving experienced by travelers between o-d pair k who subscribe to platform i compared with travelers who subscribe to platform j. Analogous to Lemma 2.5, we obtain:

$$V_k^{ij*}(\lambda) = C_k^{j*}(\lambda) - C_k^{i*}(\lambda) = -\frac{1}{D_k} \nabla \Phi_{z_k^{ij}}(q).$$

By applying sensitivity analysis, we can show that $V_k^{ij*}(\lambda)$ is positive, zero, and negative in the three regimes respectively.

Finally, we analyze how travelers between each o-d pair $k \in K$ choose platform subscription. By dropping the information impact constraints (A.13) in (OPT-F) and following (2.32) - (2.33), we obtain the extension of Theorem 2.4: characterization of the set $\Lambda^\dagger$, which is the set of size vectors induced by travelers' choice of platforms. We note that how an individual traveler between o-d pair k chooses information subscription not only depends on the choices of other travelers between the same o-d pair, but also the choices of travelers who travel between other o-d pairs. This is because travelers between different o-d pairs may take common routes and edges, and hence impact each other's costs.

Appendix B

Supplementary Material for Chapter 4

B.1 Proofs of Section 4.3

Proof of Lemma 4.1.

First, we show that for any parameter $s \in S$, the sequence $\left(\frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)}\right)_{t=1}^{\infty}$ is a non-negative martingale, and hence converges with probability 1. Note that for any $t = 1, 2, \ldots,$ and any parameter $s \in S \setminus \{s^*\}$, we have the following from (θ-update):

$$\frac{\theta^{k_{t+1}}(s)}{\theta^{k_{t+1}}(s^*)} = \frac{\theta^{k_t}(s) \cdot \prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k)}{\theta^{k_t}(s^*) \cdot \prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)}.$$

Now starting from any initial belief θ^1, consider a sequence of strategies $Q^{k_t-1} \triangleq (q^j)_{j=1}^{k_t-1}$ and a sequence of realized outcomes $Y^{k_t-1} \triangleq (y^j)_{j=1}^{k_t-1}$ before stage k_t. Then, the expected value of $\frac{\theta^{k_{t+1}}(s)}{\theta^{k_{t+1}}(s^*)}$ conditioned on θ^1, Q^{k_t-1} and Y^{k_t-1} is as follows:

$$\mathbb{E}\left[\left.\frac{\theta^{k_{t+1}}(s)}{\theta^{k_{t+1}}(s^*)}\right| \theta^1, Q^{k_t-1}, Y^{k_t-1}\right] = \frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)} \cdot \mathbb{E}\left[\frac{\prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k)}{\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)}\right] \tag{B.1}$$

where θ^{k_t} is the repeatedly updated belief from θ^1 based on Q^{k_t-1} and Y^{k_t-1} using (θ-update). Note that

$$\mathbb{E}\left[\frac{\prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k)}{\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)}\right]$$

$$= \int_{y^{k_t} y^{k_t+1} y^{k_{t+1}-1}} \left(\frac{\prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k)}{\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)} \right) \cdot \left(\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k) \right) dy^{k_t} y^{k_t+1} y^{k_{t+1}-1}$$

$$= \int_{y^{k_t} y^{k_t+1} y^{k_{t+1}-1}} \prod_{k=k_t}^{k_{t+1}-1} \phi^s(y^k|q^k) dy^{k_t} y^{k_t+1} y^{k_{t+1}-1} = 1.$$

Hence, for any $k = 1, 2, \ldots,$

$$\mathbb{E}\left[\frac{\theta^{k_{t+1}}(s)}{\theta^{k_{t+1}}(s^*)} \middle| \theta^1, Q^{k_t-1}, Y^{k_t-1} \right] = \frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)}, \quad \forall s \in S.$$

Again, from (θ-update) we know that $\frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)} \geq 0$. Hence, the sequence $\left(\frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)}\right)_{t=1}^{\infty}$ is a non-negative martingale for any $s \in S$. From the martingale convergence theorem, we conclude that $\frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)}$ converges with probability 1.

Next we show that the sequence $\left(\log \theta^{k_t}(s^*)\right)_{t=1}^{\infty}$ is a submartingale, and hence converges with probability 1. We define the estimated density function of payoffs $\left(y^k\right)_{k=k_t}^{k_{t+1}-1}$ with belief θ as $\mu\left(\left(y^k\right)_{k=k_t}^{k_{t+1}-1} \middle| \theta^{k_t}, \left(q^k\right)_{k=k_t}^{k_{t+1}-1}\right) \triangleq \sum_{s \in S} \theta^{k_t}(s) \prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)$. From ($\theta$-update), we have:

$$\mathbb{E}\left[\log \theta^{k_{t+1}}(s^*) \middle| \theta^1, Q^{k_t-1}, Y^{k_t-1}\right] = \mathbb{E}\left[\log\left(\frac{\theta^{k_t}(s^*) \prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)}{\mu\left(\left(y^k\right)_{k=k_t}^{k_{t+1}-1} \middle| \theta^{k_t}, \left(q^k\right)_{k=k_t}^{k_{t+1}-1}\right)} \right) \middle| \theta^1, Q^{k_t-1}, Y^{k_t-1} \right]$$

$$= \log \theta^{k_t}(s^*) + \mathbb{E}\left[\log\left(\frac{\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)}{\mu\left(\left(y^k\right)_{k=k_t}^{k_{t+1}-1} \middle| \theta^{k_t}, \left(q^k\right)_{k=k_t}^{k_{t+1}-1}\right)} \right) \right]$$

$$= \log \theta^{k_t}(s^*)$$
$$+ \int_{y^{k_t} y^{k_t+1} y^{k_{t+1}-1}} \left(\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k) \right) \log\left(\frac{\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)}{\mu\left(\left(y^k\right)_{k=k_t}^{k_{t+1}-1} \middle| \theta^{k_t}, \left(q^k\right)_{k=k_t}^{k_{t+1}-1}\right)} \right) dy^{k_t} y^{k_t+1} y^{k_{t+1}-1}$$

$$= \log \theta^{k_t}(s^*) + D_{KL}\left(\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k) \middle\| \mu\left(\left(y^k\right)_{k=k_t}^{k_{t+1}-1} \middle| \theta^{k_t}, \left(q^k\right)_{k=k_t}^{k_{t+1}-1}\right) \right) \geq \log \theta^{k_t}(s^*),$$

where the last inequality is due to the non-negativity of KL divergence between the distributions $\prod_{k=k_t}^{k_{t+1}-1} \phi^{s^*}(y^k|q^k)$ and $\mu\left(\left(y^k\right)_{k=k_t}^{k_{t+1}-1} \middle| \theta^{k_t}, \left(q^k\right)_{k=k_t}^{k_{t+1}-1}\right)$. Therefore, the sequence $\left(\log \theta^{k_t}(s^*)\right)_{k=1}^{\infty}$ is a submartingale. Additionally, since $\log \theta^{k_t}(s^*)$ is bounded above by zero, by the martingale convergence theorem $\log \theta^{k_t}(s^*)$ converges with probability 1. Hence,

$\theta^{k_t}(s^*)$ must also converge with probability 1.

From the convergence of $\frac{\theta^{k_t}(s)}{\theta^{k_t}(s^*)}$ and $\theta^{k_t}(s^*)$, we conclude that $\theta^{k_t}(s)$ converges with probability 1 for any $s \in S$. Since for any $k = k_t + 1, \ldots, k_{t+1} - 1$, $\theta^k = \theta^{k_t}$, we know that θ^k also converges. Let the convergent vector be denoted as $\bar{\theta} = \left(\bar{\theta}(s)\right)_{s \in S}$. We can check that for any k, $\theta^k(s) \geq 0$ for all $s \in S$ and $\sum_{s \in S} \theta^k(s) = 1$. Hence, $\bar{\theta}$ must satisfy $\bar{\theta}(s) \geq 0$ for all $s \in S$ and $\sum_{s \in S} \bar{\theta}(s) = 1$, i.e. $\bar{\theta}$ is a feasible belief vector. $\square$

Before proceeding, we show that the best response correspondence is upper hemicontinuous in the belief and the strategy profile. This result follows directly from the Berge's theorem of maximum and the fact that the expected utility function $\mathbb{E}_\theta\left[u_i^s(q_i, q_{-i})\right]$ is continuous in θ and q.

Lemma B.1. *For any $\theta \in \Delta(S)$, any $i \in I$ and any $q_{-i} \in Q_{-i}$, $\mathrm{BR}(q_{-i}, \theta)$ is upper-hemicontinuous in θ and q_{-i}.*

We are now ready to prove Lemma 4.2.

Proof of Lemma 4.2. For any stage $K \geq 1$, we construct an auxiliary sequence of strategies $\left(\hat{q}^k\right)_{k=1}^{\infty}$ as follows: First, we set $\hat{q}^k = q^k$ for all $k = 1, \ldots, K$. Then, for any $k > K$, we define the following subsequences:

- We define $\tilde{q}^{k+1} \triangleq \arg\min_{\tilde{q} \in F(\bar{\theta}, q^k)} \|\tilde{q} - q^{k+1}\|$. That is, $\tilde{q}^{k+1}$ is a strategy updated from q^k with the fixed point belief $\bar{\theta}$ (i.e. $\tilde{q}^{k+1} \in F(\bar{\theta}, q^k)$). Additionally, $\tilde{q}^{k+1}$ is the closest to q^{k+1} – the strategy in stage $k+1$ of the original sequence – among all strategies in the set $F(\bar{\theta}, q^k)$.

- We define the auxiliary strategy $\hat{q}^{k+1} \triangleq \arg\min_{q \in F(\bar{\theta}, \hat{q}^k)} \|q - \tilde{q}^{k+1}\|$. That is, $\hat{q}^{k+1}$ is a strategy updated from $\hat{q}^k$ with the fixed point belief $\bar{\theta}$ (i.e. $\hat{q}^{k+1} \in F(\bar{\theta}, \hat{q}^k)$). Additionally, $\hat{q}^{k+1}$ is the closest to $\tilde{q}^{k+1}$ among all strategies in the set $F(\bar{\theta}, \hat{q}^k)$.

Therefore, for any $k > K$, we have:

$$\|q^{k+1} - \tilde{q}^{k+1}\| = D\left(q^{k+1}, F(\bar{\theta}, q^k)\right), \ \|\tilde{q}^{k+1} - \hat{q}^{k+1}\| = D\left(\tilde{q}^{k+1}, F(\bar{\theta}, \hat{q}^k)\right). \qquad \text{(B.2)}$$

We next show by mathematical induction that for any $\ell \geq 1$, $\lim_{K \to \infty} \|q^{K+\ell} - \hat{q}^{K+\ell}\| = 0$.

To begin with, for $\ell = 1$, we have

$$\begin{aligned}\|q^{K+1} - \hat{q}^{K+1}\| &\leq \|q^{K+1} - \tilde{q}^{K+1}\| + \|\tilde{q}^{K+1} - \hat{q}^{K+1}\| \\ &\overset{(\text{B.2})}{=} D\left(q^{K+1}, F\left(\bar{\theta}, q^K\right)\right) + D\left(\tilde{q}^{K+1}, F\left(\bar{\theta}, \hat{q}^K\right)\right).\end{aligned} \tag{B.3}$$

Since θ^k converges to $\bar{\theta}$ (Lemma 4.1), $F(\theta, q)$ is upper hemicontinuous in θ (Lemma B.1), and $q^{K+1} \in F(\theta^{K+1}, q^K)$, we know that $\lim_{K\to\infty} D\left(q^{K+1}, F\left(\bar{\theta}, q^K\right)\right) = 0$. Additionally, since $\hat{q}^K = q^K$ and $\tilde{q}^{K+1} \in F(\bar{\theta}, q^K)$, $D\left(\tilde{q}^{K+1}, F\left(\bar{\theta}, \hat{q}^K\right)\right) = 0$. Therefore, $\lim_{K\to\infty} \|q^{K+1} - \hat{q}^{K+1}\| = 0$.

Now, assume that $\lim_{K\to\infty} \|q^{K+\ell} - \hat{q}^{K+\ell}\| = 0$ for some $\ell \geq 1$, we need to prove that $\lim_{K\to\infty} \|q^{K+\ell+1} - \hat{q}^{K+\ell+1}\| = 0$. Similar to (B.3), we have

$$\|q^{K+\ell+1} - \hat{q}^{K+\ell+1}\| \leq D\left(q^{K+\ell+1}, F\left(\bar{\theta}, q^{K+\ell}\right)\right) + D\left(\tilde{q}^{K+\ell+1}, F\left(\bar{\theta}, \hat{q}^{K+\ell}\right)\right)$$

Analogous to $\ell = 1$, since $F(\theta, q)$ is upper hemicontinuous in θ,

$$\lim_{K\to\infty} D\left(q^{K+\ell+1}, F\left(\bar{\theta}, q^{K+\ell}\right)\right) = 0.$$

Additionally, since $\lim_{K\to\infty} \|q^{K+\ell} - \hat{q}^{K+\ell}\| = 0$, $F(\theta, q)$ is upper hemicontinuous in q, and $\tilde{q}^{K+\ell+1} \in F\left(\bar{\theta}, q^{K+\ell}\right)$, we know that $\lim_{K\to\infty} D\left(\tilde{q}^{K+\ell+1}, F\left(\bar{\theta}, \hat{q}^{K+\ell}\right)\right) = 0$. Therefore, we have $\lim_{K\to\infty} \|q^{K+\ell+1} - \hat{q}^{K+\ell+1}\| = 0$. By mathematical induction, we conclude that for any $\ell \geq 1$, $\lim_{K\to\infty} \|q^{K+\ell} - \hat{q}^{K+\ell}\| = 0$.

Finally, Assumption 1 ensures that the strategy update with constant beliefs $\bar{\theta}$ converges to an equilibrium strategy $\bar{q} \in \mathrm{EQ}(\bar{\theta})$. That is, for any $K \geq 1$, $\lim_{\ell\to\infty} \|\hat{q}^{K+\ell} - \bar{q}\| = 0$. Therefore,

$$\lim_{k\to\infty} \|q^k - \bar{q}\| = \lim_{\ell\to\infty} \lim_{K\to\infty} \|q^{K+\ell} - \bar{q}\| \leq \lim_{\ell\to\infty} \lim_{K\to\infty} \|q^{K+\ell} - \hat{q}^{K+\ell}\| + \lim_{\ell\to\infty} \lim_{K\to\infty} \|\hat{q}^{K+\ell} - \bar{q}\| = 0.$$

Thus, $\lim_{k\to\infty} q^k = \bar{q}$. $\square$

Proof of Lemma 4.3. By iteratively applying the belief update in (θ-update), we can write:

$$\theta^{k_t}(s) = \frac{\theta^1(s)\prod_{j=1}^{k_t-1}\phi^s(y^j|q^j)}{\sum_{s'\in S}\theta^1(s')\prod_{j=1}^{k_t-1}\phi^{s'}(y^j|q^j)}, \quad \forall s \in S. \tag{B.4}$$

We define $\Phi^s(Y^{k_t-1}|Q^{k_t-1})$ as the probability density function of the history of the realized outcomes $Y^{k_t-1} = (y^j)_{j=1}^{k_t-1}$ conditioned on the history of strategies $Q^{k_t-1} = (q^j)_{j=1}^{k_t-1}$ prior to stage k_t, i.e. $\Phi^s(Y^{k_t-1}|Q^{k_t-1}) \triangleq \prod_{j=1}^{k_t-1}\phi^s(y^j|q^j)$. We rewrite (B.4) as follows:

$$\begin{aligned}\theta^{k_t}(s) &= \frac{\theta^1(s)\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\sum_{s'\in S}\theta^1(s')\Phi^{s'}(Y^{k_t-1}|Q^{k_t-1})} \le \frac{\theta^1(s)\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\theta^1(s)\Phi^s(Y^{k_t-1}|Q^{k_t-1}) + \theta^1(s^*)\Phi^{s^*}(Y^{k_t-1}|Q^{k_t-1})} \\ &= \frac{\theta^1(s)\frac{\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\Phi^{s^*}(Y^{k_t-1}|Q^{k_t-1})}}{\theta^1(s)\frac{\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\Phi^{s^*}(Y^{k_t-1}|Q^{k_t-1})} + \theta^1(s^*)}.\end{aligned} \tag{B.5}$$

For any $s \in S \setminus S^*(\bar{q})$, if we can show that the ratio $\frac{\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\Phi^{s^*}(Y^{k_t-1}|Q^{k_t-1})}$ converges to 0, then $\theta(s)$ must also converge to 0. Now, we need to consider two cases:

Case 1: $\phi^{s^*}(y|\bar{q}) \ll \phi^s(y|\bar{q})$: In this case, the log-likelihood ratio can be written as:

$$\log\left(\frac{\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\Phi^{s^*}(Y^{k_t-1}|Q^{k_t-1})}\right) = \sum_{j=1}^{k_t-1}\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right). \tag{B.6}$$

For any $s \in S$, since $\phi^s(y^j|q^j)$ is continuous in q^j, the probability density function of $\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)$ is also continuous in q^j. In Lemma 4.2, we proved that $\left(q^k\right)_{k=1}^{\infty}$ converges to $\bar{q}$. Then, the distribution of $\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)$ must converge to the distribution of $\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)$. Note that for any $s \in S \setminus S^*(\bar{q})$, the expectation of $\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)$ can be written as:

$$\mathbb{E}\left[\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)\right] = \int_y \phi^{s^*}(y|\bar{q})\cdot\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)dy = -D_{KL}\left(\phi^{s^*}(y|\bar{q})||\phi^s(y|\bar{q})\right) < 0.$$

If we can show that the equation (B.7) below holds, then we can conclude that the log-likelihood sequence defined by (B.6) converges to $-\infty$; this would in turn imply that the sequence of likelihood ratios $\frac{\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\Phi^{s^*}(Y^{k_t-1}|Q^{k_t-1})}$ must converge to 0. But first we need to show:

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{k_t-1}\log\left(\frac{\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\Phi^{s^*}(Y^{k_t-1}|Q^{k_t-1})}\right) &= \lim_{t\to\infty}\frac{1}{k_t-1}\sum_{j=1}^{k_t-1}\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)\\ &= \mathbb{E}\left[\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)\right], \quad w.p.\ 1. \end{aligned}\tag{B.7}$$

We denote the cumulative distribution function of $\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)$ as $F^j(z):\mathbb{R}\to[0,1]$, i.e. $F^j(z)=\Pr\left(\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)\leq z\right)$. The cumulative distribution function of $\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)$ is denoted $\bar{F}(z):\mathbb{R}\to[0,1]$, i.e. $\bar{F}(z)=\Pr\left(\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)\leq z\right)$. Then,

$$\lim_{j\to\infty}F^j(z)=\bar{F}(z), \quad \forall z\in\mathbb{R}. \tag{B.8}$$

For any sequence of realized outcomes $(y^j)_{j=1}^{\infty}$, we define a sequence of random variables $\Delta=(\Delta^j)_{j=1}^{\infty}$, where $\Delta^j=F^j\left(\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)\right)$. Then, we must have $\Delta^j\in[0,1]$, and for any $\delta\in[0,1]$, $\Pr(\Delta^j\leq\delta)=\Pr\left(F^j\left(\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)\right)\leq\delta\right)=\delta$. That is, Δ^j is independently and uniformly distributed on $[0,1]$. Consider another sequence of random variables $(\eta^j)_{j=1}^{\infty}$, where $\eta^j\triangleq\left(\bar{F}\right)^{-1}(\Delta^j)$. Since Δ^j is independently and identically distributed (i.i.d.) with uniform distribution, η^j is also i.i.d. distributed with the same distribution as $\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)$. Additionally, since each Δ^j is generated from the realized outcome y^j, $(\eta^j)_{j=1}^{\infty}$ is in the same probability space as $\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)$. From (B.8), we know that as $j\to\infty$, F^j converges to $\bar{F}$. Therefore, with probability 1,

$$\lim_{j\to\infty}\left|\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)-\eta^j\right|=\lim_{j\to\infty}\left|\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)-(\bar{F})^{-1}F^j\left(\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)\right)\right|=0.$$

Consequently, with probability 1,

$$\lim_{t\to\infty}\left|\frac{1}{k_t-1}\sum_{j=1}^{k_t-1}\left(\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)-\eta^j\right)\right|\leq\lim_{t\to\infty}\frac{1}{k_t-1}\sum_{j=1}^{k_t-1}\left|\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)-\eta^j\right|=0. \tag{B.9}$$

Since $(\eta^j)_{j=1}^{\infty}$ is independently and identically distributed according to the distribution of

$\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)$, from strong law of large numbers, we have:

$$\lim_{t\to\infty}\frac{1}{k_t-1}\sum_{j=1}^{k_t-1}\eta^j=\mathbb{E}\left[\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)\right]=-D_{KL}\left(\phi^{s^*}(y|\bar{q})||\phi^s(y|\bar{q})\right),\quad w.p.\ 1.$$

From (B.9), we obtain the following:

$$\lim_{t\to\infty}\frac{1}{k_t-1}\sum_{j=1}^{k_t-1}\log\left(\frac{\phi^s(y^j|q^j)}{\phi^{s^*}(y^j|q^j)}\right)=\lim_{t\to\infty}\frac{1}{k_t-1}\sum_{j=1}^{k_t-1}\eta^j=-D_{KL}\left(\phi^{s^*}(y|\bar{q})||\phi^s(y|\bar{q})\right),\quad w.p.\ 1 \tag{B.10}$$

Hence, (B.7) holds. Then, for any $s\in S\setminus S^*(\bar{q})$, $\lim_{t\to\infty}\frac{\Phi^s(Y^{k_t-1}|Q^{k_t-1})}{\Phi^{s^*}(Y^{k_t-1}|Q^{k_t-1})}=0$. Thus, from (B.5), we know that $\lim_{t\to\infty}\theta^{k_t}(s)=0$ for all $s\in S\setminus S^*(\bar{q})$. Since for any $k=k_t+1,\dots,k_{t+1}-1$, $\theta^k=\theta^{k_t}$, we know that $\lim_{k\to\infty}\frac{\Phi^s(Y^{k-1}|Q^{k-1})}{\Phi^{s^*}(Y^{k-1}|Q^{k-1})}=0$ and $\lim_{k\to\infty}\theta^k(s)=0$ for all $s\in S\setminus S^*(\bar{q})$.

Finally, since $\theta^1(s)>0$ for all $s\in S$, the true parameter s^* is never excluded from the belief. Therefore, $\lim_{k\to\infty}\frac{1}{k}\log\left(\theta^k(s^*)\right)=0$. For any $s\in S\setminus S^*(\bar{q})$, we have the following:

$$\begin{aligned}
&\lim_{k\to\infty}\frac{1}{k}\log\left(\theta^k(s)\right)=\lim_{k\to\infty}\frac{1}{k}\log\left(\theta^k(s^*)\right)+\lim_{k\to\infty}\frac{1}{k}\log\left(\frac{\theta^k(s)}{\theta^k(s^*)}\right)\\
&=\lim_{k\to\infty}\frac{1}{k}\log\left(\frac{\theta^k(s)}{\theta^k(s^*)}\right)=\lim_{k\to\infty}\frac{1}{k}\log\left(\frac{\theta^1(s)}{\theta^1(s^*)}\right)+\lim_{k\to\infty}\frac{1}{k}\log\left(\frac{\Phi^s(Y^{k-1}|Q^{k-1})}{\Phi^{s^*}(Y^{k-1}|Q^{k-1})}\right)\\
&=\mathbb{E}\left[\log\left(\frac{\phi^s(y|\bar{q})}{\phi^{s^*}(y|\bar{q})}\right)\right]=-D_{KL}\left(\phi^{s^*}(y|\bar{q})||\phi^s(y|\bar{q})\right),\quad w.p.\ 1.
\end{aligned}$$

Case 2: $\phi^{s^*}(y|\bar{q})$ *is not absolutely continuous in* $\phi^s(y|\bar{q})$.
In this case, $\phi^s(y|\bar{q})=0$ does not imply $\phi^{s^*}(y|\bar{q})=0$ with probability 1, i.e. $\Pr\left(\phi^s(y|\bar{q})=0\right)>0$, where $\Pr(\cdot)$ is the probability of y with respect to the true distribution $\phi^{s^*}(y|\bar{q})$. Since the distributions $\phi^s(y|q)$ and $\phi^{s^*}(y|q)$ are continuous in q, the probability $\Pr\left(\phi^s(y|q)=0\right)$ must also be continuous in q. Therefore, for any $\epsilon\in(0,\Pr\left(\phi^s(y|\bar{q})=0\right))$, there exists $\delta>0$ such that $\Pr\left(\phi^s(y|q)=0\right)>\epsilon$ for all $q\in\{q|\|q-\bar{q}\|<\delta\}$.

From Lemma 4.2, we know that $\lim_{k\to\infty}q^k=\bar{q}$. Hence, we can find a positive number $K_1>0$ such that for any $k>K_1$, $\|q^k-\bar{q}\|<\delta$, and hence $\Pr\left(\phi^s(y^k|q^k)=0\right)>\epsilon$. We then have $\sum_{k=1}^{\infty} \Pr\left(\phi^s(y^k|q^k) = 0\right) = \infty$. Moreover, since the event $\phi^s(y^k|q^k) = 0$ is independent from the event $\phi^s(y^{k'}|q^{k'}) = 0$ for any k, k', we can conclude that $\Pr\big(\phi^s(y^k|q^k) = 0,$ infinitely often$\big) = 1$ based on the second Borel-Cantelli lemma. Hence, we must have $\Pr\left(\phi^s(y^k|q^k) > 0,\ \forall k\right) = 0$. From the Bayesian update (θ-update), we know that if $\phi^s(y^k|q^k) = 0$ for some stage k, then any belief of s updated after stage k is 0. Therefore, we can conclude that $\Pr\left(\theta^k(s) > 0,\ \forall k\right) = 0$ with probability 1, i.e. there exists a positive number $K^* > K_1$ with probability 1 such that $\theta^k(s) = 0$ for any $k > K^*$. □

Proof of Proposition 4.1. Firstly, if $[\theta] \setminus S^*(q)$ is a non-empty set for any $\theta \in \Delta(S) \setminus [\theta^*]$ and any $q \in \mathrm{EQ}(\theta)$, then no belief with imperfect information $\theta \in \Delta(S) \setminus [\theta^*]$ satisfies (4.2a). That is, only the complete information belief vector θ^* can be a fixed point belief. Therefore, all fixed point must be complete information fixed points.

On the other hand, assume for the sake of contradiction that there exists a belief $\theta^\dagger \in \Delta(S) \setminus \{\theta^*\}$ such that $[\theta^\dagger] \subseteq S^*(q^\dagger)$ for an equilibrium strategy $q^\dagger \in \mathrm{EQ}(\theta^\dagger)$, then $\left(\theta^\dagger, q^\dagger\right)$, which is not a complete information fixed point, is in the set Ω. Thus, we arrive at a contradiction.

Secondly, from condition *(i)* that $[\bar{\theta}] \subseteq S^*(q)$ for any $\|q - \bar{q}\| < \xi$, we have:

$$\mathbb{E}_{\bar{\theta}}[u_i^s(q)] = u_i^{s^*}(q), \quad \forall i \in I. \tag{B.11}$$

For any $\bar{q} \in \mathrm{EQ}(\bar{\theta})$, from condition *(ii)* that $\bar{q}_i$ is a best response to $\bar{q}_{-i}$, $\bar{q}_i$ must be a local maximizer of $\mathbb{E}_{\bar{\theta}}[u_i^s(q_i, \bar{q}_{-i})]$. From (B.11), $\bar{q}_i$ is a local maximizer of $u_i^{s^*}(q_i, \bar{q}_{-i})$. Since the function $u_i^{s^*}(q_i, \bar{q}_{-i})$ is concave in q_i, $\bar{q}_i$ is also a global maximizer of $u_i^{s^*}(q_i, \bar{q}_{-i})$, and hence is a best response of $\bar{q}_{-i}$ with complete information of s^*. Since this argument holds for all $i \in I$, $\bar{q}$ is a complete information equilibrium. □

Proof of Proposition 4.2. On one hand, if $\Omega = \{(\theta^*, \mathrm{EQ}(\theta^*))\}$, then for any initial state, the learning dynamics converges to a complete information fixed point with belief θ^* and strategy in $\mathrm{EQ}(\theta^*)$. That is, $(\theta^*, \mathrm{EQ}(\theta^*))$ is globally stable. On the other hand, if there exists another fixed point $\left(\theta^\dagger, q^\dagger\right) \in \Omega \setminus \{(\theta^*, \mathrm{EQ}(\theta^*))\}$, then learning that starts with the initial belief $\theta^1 = \theta^\dagger$ (resp. $\theta^1 = \theta^*$) and strategy $q^1 = q^\dagger$ (resp. $q^1 = q^*$) remains at $\left(\theta^\dagger, q^\dagger\right)$ (resp. (θ^*, q^*)) for all stages w.p.1. Thus, in this case, globally stable fixed points do not exist. $\square$

Proof of Lemma 4.4.

(i) From Assumption *(A2a)*, we know that such ϵ' must exist.

(ii) Since $\hat{\epsilon} \leq \epsilon$, we know from Assumption *(A2b)* that $\mathrm{BR}(\theta, q) \subseteq N_{\delta}\left(\mathrm{EQ}(\bar{\theta})\right)$ for any $\theta \in N_{\hat{\epsilon}}\left(\bar{\theta}\right)$ and any $q \in N_{\delta}\left(\mathrm{EQ}(\bar{\theta})\right)$.

(iii) Under Assumption 1, we know from Theorem 4.1 that the sequence of the beliefs and strategies converges to a fixed point $\left(\theta^{\dagger}, q^{\dagger}\right)$. If $\theta^k \in N_{\hat{\epsilon}}(\bar{\theta})$ for all k, then $\lim_{k\to\infty} \theta^k = \theta^{\dagger} \in N_{\hat{\epsilon}}(\bar{\theta}) \subseteq N_{\bar{\epsilon}}(\bar{\theta})$. Additionally, from *(i)* and the fact that $\hat{\epsilon} \leq \epsilon'$, we know that $\lim_{k\to\infty} q^k = q^{\dagger} \in \mathrm{EQ}(\theta^{\dagger}) \subseteq N_{\bar{\delta}}\left(\mathrm{EQ}(\bar{\theta})\right)$. Therefore,

$$\begin{aligned}\lim_{k\to\infty} \Pr\left(\theta^k \in N_{\bar{\epsilon}}(\bar{\theta}) \geq q^k \in N_{\bar{\delta}}(\mathrm{EQ}(\bar{\theta}))\right) \geq & \Pr\left(\theta^k \in N_{\hat{\epsilon}}(\bar{\theta}),\ \forall k\right) \\ \geq & \Pr\left(\theta^k \in N_{\hat{\epsilon}}(\bar{\theta}), q^k \in N_{\delta}(\mathrm{EQ}(\bar{\theta})),\ \forall k\right).\end{aligned}$$

$\square$

In the proofs of Lemmas 4.5 – 4.6, we denote $\left(\tilde{\theta}^k\right)_{k=1}^{\infty}$ as an auxiliary belief sequence that is updated in every stage (instead of just updated at $(k_t)_{k=1}^{\infty}$). That is,

$$\tilde{\theta}^1 = \theta^1, \text{ and } \tilde{\theta}^{k+1}(s) = \frac{\tilde{\theta}^k(s)\phi^s(y^k|q^k)}{\sum_{s'\in S}\tilde{\theta}^k(s')\phi^{s'}(y^k|q^k)}, \quad \forall s \in S, \quad \forall k = 1, 2, \ldots \tag{B.12}$$

From (θ-update), we know that

$$\theta^k = \begin{cases} \tilde{\theta}^k, & \forall k = k_t, \quad k = 1, 2, \ldots, \\ \theta^{k-1}, & \text{otherwise.} \end{cases} \tag{B.13}$$

Proof of Lemma 4.5. First, note that $0 < \rho^1 < \rho^2 < \frac{\hat{\epsilon}}{|S|}$. For any $s \in S \setminus [\bar{\theta}]$ and any $k > 1$, we denote $U^k(s)$ the number of upcrossings of the interval $[\rho^1, \rho^2]$ that the belief $\tilde{\theta}^j(s)$ completes by stage k. That is, $U^k(s)$ is the maximum number of intervals $\left([\underline{k}_i, \overline{k}_i]\right)_{i=1}^{U^k(s)}$ with $1 \leq \underline{k}_1 < \overline{k}_1 < \underline{k}_2 < \overline{k}_2 < \cdots < \underline{k}_{U^k(s)} < \overline{k}_{U^k(s)} \leq k$, such that $\tilde{\theta}^{\underline{k}_i}(s) < \rho^1 < \rho^2 < \tilde{\theta}^{\overline{k}_i}(s)$ for $i = 1, \ldots U^k(s)$. Since the beliefs $\left(\tilde{\theta}^j(s)\right)_{j=1}^{k}$ are updated based on randomly realized payoffs $(y^j)_{j=1}^k$ as in (B.12), $U^k(s)$ is also a random variable. For any $k > 1$, $U^k(s) \geq 1$ if and only if $\tilde{\theta}^1(s) < \rho^1$ and there exists a stage $j \leq k$ such that $\tilde{\theta}^j(s) > \rho^2$. Equivalently, $\lim_{k\to\infty} U^k(s) \geq 1$ if and only if $\tilde{\theta}^1(s) < \rho^1$ and there exists a stage $k > 1$ such that $\tilde{\theta}^k(s) > \rho^2$. Therefore, if $\tilde{\theta}^1(s) < \rho^1$ for all $s \in S \setminus [\bar{\theta}]$, then:

$$\Pr\left(\tilde{\theta}^k(s) \leq \rho^2,\ \forall s \in S \setminus [\bar{\theta}],\ \forall k\right) = 1 - \Pr\left(\exists s \in S \setminus [\bar{\theta}] \text{ and } k,\ s.t.\ \tilde{\theta}^k(s) > \rho^2\right)$$
$$\geq 1 - \sum_{s\in S\setminus[\bar{\theta}]} \Pr\left(\exists k,\ s.t.\ \ \tilde{\theta}^k(s) > \rho^2\right) = 1 - \sum_{s\in S\setminus[\bar{\theta}]} \lim_{k\to\infty} \Pr\left(U^k(s) \geq 1\right). \tag{B.14}$$

Next, we define $\alpha \triangleq \bar{\theta}(s^*) - \rho^1$. Since $0 < \rho^1 < \min_{s\in[\bar{\theta}]}\{\bar{\theta}(s)\}$ and s^* is in the support set, we have $\alpha \in (0, \bar{\theta}(s^*))$. If $\tilde{\theta}^1(s)$ satisfies (4.8a) – (4.8b), then $\frac{\tilde{\theta}^1(s)}{\tilde{\theta}^1(s^*)} < \frac{\rho^1}{\alpha}$ for all $s \in S \setminus [\bar{\theta}]$. Additionally, for any stage k and any $s \in S \setminus [\bar{\theta}]$, if $\tilde{\theta}^k(s) > \rho^2$, then $\frac{\tilde{\theta}^k(s)}{\tilde{\theta}^k(s^*)} \geq \rho^2$ because $\tilde{\theta}^k(s^*) \leq 1$. Hence, whenever $\tilde{\theta}^k(s)$ completes an upcrossing of the interval $[\rho^1, \rho^2]$, $\frac{\tilde{\theta}^k(s)}{\tilde{\theta}^k(s^*)}$ must also have completed an upcrosssing of the interval $\left[\frac{\rho^1}{\alpha}, \rho^2\right]$. From (4.7a) – (4.7b), we can check that $\frac{\rho^1}{\alpha} < \rho^2$ so that the interval $\left[\frac{\rho^1}{\alpha}, \rho^2\right]$ is valid. We denote $\hat{U}^k(s)$ as the number of upcrossings of the sequence $\left(\frac{\tilde{\theta}^j(s)}{\tilde{\theta}^j(s^*)}\right)_{j=1}^k$ with respect to the interval $\left[\frac{\rho^1}{\alpha}, \rho^2\right]$ until stage k. Then, $U^k(s) \leq \hat{U}^k(s)$ for all k. Therefore, we can write:

$$\Pr\left(U^k(s) \geq 1\right) \leq \Pr\left(\hat{U}^k(s) \geq 1\right) \leq \mathbb{E}\left[\hat{U}^k(s)\right], \tag{B.15}$$

where the last inequality is due to Makov inequality.

From the proof of Lemma 4.1, we know that the sequence $\left(\frac{\tilde{\theta}^k(s)}{\tilde{\theta}^k(s^*)}\right)_{k=1}^\infty$ is a martingale. Therefore, we can apply the Doob's upcrossing inequality as follows:

$$\mathbb{E}\left[\hat{U}^k(s)\right] \leq \frac{\mathbb{E}\left[\max\{\frac{\rho^1}{\alpha} - \frac{\tilde{\theta}^k(s)}{\tilde{\theta}^k(s^*)}, 0\}\right]}{\rho^2 - \frac{\rho^1}{\alpha}} \leq \frac{\frac{\rho^1}{\alpha}}{\rho^2 - \frac{\rho^1}{\alpha}}, \quad \forall k. \tag{B.16}$$

From (B.13), (B.14) – (B.16), and (4.7a) – (4.7b), we can conclude that:

$$\Pr\left(\theta^k(s) \leq \rho^2,\ \forall s \in S \setminus [\bar{\theta}],\ \forall k\right) = \Pr\left(\tilde{\theta}^k(s) \leq \rho^2,\ \forall s \in S \setminus [\bar{\theta}],\ \forall k\right)$$
$$\geq 1 - \frac{\frac{\rho^1}{\alpha}|S \setminus [\bar{\theta}]|}{\rho^2 - \frac{\rho^1}{\alpha}} = 1 - \frac{\frac{\rho^1}{\bar{\theta}(s^*)-\rho^1}|S \setminus [\bar{\theta}]|}{\rho^2 - \frac{\rho^1}{\bar{\theta}(s^*)-\rho^1}} > \gamma.$$

$\square$

Proof of Lemma 4.6. From Assumption *(A2c)*, we know that $[\bar{\theta}] \subseteq S^*(q^1)$ if $q^1 \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$. Hence, $\phi^s(y^1|q^1) = \phi^{s^*}(y^1|q^1)$ for any $s \in [\bar{\theta}]$ and any realized payoff y^1. Therefore,

$$\frac{\tilde{\theta}^2(s)}{\tilde{\theta}^2(s^*)} = \frac{\tilde{\theta}^1(s)}{\tilde{\theta}^1(s^*)}\frac{\phi^s(y^1|q^1)}{\phi^{s^*}(y^1|q^1)} = \frac{\tilde{\theta}^1(s)}{\tilde{\theta}^1(s^*)}, \quad w.p.\ 1, \quad \forall s \in [\bar{\theta}]. \tag{B.17}$$

This implies that $\frac{\sum_{s\in[\bar{\theta}]}\tilde{\theta}^2(s)}{\tilde{\theta}^2(s^*)} = \frac{\sum_{s\in[\bar{\theta}]}\tilde{\theta}^1(s)}{\tilde{\theta}^1(s^*)}$, and for all $s \in [\bar{\theta}]$:

$$\frac{\tilde{\theta}^2(s)}{\sum_{s\in[\bar{\theta}]}\tilde{\theta}^2(s)} = \frac{\tilde{\theta}^2(s)}{\tilde{\theta}^2(s^*)}\frac{\tilde{\theta}^2(s^*)}{\sum_{s\in[\bar{\theta}]}\tilde{\theta}^2(s)} = \frac{\tilde{\theta}^1(s)}{\tilde{\theta}^1(s^*)}\frac{\tilde{\theta}^1(s^*)}{\sum_{s\in[\bar{\theta}]}\tilde{\theta}^1(s)} = \frac{\tilde{\theta}^1(s)}{\sum_{s\in[\bar{\theta}]}\tilde{\theta}^1(s)}.$$

Thus, we have

$$\frac{\tilde{\theta}^2(s)}{\tilde{\theta}^1(s)} = \frac{\sum_{s\in[\bar{\theta}]}\tilde{\theta}^2(s)}{\sum_{s\in[\bar{\theta}]}\tilde{\theta}^1(s)}, \quad w.p.\ 1, \quad \forall s \in [\bar{\theta}].$$

Since $\sum_{s\in[\bar{\theta}]}\tilde{\theta}^1(s) \leq 1$, if $\tilde{\theta}^2(s) < \rho^2$ for all $s \in S \setminus [\bar{\theta}]$, then we have $\frac{\tilde{\theta}^2(s)}{\tilde{\theta}^1(s)} > 1 - |S \setminus [\bar{\theta}]|\rho^2$. Additionally, since $\sum_{s\in[\bar{\theta}]}\tilde{\theta}^2(s) < 1$ and $\tilde{\theta}^1(s) < \rho^3$ for all $s \in [\bar{\theta}]$, we have $\frac{\tilde{\theta}^2(s)}{\tilde{\theta}^1(s)} < \frac{1}{1-|S\setminus[\bar{\theta}]|\rho^3}$. Since by (4.7c), $\rho^3 \leq \bar{\theta}(s)$ for all $s \in S \setminus [\bar{\theta}]$, any $\tilde{\theta}^1(s) \in \left(\bar{\theta}(s) - \rho^3, \bar{\theta}(s) + \rho^3\right)$ is a non-negative number for all $s \in [\bar{\theta}]$. Therefore, we have the following bounds:

$$\left(\bar{\theta}(s) - \rho^3\right)\left(1 - |S \setminus [\bar{\theta}]|\rho^2\right) < \tilde{\theta}^2(s) < \frac{\bar{\theta}(s) + \rho^3}{1 - |S \setminus [\bar{\theta}]|\rho^3}. \tag{B.18}$$

Since

$$\rho^3 \overset{(4.7c)}{\leq} \frac{\hat{\epsilon} - |S \setminus [\bar{\theta}]||S|\rho^2\bar{\theta}(s)}{|S| - |S \setminus [\bar{\theta}]||S|\rho^2}, \quad \forall s \in [\bar{\theta}], \tag{B.19}$$

we can check that $\left(\bar{\theta}(s) - \rho^3\right)\left(1 - |S \setminus [\bar{\theta}]|\rho^2\right) \geq \bar{\theta}(s) - \frac{\hat{\epsilon}}{|S|}$ for all $s \in [\bar{\theta}]$. To ensure the right-hand-side of (B.19) is positive, we need to have $\rho^2 < \frac{\hat{\epsilon}}{|S\setminus[\bar{\theta}]||S|\bar{\theta}(s)}$ for all $s \in [\bar{\theta}]$, which is satisfied by (4.7b). Also, since $\rho^3 \overset{(4.7c)}{\leq} \frac{\hat{\epsilon}}{|S|+|S\setminus[\bar{\theta}]|\left(\bar{\theta}(s)|S|+\hat{\epsilon}\right)}$ for all $s \in [\bar{\theta}]$, we have $\frac{\bar{\theta}(s)+\rho^3}{1-|S\setminus[\bar{\theta}]|\rho^3} < \bar{\theta}(s) + \frac{\hat{\epsilon}}{|S|}$ for all $s \in [\bar{\theta}]$. Therefore, we can conclude that $\tilde{\theta}^2(s) \in \left(\bar{\theta}(s) - \frac{\hat{\epsilon}}{|S|}, \bar{\theta}(s) + \frac{\hat{\epsilon}}{|S|}\right)$ for all $s \in [\bar{\theta}]$. Additionally, if $\tilde{\theta}^2(s) \leq \rho^2 < \frac{\hat{\epsilon}}{|S|}$ for all $s \in S \setminus [\bar{\theta}]$, then $\tilde{\theta}^2 \in N_{\hat{\epsilon}}\left(\bar{\theta}\right)$. From *(ii)* in Lemma 4.4, we know that $\mathrm{BR}(\tilde{\theta}^k, q^1) \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$ for both $k = 1, 2$. From (B.13), we have $\theta^2 \in N_{\hat{\epsilon}}\left(\bar{\theta}\right)$, and $\mathrm{BR}(\theta^2, q^1) \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$. Since $q^1 \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$, the updated strategy q^2 given by (q-update) must also be in the neighborhood $N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$.

We now use mathematical induction to prove that the belief of any $s \in [\bar{\theta}]$ satisfies $\tilde{\theta}^k(s) \in \left(\bar{\theta}(s) - \frac{\hat{\epsilon}}{|S|}, \bar{\theta}(s) + \frac{\hat{\epsilon}}{|S|}\right)$ for stages $k > 2$. If in stages $j = 1, \ldots, k$, $|\tilde{\theta}^j(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}$ for all $s \in [\bar{\theta}]$ and $\tilde{\theta}^j(s) < \rho^2 < \frac{\hat{\epsilon}}{|S|}$ for all $s \in S \setminus [\bar{\theta}]$, then $\tilde{\theta}^j \in N_{\hat{\epsilon}}\left(\bar{\theta}\right)$ for all $j = 1, \ldots, k$. Thus, from (B.13) and *(ii)* in Lemma 4.4, we have $\theta^j \in N_{\hat{\epsilon}}\left(\bar{\theta}\right)$ and $\mathrm{BR}\left(\theta^j, q^{j-1}\right) \subseteq N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$. Therefore, $q^j \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$ for all $j = 2, \ldots, k$.

From Assumption *(A2c)*, we know that $[\bar{\theta}] \subseteq S^*(q^j)$ for all $j = 1, \ldots, k$. Therefore, for any $s \in [\bar{\theta}]$ and any $j = 1, \ldots, k$, $\phi^s(y^j|q^j) = \phi^{s^*}(y^j|q^j)$ with probability 1. Then, by iteratively applying (B.17), we have $\frac{\tilde{\theta}^{k+1}(s)}{\tilde{\theta}^1(s)} = \frac{\sum_{s \in [\bar{\theta}]} \tilde{\theta}^{k+1}(s)}{\sum_{s \in [\bar{\theta}]} \tilde{\theta}^1(s)}$ for all $s \in [\bar{\theta}]$ with probability 1. Analogous to $k = 2$, we can prove that $|\tilde{\theta}^{k+1}(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}$ for all $s \in [\bar{\theta}]$. From (B.13), we also have $|\theta^{k+1}(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}$ for all $s \in [\bar{\theta}]$. From the principle of mathematical induction, we conclude that in all stages k, $|\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}$ for all $s \in [\bar{\theta}]$, and $q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right)$ for all k. Therefore, we have proved (4.10).

$\square$

Finally, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We combine Lemmas 4.4 – 4.6. For any $\gamma \in (0, 1)$, and any $\bar{\epsilon}, \bar{\delta} > 0$, consider $\delta^1 = \delta$ and $\epsilon^1 \triangleq \min\{\rho^1, \rho^3\}$ given by (4.7a) and (4.7c). If $\theta^1 \in N_{\epsilon^1}(\bar{\theta})$, then $|\theta^1(s) - \bar{\theta}(s)| < \epsilon^1$ for all $s \in S$. Recall from *(iii)* in Lemma 4.4, $\lim_{k\to\infty} \Pr\left(\theta^k \in N_{\bar{\epsilon}}(\theta),\ q^k \in N_{\bar{\delta}}(\mathrm{EQ}(\bar{\theta}))\right) \geq \Pr\left(\theta^k \in N_{\hat{\epsilon}}(\bar{\theta}),\ q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right),\ \forall k\right)$. Since $\rho^2 \leq \hat{\epsilon}/|S|$, we further have:

$$\Pr\left(\theta^k \in N_{\hat{\epsilon}}(\bar{\theta}),\ q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right),\ \forall k\right) \geq \Pr\begin{pmatrix} |\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|},\ \forall s \in [\bar{\theta}],\ \forall k \text{ and} \\ \theta^k < \rho^2,\ \forall s \in S \setminus [\bar{\theta}],\ q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right),\ \forall k \end{pmatrix}$$

$$= \Pr\left(\theta^k(s) < \rho^2, \forall s \in S \setminus [\bar{\theta}], \forall k\right) \cdot \Pr\left(\begin{array}{l|l} |\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|}, \forall s \in [\bar{\theta}], \forall k & \theta^k(s) < \rho^2. \\ \text{and } q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right),\ \forall k & \forall s \in S \setminus [\bar{\theta}], \forall k \end{array}\right)$$

For any $\theta^1 \in N_{\epsilon^1}(\bar{\theta})$ and any $q^1 \in N_{\delta^1}\left(\mathrm{EQ}(\bar{\theta})\right)$, we know from Lemmas 4.5 – 4.6 that:

$$\Pr\left(\theta^k(s) < \rho^2,\ \forall s \in S \setminus [\bar{\theta}],\ \forall k\right) > \gamma, \text{ and}$$
$$\Pr\left(\begin{array}{l} |\theta^k(s) - \bar{\theta}(s)| < \frac{\hat{\epsilon}}{|S|},\ \forall s \in [\bar{\theta}],\ \forall k \\ \text{and } q^k \in N_\delta\left(\mathrm{EQ}(\bar{\theta})\right),\ \forall k \end{array} \middle| \theta^k(s) < \rho^2,\ \forall s \in S \setminus [\bar{\theta}],\ \forall k \right) = 1$$

Therefore, for any $\theta^1 \in N_{\epsilon^1}(\bar{\theta})$ and any $q^1 \in N_{\delta^1}\left(\mathrm{EQ}(\bar{\theta})\right)$, the states of learning dynamics satisfy $\lim_{k\to\infty} \Pr\left(\theta^k \in N_{\bar{\epsilon}}(\theta),\ q^k \in N_{\bar{\delta}}\left(\mathrm{EQ}(\bar{\theta})\right)\right) > \gamma$. Thus, $\left(\bar{\theta}, \bar{q}\right)$ is locally stable. □

B.2 Proofs of Section 4.5

Proof of Proposition 4.3. Under condition (1), since the true parameter s^* is identifiable, we know any fixed point belief must have complete complete information of s^*. Thus, the fixed point strategy must be a complete information Wardrop equilibrium.

Under conditions (2) and (3), we know that at any fixed point, the belief $\bar{\theta}$ must consistently estimate the cost of all edges, i.e. $\mathbb{E}_{\bar{\theta}}[\ell_e^s(\bar{w}_e)] = \ell_e^{s^*}(\bar{w}_e)$ for all $e \in E$ given any $\left(\bar{\theta}, \bar{w}\right) \in \Omega$. Therefore, we have $\mathbb{E}_{\bar{\theta}}[\ell_r^s(\bar{q})] = \ell_r^{s^*}(\bar{q})$ for all $r \in R$.

From the variational inequality, we know that

$$\sum_{r\in R} \mathbb{E}_{\bar{\theta}}[\ell_r^s(\bar{q})] \cdot (q_r - \bar{q}_r) \geq 0, \quad \forall q \in Q,$$
$$\Rightarrow \qquad \sum_{r\in R} \ell_r^{s^*}(\bar{q}) \cdot (q_r - \bar{q}_r) \geq 0, \quad \forall q \in Q.$$

That is, $\bar{q}$ satisfies the variational inequality with complete information of s^*. Therefore, we can conclude that $\bar{q}$ must be a complete information equilibrium, and the induced edge load vector must be $\bar{w} = w^*$. □

Before presenting the proof of Proposition 4.4, we first introduce the definition of series-parallel networks and Braess's paradox.

Definition B.1. *Series-parallel networks A network is series-parallel if and only if it can be constructed recursively from single edges by connecting networks in series or in parallel.*

Definition B.2. *Braess's paradox occurs in a network if there are two sets of latency functions $\tilde{L} \triangleq \{\tilde{\ell}_e(w_e)\}_{e\in E}$ and $L \triangleq \{\ell_e(w_e)\}_{e\in E}$ such that $\tilde{\ell}_e(w_e) \geq \ell_e(w_e)$ for all $e \in E$ and all $w_e \geq 0$, but the equilibrium social cost associated with $\tilde{L}$ is lower than that with L.*

Lemma B.2 (Theorem 1 in Milchtaich [2006])**.** *Braess's paradox does not occur in a network with single origin-destination pair if and only if the network is series-parallel.*

We are now ready to prove Proposition 4.4:

Proof of Proposition 4.4. For any fixed point $(\bar{\theta}, \bar{w})$, we know from Theorem 4.1 that for the utilized resources $e \in \bar{E}$, the average cost is accurately estimated, i.e. $\mathbb{E}_{\bar{\theta}}[\ell_e^s(\bar{w}_e)] = \ell_e^{s^*}(\bar{w}_e)$. Consider another congestion game $\tilde{G}$, in which the latency functions of resources $e \in E \backslash \bar{E}$ are $\tilde{\ell}_e^{s^*}(w_e) = \infty$ for any $w_e \geq 0$, and the costs of resources $e \in \bar{E}$ do not change, i.e. $\tilde{\ell}_e^{s^*}(w_e) = \ell_e^{s^*}(w_e)$. Then, $\bar{w}$ is the equilibrium load vector of $\tilde{G}$ with complete information of s^*, and $C(\bar{w})$ is the equilibrium social cost. Note that w^* is the complete information equilibrium load vector in the original congestion game G, where $\ell_e^{s^*}(w_e) \leq \tilde{\ell}_e^{s^*}(w_e)$ for all $e \in E$ and all $w_e \geq 0$. Since the network is series parallel with single origin-destination pair, following from Lemma B.2, we know that Baraess paradox does not occur. Hence, we can conclude that $C(\bar{w}) \geq C(w^*)$ for any $\bar{w}$.

$\square$

B.3 Learning in Games with Finite Strategy Set

Our results in Sec. 4.3 can be extended to learning in games where strategy sets are finite and players can choose mixed strategies. In this game, each player i's action set (pure strategies) is a finite set A_i, and the action profile (pure strategy profile) is denoted as $a = (a_i)_{i\in I} \in A = \prod_{i\in I} A_i$. Given any parameter s and any action profile a, the distribution of players' payoff y is $\phi^s(y|a)$.

We denote player i's mixed strategy as $q_i = (q_i(a_i))_{a_i\in A_i} \in Q_i = \Delta(A_i)$, where $q_i(a_i)$ is the probability of choosing the action a_i. The strategy set Q_i is bounded and convex. Players' action profile in each stage k, denoted as $a^k = \left(a_i^k\right)_{i\in I}$, is realized from the mixed strategy profile q^k.

Analogous to (θ-update), the information system updates the belief θ^{k_t} based on actions $\left(a^k\right)_{k=k_t}^{k_{t+1}-1}$ and the realized payoff vectors $\left(y^k\right)_{k=k_t}^{k_{t+1}-1}$ as follows:

$$\theta^{k_{t+1}}(s) = \frac{\theta^{k_t}(s)\prod_{k=k_t}^{k_{t+1}-1}\phi^s(y^k|a^k)}{\sum_{s'\in S}\theta^{k_t}(s')\prod_{k=k_t}^{k_{t+1}-1}\phi^{s'}(y^k|a^k)}, \quad \forall s \in S.$$

Similar to Sec. 4.2, we consider three types of best-response updates:

1. *Simultaneous best-response dynamics.* All players choose an action that is a best response to their opponents' actions given the updated belief and their opponents' action profile:

$$a_i^{k+1} \in \mathrm{BR}_i(\theta^{k+1}, a_{-i}^k), \quad \forall i \in I.$$

2. *Sequential best-response dynamics.* Players change their actions to be a best-response strategy of other opponents' actions one by one:

$$a_i^{k+1} \begin{cases} \in \mathrm{BR}_i(\theta^{k+1}, a_{-i}^k), & \text{if } k \ mod \ |I| = i. \\ = a_i^k, & \text{otherwise.} \end{cases}$$

3. *Fictitious play.* The mixed strategy q_i^k represents player i's empirical frequency of actions in previous stages $1, \ldots, k$. In each stage k, all players best respond to their opponents' empirical frequency q_{-i}^k:

$$a_i^k \in \mathrm{BR}_i(\theta^k, q_{-i}^k), \quad q_i^{k+1} = \frac{k}{k+1}q_i^k + \frac{1}{k+1}a_i^k, \quad \forall i \in I, \quad \forall k.$$

We extend the definition of payoff equivalent parameters in Definition 4.2 as follows: Parameter s is payoff-equivalent to s^* given $q \in Q$ if the distribution of payoffs under s is identical to that under s^* for all actions that are assigned with positive probability given q. Therefore, the payoff-equivalent parameter set given q is defined as $S^*(q) \triangleq \{S\,|D_{KL}\left(\phi^{s*}\left(y|a\right)||\phi^s\left(y|a\right)\right) = 0, \ \forall a \in [q]\}$, where $[q] = \{A|q(a) > 0\}$ is the support set of the mixed strategy profile q. In addition, the set of payoff-equivalent parameters on a strategy set $\bar{Q} \subseteq Q$ is $S^*(\bar{Q}) = \{S | s \in S^*(q), \ \forall q \in \bar{Q}\}$.

The convergence result in Theorem 4.1 can be readily extended to games with finite strategy sets: Under Assumption 1, the beliefs $\left(\theta^k\right)_{k=1}^{\infty}$ converge to a fixed point belief $\bar{\theta}$ that accurately estimates the payoff distribution for all action profiles that are taken with positive probability, and the strategies $\left(q^k\right)_{k=1}^{\infty}$ converge to the equilibrium set $\mathrm{EQ}(\bar{\theta})$.

The results on global and local stability properties in Proposition 4.2 and Theorem 4.2 also hold for games with finite strategy set. Moreover, for games with a finite strategy set, any fixed point that satisfies the sufficient condition of local stability must be a complete information fixed point. This is because any local perturbation of a fixed point strategy profile can lead to a mixed strategy with full support on all action profiles, and these mixed strategies can distinguish any parameter $s \neq s^*$ from s^*. Thus, local consistency condition in Assumption *(A2c)* is only satisfied by the complete information belief θ^*.

Appendix C

Supplementary Material for Chapter 5

C.1 Proofs of Section 5.3

Proof of Theorem 5.1. First, we proof that the four conditions of market equilibrium (x^*, p^*, τ^*) ensures that x^* satisfies the feasibility constraints of the primal (LP), (u^*, τ^*) satisfies the dual (D), and (x^*, u^*, τ^*) satisfies the complementary slackness conditions. The vector u^* is the utility vector computed from (5.4).

(i) Feasibility constraints of (LP): Since x^* is a feasible trip vector, x^* must satisfy the feasibility constraints of (LP).

(ii) Feasibility constraints of (D): From the stability condition (5.6), individual rationality (5.5), and the fact that toll prices are non-negative, we know that (u^*, τ^*) satisfies the feasibility constraints of (D).

(iii) Complementary slackness condition with respect to (LP.a): If rider m is not assigned, then (LP.a) is slack with the integer trip assignment x^* for some rider m. The budget balanced condition (5.7b) shows that $p^*_m = 0$. Since rider m is not in any trip and the payment is zero, the dual variable (i.e. rider m's utility) $u^{m*} = 0$. On the other hand, if $u^{m*} > 0$, then rider m must be in a trip, and constraint (LP.a) must be tight. Thus, we can conclude that the complementary slackness condition with respect to the primal constraint (LP.a) is satisfied.

(iv) Complementary slackness condition with respect to (LP.b): Since the mechanism is market clearing, toll price τ_e is nonzero if and only if the load on edge e is below the capacity, i.e. the primal constraint (LP.b) is slack for edge $e \in E$. Therefore, the complementary slackness condition with respect to the primal constraint (LP.b) is satisfied.

(v) Complementary slackness condition with respect to (D.a): From (5.7a), we know that for any organized trip, the corresponding dual constraint (D.a) is tight. If constraint (D.a) is slack for a trip (b, r), then the budget balance constraint ensures that trip is not organized. Therefore, the complementary slackness condition with respect to the primal constraint (D.a) is satisfied.

We can analogously show that the inverse of (i) – (v) are also true: the feasibility constraints of (LP) and (D), and the complementary slackness conditions ensure that (x^*, p^*, τ^*) is a market equilibrium. Thus, we can conclude that (x^*, p^*, τ^*) is a market equilibrium if and only if (x^*, u^*, τ^*) satisfies the feasibility constraints of (LP) and (D), and the complementary slackness conditions.

From strong duality theory, we know that the equilibrium trip vector x^* must be an optimal integer solution of (LP). Therefore, the existence of market equilibrium is equivalent to the existence of an integer optimal solution of (LP). The optimal trip assignment is an optimal integer solution of (LP), and (u^*, τ^*) is an optimal solution of the dual problem (D). The payment p^* can be computed from (5.4). $\square$

Proof of Corollary 5.1. Consider any two routes $r, r' \in R$ such that $t_r \geq t_{r'}$. Given x^*, we denote the rider group that takes route r as b_r. If no rider group is assigned to route r, then we denote $b_r = \emptyset$. From (5.7a), we have

$$\sum_{m \in b_r} u_m^* + \sum_{e \in r} \tau_e^* = V_r(b_r).$$

Additionally, since (u^*, τ^*) satisfies constraint (D.a), we know that

$$\sum_{m \in b_r} u_m^* + \sum_{e \in r'} \tau_e^* \geq V_{r'}(b_r).$$

Therefore, we must have:

$$\sum_{e \in r'} \tau_e^* - \sum_{e \in r} \tau_e^* \geq V_{r'}(b_r) - V_r(b_r) = \left(\sum_{m \in b_r} \beta^m + \sum_{m \in b_r} \gamma^m(|b_r|) + \delta |b_r| \right) (t_r - t_{r'}) \geq 0.$$

□

C.2 Proofs of Section 5.4.

Proof of Lemma 5.1. Consider any (fractional) optimal solution of (LP), denoted as $\hat{x}^*$. We denote $\hat{f}(b) = \sum_{r \in R} \hat{x}_r^*(b)$ as the flow of group b, and $\widehat{F} = \sum_{b \in B} \hat{f}(b)$ is the total flows. Since $\hat{x}^*$ is feasible, we know that $\widehat{F} \leq C$, where C is the maximum capacity of the network. The set of all groups with positive flow in $\hat{x}$ is $\widehat{B} \triangleq \{\hat{b} \in B | \hat{f}(\hat{b}) > 0\}$. For each $\hat{b} \in \hat{B}$, we re-write the trip value function as follows:

$$V_r(\hat{b}) = z(\hat{b}) - g(\hat{b}) t_r, \quad \forall \left(\hat{b}, r\right) \in \hat{B} \times R,$$

where $z(\hat{b}) = \sum_{m \in \hat{b}} (\alpha^m - \pi^m) - \sigma |\hat{b}|$, and $g(\hat{b}) = \sum_{m \in \hat{b}} (\beta^m + \gamma(|b|)) + \delta |b|$.

We denote the number of rider groups in $\hat{B}$ as n, and re-number these rider groups in decreasing order of $g(\hat{b})$, i.e.

$$g(\hat{b}_1) \geq g(\hat{b}_2) \geq \cdots \geq g(\hat{b}_n).$$

We now construct another trip vector x^* by the following procedure:

Initialization: Zero assignment vector $x_r^*(b) \leftarrow 0$ for all $r \in R$ and all $b \in B$

For $j = 1, \ldots, n$*:*

(a) Assign rider group $\hat{b}_j$ to a route $\hat{r}$ in R^*, which has the minimum travel time among all routes with flow less than the capacity, i.e. $\hat{r} \in \arg\min_{r \in \{R^* | \sum_{b \in B} x_r^*(b) < k_r^*\}} \{t_r\}$.

(b) If $\sum_{b \in B} x_{\hat{r}}^*(b) + \hat{f}(\hat{b}_j) \leq k_{\hat{r}}^*$, then $x_{\hat{r}}^*(\hat{b}_j) = \hat{f}(\hat{b}_j)$.

(c) Otherwise, assign $x_{\hat{r}}^*(\hat{b}_j) = k_{\hat{r}}^* - \sum_{b \in B} x_{\hat{r}}^*(b)$, and continue to assign the remaining weight of rider group $\hat{b}_j$ to the next unsaturated route with the minimum cost. Repeat this process until the condition in (b) is satisfied, i.e. the total weight $\hat{f}(b_j)$ is assigned.

We can check that $\sum_{b\ni m}\sum_{r\in R} x_r^*(b) = \sum_{b\ni m}\hat{f}(b) \leq 1$ so that (LPk^*.a) is satisfied. Additionally, since in the assignment procedure, the total weight assigned to route r is less than or equal to k_r^*, we must have $\sum_{b\in B} x_r^*(b) \leq k_r^*$ for all $r \in R$, i.e. (LPk^*.b) is satisfied. Thus, x^* is a feasible solution of (LPk^*).

It remains to prove that x^* is optimal of (LPk^*). We prove this by showing that $V(x^*) \geq V(\hat{x}^*)$. The objective function $S(x^*)$ can be written as follows:

$$\sum_{r\in R}\sum_{b\in B} V_r(b)x_r^*(b) = \sum_{r\in R}\sum_{b\in B} z(b)x_r^*(b) - \sum_{r\in R}\sum_{b\in B} g(b)t_r x_r^*(b). \tag{C.1}$$

We note that since $\sum_{r\in R} k_r^* = C$ and $\sum_{r\in R}\sum_{b\in B}\hat{x}_r^*(b) \leq C$, the algorithm must terminate with all groups in $\hat{B}$ being assigned. Therefore, $\sum_{r\in R} x_r^*(b) = \hat{f}(b) = \sum_{r\in R}\hat{x}_r^*(b)$ for all $b \in B$. Therefore,

$$\sum_{r\in R}\sum_{b\in B} z(b)x_r^*(b) = \sum_{b\in B} z(b)\hat{f}(b) = \sum_{r\in R}\sum_{b\in B} z(b)\hat{x}_r^*(b) \tag{C.2}$$

Then, $V(x^*) \geq V(\hat{x}^*)$ is equivalent to $\sum_{r\in R}\sum_{b\in B} g(b)t_r x_r^*(b) \leq \sum_{r\in R}\sum_{b\in B} g(b)t_r\hat{x}_r^*$. To prove this, we show that x^* minimizes the term $\sum_{r\in R}\sum_{b\in B} g(b)t_r x_r^*(b)$ among all feasible x that induces the same flow of groups as $\hat{x}^*$, i.e.

$$x^* \in \arg\min_{x\in X(\hat{f})} \sum_{r\in R}\sum_{b\in B} g(b)t_r x_r(b), \tag{C.3}$$

where

$$X(\hat{f}) \triangleq \left\{ (x_r(b))_{r\in R, b\in B} \;\middle|\; \begin{array}{l} \sum_{r\in R} x_r(b) = \hat{f}(b), \quad \forall b \in B, \\ \sum_{b\in B}\sum_{r\ni e} x_r(b) \leq q_e, \quad \forall e \in E, \\ x_r(b) \geq 0, \quad \forall r \in R, \quad \forall b \in B. \end{array} \right\} \tag{C.4}$$

We prove (C.3) by mathematical induction. To begin with, (C.3) holds trivially on any single-link network. We ext prove that if (C.3) holds on two series-parallel sub-networks G^1

and G^2, then (C.3) holds on the network G that connects G^1 and G^2 in series or in parallel. In particular, we analyze the cases of series connection and parallel connection separately:

(Case 1) Series-parallel Network G is formed by connecting two series-parallel sub-networks G^1 and G^2 in series.

We denote the set of routes in subnetwork G^1 and G^2 as R^1 and R^2, respectively. Since G^1 and G^2 are connected in series, the set of routes in network G is $R \triangleq R^1 \times R^2$. Since the two sub-networks are connected in sequence, the group flow vector in G^1 (resp. G^2) is $\hat{f}^1_{r^1}(b) = \sum_{r^2 \in R^2} \hat{f}_{r^1 r^2}(b)$ (resp. $\hat{f}^2_{r^2}(b) = \sum_{r^1 \in R^1} \hat{f}_{r^1 r^2}(b)$) for all $b \in B$ and all $r^1 \in R^1$ (resp.$r^2 \in R^2$). Analogously, we define the set of trip vectors on sub-network G^1 (resp. G^2) that satisfies the constraint in (C.3) as $X^1(\hat{f}^1)$ (resp. $X^2(\hat{f}^2)$). We can check that $X^1(\hat{f}^1)$ (resp. $X^2(\hat{f}^2)$) is the set of trip vectors in $X(\hat{f})$ that is restricted on network G^1 (resp. G^2). That is, for any $x \in X(\hat{f})$, we can find $x^1 \in X^1(\hat{f}^1)$ (resp. $x^2 \in X^2(\hat{f}^2)$) such that $\sum_{r^2 \in R^2} x_{r^1 r^2}(b) = x^1_{r^1}(b)$ (resp. $\sum_{r^1 \in R^1} x_{r^1 r^2}(b) = x^2_{r^2}(b)$) for all $b \in B$ and all $r^1 \in R^1$ (resp. $r^2 \in R^2$). Since the two subnetworks are connected sequentially, we have the follows:

$$\sum_{r \in R} \sum_{b \in B} g(b) t_r x_r(b) = \sum_{r^1 \in R^1} \sum_{b \in B} g(b) t_{r^1} \left(\sum_{r^2 \in R^2} x_{r^1 r^2}(b) \right) + \sum_{r^2 \in R^2} \sum_{b \in B} g(b) t_{r^2} \left(\sum_{r^1 \in R^1} x_{r^1 r^2}(b) \right)$$
$$= \sum_{r^1 \in R^1} \sum_{b \in B} g(b) t_{r^1} x^1_{r^1}(b) + \sum_{r^2 \in R^2} \sum_{b \in B} g(b) t_{r^2} x^2_{r^2}(b). \tag{C.5}$$

We also denote the trip vector that is obtained from procedure (a) – (c) based on $\hat{f}$ in G^1 (resp. G^2) as x^{1*} (resp. x^{2*}). We now argue that $\sum_{r^2 \in R^2} x^*_{r^1 r^2}(b) = x^{1*}_{r^1}(b)$ for all $b \in B$ and all $r^1 \in R^1$. For the sake of contradiction, assume that there exists $b \in B$ such that $\sum_{r^2 \in R^2} x^*_{r^1 r^2}(b) \neq x^{1*}_{r^1}(b)$ for at least one $r^1 \in R^1$. We denote $\hat{b}$ as one such group with the maximum $g(\hat{b})$. Since the total flow of $\hat{b}$ is $\hat{f}(\hat{b})$ in both x^* and x^{1*}, if $\sum_{r^2 \in R^2} x^*_{r^1 r^2}(\hat{b}) \neq x^{1*}_{r^1}(\hat{b})$ on one $r^1 \in R^1$, the same inequality must hold for another $r^{1'} \in R^1$. Without loss of generality, we assume that $t_{r^1} < t_{r^{1'}}$. Since any group $\hat{b}'$ that is assigned before $\hat{b}$ ($g(\hat{b}') > g(\hat{b})$) satisfy $\sum_{r^2 \in R^2} x^*_{r^1 r^2}(\hat{b}') = x^{1*}_{r^1}(\hat{b}')$ for all $r^1 \in R^1$, if $\sum_{r^2 \in R^2} x^*_{r^1 r^2}(\hat{b}) < x^{1*}_{r^1}(\hat{b})$, then x^{1*} is not obtained by procedure (a) – (c) on G^1 because r^1 is not saturated with x^{1*} in the round of assigning $\hat{b}$, and more flow of $\hat{b}$ should be moved from $r^{1'}$ to r^1 to saturate route r^1. We can analogously argue that if $\sum_{r^2 \in R^2} x^*_{r^1 r^2}(\hat{b}) > x^{1*}_{r^1}(\hat{b})$, then x^* is not obtained from the procedure (a) – (c) for G. In either case, we have arrived at a contradiction. We can analogously argue that $\sum_{r^1 \in R^1} x^*_{r^1 r^2}(b) = x^{2*}_{r^2}(b)$ for all $b \in B$ and all $r^2 \in R^2$. Therefore,

$$\sum_{r \in R} \sum_{b \in B} g(b) t_r x^*_r(b) = \sum_{r^1 \in R^1} \sum_{b \in B} g(b) t_{r^1} \left(\sum_{r^2 \in R^2} x^*_{r^1 r^2}(b) \right) + \sum_{r^2 \in R^2} \sum_{b \in B} g(b) t_{r^2} \left(\sum_{r^1 \in R^1} x^*_{r^1 r^2}(b) \right)$$
$$= \sum_{r^1 \in R^1} \sum_{b \in B} g(b) t_{r^1} x^{1*}_{r^1}(b) + \sum_{r^2 \in R^2} \sum_{b \in B} g(b) t_{r^2} x^{2*}_{r^2}(b) \tag{C.6}$$

If (C.3) holds on both sub-networks (i.e. $x^{1*} \in \arg\min_{x \in X^1(\hat{f}^1)} \sum_{r^1 \in R^1} \sum_{b \in B} g(b) t_{r^1} x^1_{r^1}(b)$ and $x^{2*} \in \arg\min_{x \in X^2(\hat{f}^2)} \sum_{r^2 \in R^2} \sum_{b \in B} g(b) t_{r^2} x^2_{r^2}(b)$), then from (C.5) – (C.6), we know that (C.3) also holds in network G.

(Case 2) Series-parallel Network G is formed by connecting two series-parallel networks G_1 and G_2 in parallel.
Same as case 1, we denote R^1 (resp. R^2) as the set of routes in G^1 (resp. G^2). Then, the set of all routes in G is $R = R^1 \cup R^2$.

Given any $\hat{f}$, we compute x^* from the procedure (a) – (c) in network G. We denote $f^{1*} = \sum_{r^1 \in R^1} \sum_{b \in B} x^*_r(b)$ (resp. $f^{2*} = \sum_{r^2 \in R^2} \sum_{b \in B} x^*_r(b)$) as the total flow assigned to subnetwork G^1 (resp. G^2) in x^*. We now denote x^{1*} (resp. x^{2*}) as the trip vector x^* restricted on sub-network G^1 (resp. G^2), i.e. $x^{1*} = \left(x^*_{r^1}(b)\right)_{r^1 \in R^1, b \in B}$ (resp. $x^{2*} = \left(x^*_{r^2}(b)\right)_{r^2 \in R^2, b \in B}$). We can check that x^{1*} (resp. x^{2*}) is the trip vector obtained by the procedure (a) – (c) given the total flow f^{1*} (resp. f^{2*}) on network G^1 (resp. G^2).

Consider any arbitrary split of the total flow $\hat{f}$ to the two sub-networks, denoted as $\left(\hat{f}^1, \hat{f}^2\right)$, such that $\hat{f}^1(b) + \hat{f}^2(b) = \hat{f}(b)$ for all $b \in B$. Given $\hat{f}^1$ (resp. $\hat{f}^2$), we denote the trip vector obtained by procedure (i) – (iii) on sub-network G^1 (resp. G^2) as $\hat{x}^{1*}$ (resp. $\hat{x}^{2*}$). We also define the set of feasible trip vectors on sub-network G^1 (resp. G^2) that induce the total flow $\hat{f}^1$ (resp. $\hat{f}^2$) given by (C.4) as $X^1(\hat{f}^1)$ (resp. $X^2(\hat{f}^2)$). Then, the set of all trip vectors that induce $\hat{f}$ on network G is $X(\hat{f}) = \cup_{\left(\hat{f}^1, \hat{f}^2\right)} (X^1(\hat{f}^1), X^2(\hat{f}^2))$.

Under our assumption that (C.3) holds on sub-network G^1 and G^2 with any total flow, we know that given any flow split $\left(\hat{f}^1, \hat{f}^2\right)$,

$$\sum_{r^1 \in R^1} \sum_{b \in B} g(b) t_r \hat{x}^{1*}_{r^1}(b) + \sum_{r^2 \in R^2} \sum_{b \in B} g(b) t_r \hat{x}^{2*}_{r^2}(b) \leq \sum_{r^1 \in R^1} \sum_{b \in B} g(b) t_r \hat{x}^{1}_{r^1}(b) + \sum_{r^2 \in R^2} \sum_{b \in B} g(b) t_r \hat{x}^{2}_{r^2}(b),$$
$$\forall \hat{x}^1 \in X(\hat{f}^1), \hat{x}^2 \in X(\hat{f}^2).$$

Therefore, the optimal solution of (C.3) must be a trip vector $(\hat{x}^{1*}, \hat{x}^{2*})$ associated with a flow split $\left(\hat{f}^1, \hat{f}^2\right)$. It thus remains prove that any $(\hat{x}^{1*}, \hat{x}^{2*})$ associated with flow split $\left(\hat{f}^1, \hat{f}^2\right) \neq (f^{1*}, f^{2*})$ cannot be an optimal solution (i.e. can be improved by re-arranging flows).

For any $\left(\hat{f}^1, \hat{f}^2\right) \neq (f^{1*}, f^{2*})$, we can find a group b_j such that $\hat{f}^1(b_j) \neq f^{1*}(b_j)$ (henceforth $\hat{f}^2(b_j) \neq f^{2*}(b_j)$). We denote $b_{\hat{j}}$ as one such group with the maximum $g(b)$, i.e. $\hat{f}^1(b_j) = f^{1*}(b_j)$ for any $1, \ldots, \hat{j}-1$. Since groups $b_1, \ldots, b_{\hat{j}-1}$ are assigned before group $b_{\hat{j}}$ according to procedure (a) – (c), we know that $\hat{x}^{1*}_{r^1}(b_j) = x^*_{r^1}(b_j)$ and $\hat{x}^{2*}_{r^2}(b_j) = x^*_{r^2}(b_j)$ for all $r^1 \in R^1$, all $r^2 \in R^2$ and all $j = 1, \ldots, \hat{j}-1$. Since $\hat{f}^1(b_{\hat{j}}) \neq f^{1*}(b_{\hat{j}})$, the trip vector in $\hat{x}^{1*}$ and $\hat{x}^{2*}$ must be different from that in x^*. Without loss of generality, we assume that $\hat{f}^1(b_{\hat{j}}) > f^{1*}(b_{\hat{j}})$ and $\hat{f}^2(b_{\hat{j}}) < f^{2*}(b_{\hat{j}})$. Then, there must exist routes $\hat{r}^1 \in R^1$ and $\hat{r}^2 \in R^2$ such that $\hat{x}^{1*}_{\hat{r}^1}(b_{\hat{j}}) > x^*_{\hat{r}^1}(b_{\hat{j}})$ and $\hat{x}^{2*}_{\hat{r}^2}(b_{\hat{j}}) < x^*_{\hat{r}^2}(b_{\hat{j}})$. Moreover, since x^* assigns group $b_{\hat{j}}$ to routes with the minimum travel time cost that are unsaturated after assigning groups $b_1, \ldots, b_{\hat{j}-1}$, we have $t_{\hat{r}^2} < t_{\hat{r}^1}$. If route $\hat{r}^2$ is unsaturated given $\hat{x}^{2*}$, then we decrease $\hat{x}^{1*}_{\hat{r}^1}(b_{\hat{j}})$ and increase $\hat{x}^{2*}_{\hat{r}^2}(b_{\hat{j}})$ by a small positive number $\epsilon > 0$. We can check that the objective function of (C.3) is reduced by $\epsilon(t_{\hat{r}^1} - t_{\hat{r}^2})\epsilon g(b_{\hat{j}}) > 0$. On the other hand, if route $\hat{r}^2$ is saturated, then group $b_{\hat{j}+1}$ must be assigned to $\hat{r}^2$ because it is assigned right after group $b_{\hat{j}}$. Then, we decrease $x^{1*}_{\hat{r}^1}(b_{\hat{j}})$ and $x^{2*}_{\hat{r}^2}(b_{\hat{j}+1})$ by $\epsilon > 0$, increases $x^{1*}_{\hat{r}^1}(b_{\hat{j}+1})$ and $x^{2*}_{\hat{r}^2}(b_{\hat{j}})$ by ϵ (i.e. exchange a small fraction of group $b_{\hat{j}}$ with group $b_{\hat{j}+1}$). Note that $g(b_{\hat{j}}) > g(b_{\hat{j}+1})$ and $t_{\hat{r}^1} > t_{\hat{r}^2}$. We can thus check that the objective function of (C.3) is reduced by $\epsilon(t_{\hat{r}^1} g(b_{\hat{j}}) - t_{\hat{r}^2} g(b_{\hat{j}+1}))\epsilon > 0$. Therefore, we have found an adjustment of trip vector $(\hat{x}^{1*}, \hat{x}^{2*})$ that reduces the objective function of (C.3). Hence, for any flow split $\left(\hat{f}^1, \hat{f}^2\right) \neq (f^{1*}, f^{2*})$, the associated trip vector $(\hat{x}^{1*}, \hat{x}^{2*})$ is not the optimal solution of (C.3). The optimal solution of (C.3) must be constructed by procedure (i) – (iii) with flow split (f^{1*}, f^{2*}), i.e. must be x^*.

We have shown from cases 1 and 2 that if x^* is an optimal solution of (C.3) on two series-parallel sub-networks, then x^* is an optimal solution on the connected series-parallel network. Moreover, since (C.3) holds trivially when the network is a single edge, and any series-parallel network is formed by connecting series-parallel sub-networks in series or parallel, we can conclude that x^* obtained from procedure (a) – (c) minimizes the objective function in (C.3) for any flow vector $\hat{f}$ on any series-parallel network.

From (C.1), (C.2) and (C.3), we can conclude that $V(x^*) \geq V(\hat{x}^*)$. Hence, x^* must be an optimal solution in (LPk^*). $\square$

Proof of Lemma 5.2. First, for any feasible x in (LPk^*), consider a vector y such that for any $(r, b) \in \{B \times R | x_r(b) = 1\}$, $y_l(b) = 1$ for one $l \in L_r$ and $y_l(\bar{b}) = 0$ for any other $(\bar{b}, l)$. We can check that y is feasible in (LP-y) and $S(x) = S(y)$. On the other hand, for any feasible y in (LP-y), there exists $x = \chi(y)$ as in (5.15) such that x is feasible in (LPk^*) and $S(x) = S(y)$. Thus, (LPk^*) and (LP-y) are equivalent in that for any feasible solution of one linear program, there exists a feasible solution that achieves the same social welfare in the other linear program.

Therefore, (LPk^*) has an integer optimal solution if and only if (LP-y) has an integer optimal solution, and for any integer optimal solution y^* of (LP-y), $x = \chi(y^*)$ as in (5.15) is an optimal solution of (LPk^*). $\square$

Proof of Lemma 5.3. We write the dual program of (LP-y) as follows:

$$\begin{aligned}
\min_{u,\mu} \quad & \sum_{m \in M} u^m + \sum_{l \in L} \mu_l, \\
s.t. \quad & \sum_{m \in \bar{b}} u^m + \mu_l \geq W_l(\bar{b}) \quad \forall \bar{b} \in \bar{B}, \quad \forall l \in L, && \text{(D-y.a)} \\
& u^m \geq 0, \mu_l \geq 0, \quad \forall m \in M, \quad \forall l \in L. && \text{(D-y.b)}
\end{aligned}$$

For any Walrasian equilibrium (y^*, u^*), we consider the vector $\mu^* = (\mu_l^*)_{l \in L}$ as follows:

$$\mu_l^* = \max_{\bar{b} \in \bar{B}} W_l(\bar{b}) - \sum_{m \in \bar{b}} u^{m*}, \quad \forall l \in L. \tag{C.8}$$

From the definition of Walrasian equilibrium, we know that y^* is a feasible solution of (LP-y), and (u^*, μ^*) is a feasible solution of (D-y). We now show that (y^*, u^*, μ^*) satisfies complementary slackness condition of (LP-y) and (D-y).

- Complementary slackness condition for (LP-y.a): Condition *(ii)* in Definition 5.5 ensures that rider m's utility is positive if and only if (LP-y.a) is tight (i.e. rider m joins a trip).

- Complementary slackness condition for (LP-y.b): If no rider group takes route $l \in L$, i.e. (LP-y.b) is slack and $\bar{b}_l = \emptyset$, then μ_l^* as in (C.8) is zero. On the other hand, $\mu_l^* > 0$, then $\bar{b}_l \neq \emptyset$. Hence, (LP-y.b) must be tight.

- Complementary slackness condition for (D-y.a): From condition (i) in Definition 5.5, we know that $y_l^*(\bar{b}_l) = 1$ if and only if $\bar{b}_l \in \arg\max_{\bar{b} \in \bar{B}} W_l(\bar{b}) - \sum_{m \in \bar{b}} u^{m*}$, i.e. constraint (D-y.a) is tight.

From strong duality, we know that y^* must be an integer optimal solution of (LP-y) and (u^*, μ^*) must be an optimal solution of (D-y). Therefore, we can conclude that a Walrasian equilibrium $(y^* t, u^*)$ exists in the equivalent economy $\mathcal{G}$ if and only if (LP-y) has an optimal integer solution. □

Proof of Lemma 5.4. Since all riders have homogeneous carpool disutility, we can simplify the trip value function $\overline{V}_r(\bar{b})$ as follows:

$$\overline{V}_r(\bar{b}) = \sum_{m \in h_r(\bar{b})} \eta_r^m - \theta(|h_r(\bar{b})|),$$

where $\eta_r^m \triangleq \alpha^m - \beta^m t_r$ and $\theta(|h_r(\bar{b})|) = \left(\pi(|h_r(\bar{b})|) + \sigma\right) |h_r(\bar{b})| + \left(\gamma(|h_r(\bar{b})|) + \delta\right) |h_r(\bar{b})| t_r$.

Before proving that the augmented trip value function $\overline{V}_r(\bar{b})$ satisfies (a) and (b) in Definition 5.5, we first provide the following statements that will be used later:

(i) The function $\theta(|h_r(\bar{b})|)$ is non-decreasing in $|h_r(\bar{b})|$ because the marginal carpool disutility is non-decreasing in the group size.

(ii) The representative rider group for any trip $\left(\bar{b}, r\right) \in \bar{B} \times R$ can be constructed by selecting riders from $\bar{b}$ in decreasing order of η_r^m. The last selected rider ℓ (i.e. the rider in

$h_r(\bar{b})$ with the minimum value of η_r^m) satisfies:

$$\eta_r^\ell \geq \theta(|h_r(\bar{b})|) - \theta(|h_r(\bar{b})| - 1). \tag{C.9}$$

That is, adding rider ℓ to the set $h_r(\bar{b}) \setminus \{\ell\}$ increases the trip valuation. Additionally,

$$\eta_r^m < \theta(|h_r(\bar{b})| + 1) - \theta(|h_r(\bar{b})|), \quad \forall m \in \bar{b} \setminus h_r(\bar{b}). \tag{C.10}$$

Then, adding any rider in $\bar{b} \setminus h_r(\bar{b})$ to $h_r(\bar{b})$ no longer increases the trip valuation.

(iii) $|h_r(\bar{b}')| \geq |h_r(\bar{b})|$ for any two rider groups $\bar{b}', \bar{b} \in B$ such that $\bar{b}' \supseteq \bar{b}$.

Proof of (iii). Assume for the sake of contradiction that $|h_r(\bar{b}')| < |h_r(\bar{b})|$. Consider the rider $\ell \in \arg\min_{m \in h_r(\bar{b})} \eta_r^m$. The value η_r^ℓ satisfies (C.9). Since $|h_r(\bar{b}')| < |h_r(\bar{b})|$, $\bar{b}' \supseteq \bar{b}$, and we know that riders in the representative rider group $h_r(\bar{b}')$ are the ones with $|h_r(\bar{b}')|$ highest η_r^m in $\bar{b}'$, we must have $\ell \notin h_r(\bar{b}')$. From (C.10), we know that $\eta_r^\ell < \theta(|h_r(\bar{b}')| + 1) - \theta(|h_r(\bar{b}')|)$. Since the marginal carpool disutility is non-decreasing in the rider group size, we can check that $\theta(|h_r(\bar{b})| + 1) - \theta(|h_r(\bar{b})|)$ is non-decreasing in $|h_r(\bar{b})|$. Since $|h_r(\bar{b}')| < |h_r(\bar{b})|$, we have $|h_r(\bar{b}')| \leq |h_r(\bar{b})| - 1$. Therefore,

$$\eta_r^\ell < \theta(|h_r(\bar{b}')| + 1) - \theta(|h_r(\bar{b}')|) \leq \theta(|h_r(\bar{b})|) - \theta(|h_r(\bar{b})| - 1),$$

which contradicts (C.9) and the fact that $\ell \in h_r(\bar{b})$. Hence, $|h_r(\bar{b}')| \geq |h_r(\bar{b})|$.

We now prove that $\overline{V}$ satisfies *(i)* in Definition 5.5. For any $\bar{b}, \bar{b}' \subseteq M$ and $\bar{b} \subseteq \bar{b}'$, consider two cases:

Case 1: $i \notin h_r(\{i\} \cup \bar{b}')$. In this case, $h_r(\bar{b}' \cup i) = h_r(\bar{b}')$, and $\overline{V}(i|\bar{b}') = \overline{V}(\bar{b}' \cup i) - \overline{V}(\bar{b}') = 0$. Since $\overline{V}$ satisfies monotonicity condition, we have $\overline{V}(i|\bar{b}) \geq 0$. Therefore, $\overline{V}(i|\bar{b}) \geq \overline{V}(i|\bar{b}')$.

Case 2: $i \in h_r(\{i\} \cup \bar{b}')$. We argue that $i \in h_r(\{i\} \cup \bar{b})$. From (C.9), $\eta_r^i \geq \theta(|h_r(\bar{b}')|) - \theta(|h_r(\bar{b}')| - 1)$. Since $\bar{b}' \supseteq \bar{b}$, we know from (iii) that $|h_r(\bar{b}')| \geq |h_r(\bar{b})|$. Hence, $\eta_r^i \geq \theta(|h_r(\bar{b})|) - \theta(|h_r(\bar{b})| - 1)$, and thus $i \in h_r(\{i\} \cup \bar{b})$.

We define $\ell' \triangleq \arg\min_{m \in h_r(\bar{b}')} \eta_r^m$ and $\ell \triangleq \arg\min_{m \in h_r(\bar{b})} \eta_r^m$. We also consider two thresholds $\mu' = \theta(|h_r(\bar{b}')| + 1) - \theta(|h_r(\bar{b}')|)$, and $\mu = \theta(|h_r(\bar{b})| + 1) - \theta(|h_r(\bar{b})|)$. Since $\bar{b}' \supseteq \bar{b}$, from (iii), we have $|h_r(\bar{b}')| \geq |h_r(\bar{b})|$ and thus $\mu' \geq \mu$. We further consider four sub-cases:

(2-1) $\eta_r^{\ell'} \geq \mu'$ and $\eta_r^{\ell} \geq \mu$. From (C.9) and (C.10), $h_r(\{i\} \cup \bar{b}') = h_r(\bar{b}') \cup \{i\}$ and $h_r(\{i\} \cup \bar{b}) = h_r(\bar{b}) \cup \{i\}$. The marginal value of i is $\overline{V}_r(i|\bar{b}') = \eta_r^i - \mu'$, and $\overline{V}_r(i|\bar{b}) = \eta_r^i - \mu$. Since $\mu' \geq \mu$, $\overline{V}_r(i|\bar{b}') \leq \overline{V}_r(i|\bar{b})$.

(2-2) $\eta_r^{\ell'} < \mu'$ and $\eta_r^{\ell} \geq \mu$. Since $i \in h_r(\{i\} \cup \bar{b}')$ in *Case 2*, we know from (C.9) and (C.10) that $h_r(\{i\} \cup \bar{b}') = h_r(\bar{b}') \setminus \{\ell'\} \cup \{i\}$ and $h_r(\{i\} \cup \bar{b}) = h_r(\bar{b}) \cup \{i\}$. Therefore, $\overline{V}_r(i|\bar{b}') = \eta_r^i - \eta_r^{\ell'}$ and $\overline{V}_r(i|\bar{b}) = \eta_r^i - \mu$. We argue in this case, we must have $|h_r(\bar{b}')| > |h_r(\bar{b})|$. Assume for the sake of contradiction that $|h_r(\bar{b}')| = |h_r(\bar{b})|$, then $\mu' = \mu$ and $\eta_r^{\ell'} \geq \eta_r^{\ell}$ because $\bar{b}' \supseteq \bar{b}$. However, this contradicts the assumption of this subcase that $\eta_r^{\ell'} < \mu' = \mu \leq \eta_r^{\ell}$. Hence, we must have $|h_r(\bar{b}')| \geq |h_r(\bar{b})| + 1$. Then, from (C.9), we have $\eta_r^{\ell'} \geq \theta(|h_r(\bar{b}')|) - \theta(|h_r(\bar{b}')| - 1) \geq \mu$. Hence, $\overline{V}_r(i|\bar{b}') \leq \overline{V}_r(i|\bar{b})$.

(2-3) $\eta_r^{\ell'} \geq \mu'$ and $\eta_r^{\ell} < \mu$. From (C.9) and (C.10), $h_r(i \cup \bar{b}') = h_r(\bar{b}') \cup \{i\}$ and $h_r(\{i\} \cup \bar{b}) = h_r(\bar{b}) \setminus \{\ell'\} \cup \{i\}$. Therefore, $\overline{V}_r(i|\bar{b}') = \eta_r^i - \mu'$ and $\overline{V}_r(i|\bar{b}) = \eta_r^i - \eta_r^{\ell}$. Since $\mu' \geq \mu \geq \eta_r^{\ell}$, we know that $\overline{V}_r(i|\bar{b}') \leq \overline{V}_r(i|\bar{b})$.

(2-4) $\eta_r^{\ell'} < \mu'$ and $\eta_r^{\ell} < \mu$. From (C.9) and (C.10), $h_r(\{i\} \cup \bar{b}') = h_r(\bar{b}') \setminus \{\ell'\} \cup \{i\}$, and $h_r(\{i\} \cup \bar{b}) = h_r(\bar{b}) \setminus \{\ell\} \cup \{i\}$. Therefore, $\overline{V}_r(i|\bar{b}') = \eta_r^i - \eta_r^{\ell'}$ and $\overline{V}_r(i|\bar{b}) = \eta_r^i - \eta_r^{\ell}$. If $|h_r(\bar{b}')| = |h_r(\bar{b})|$, then we must have $\eta_r^{\ell'} \geq \eta_r^{\ell}$, and hence $\overline{V}_r(i|\bar{b}') \leq \overline{V}_r(i|\bar{b})$. On the other hand, if $|h_r(\bar{b}')| \geq |h_r(\bar{b})| + 1$, then from (C.9) we have $\eta_r^{\ell} \geq \theta(|h_r(\bar{b}')|) - \theta(|h_r(\bar{b}')| - 1) \geq \mu > \eta_r^{\ell}$. Therefore, we can also conclude that $\overline{V}_r(i|\bar{b}') \leq \overline{V}_r(i|\bar{b})$.

From all four subcases, we can conclude that in case 2, $\overline{V}_r(i|\bar{b}) \geq \overline{V}_r(i|\bar{b}')$.

We now prove that $\overline{V}$ satisfies condition (ii) of Definition 5.5 by contradiction. Assume for the sake of contradiction that (5.17) is not satisfied. Then, there must exist a group $\bar{b} \in \bar{B}$, and $i, j, k \in M \setminus \bar{b}$ such that:

$$\overline{V}_r(i,j|\bar{b}) + \overline{V}_r(k|\bar{b}) > \overline{V}_r(i|\bar{b}) + \overline{V}_r(j,k|\bar{b}), \quad \Rightarrow \quad \overline{V}_r(j|i,\bar{b}) > \overline{V}_r(j|k,\bar{b}), \tag{C.11a}$$

$$\overline{V}_r(i,j|\bar{b}) + \overline{V}_r(k|\bar{b}) > \overline{V}_r(j|\bar{b}) + \overline{V}_r(i,k|\bar{b}), \quad \Rightarrow \quad \overline{V}_r(i|j,\bar{b}) > \overline{V}_r(i|k,\bar{b}). \tag{C.11b}$$

We consider the following four cases:

Case A: $h_r\left(\bar{b} \cup \{i,j\}\right) = h_r\left(\bar{b} \cup \{i\}\right) \cup \{j\}$ and $h_r\left(\bar{b} \cup \{j,k\}\right) = h_r\left(\bar{b} \cup \{k\}\right) \cup \{j\}$. In this case, if $|h_r\left(\bar{b} \cup \{i\}\right)| \geq |h_r\left(\bar{b} \cup \{k\}\right)|$, then $\overline{V}_r(j|i,\bar{b}) \leq \overline{V}_r(j|k,\bar{b})$, which contradicts (C.11a). On the other hand, if $|h_r\left(\bar{b} \cup \{i\}\right)| < |h_r\left(\bar{b} \cup \{k\}\right)|$, then we must have

$h_r\left(\bar{b}\cup\{i\}\right) = h_r(\bar{b})$ and $h_r\left(\bar{b}\cup\{k\}\right) = h_r(\bar{b})\cup\{k\}$. Therefore, $\overline{V}_r(i|j,\bar{b}) = 0$, and (C.11b) cannot hold. We thus obtain the contradiction.

Case B: $|h_r\left(\bar{b}\cup\{i,j\}\right)| = |h_r\left(\bar{b}\cup\{i\}\right)|$ and $|h_r\left(\bar{b}\cup\{j,k\}\right)| = |h_r\left(\bar{b}\cup\{k\}\right)|$. We further consider the following four sub-cases:

(B-1). $h_r\left(\bar{b}\cup\{i,j\}\right) = h_r\left(\bar{b}\cup\{i\}\right)$ and $h_r\left(\bar{b}\cup\{j,k\}\right) = h_r\left(\bar{b}\cup\{k\}\right)$. In this case, $\overline{V}_r(j|i,\bar{b}) = \overline{V}_r(j|k,\bar{b}) = 0$. Hence, we arrive at a contradiction against (C.11a).

(B-2). $h_r\left(\bar{b}\cup\{i,j\}\right) \neq h_r\left(\bar{b}\cup\{i\}\right)$ and $h_r\left(\bar{b}\cup\{j,k\}\right) = h_r\left(\bar{b}\cup\{k\}\right)$. In this case, when j is added to the set $\bar{b}\cup\{i\}$, j replaces a rider, denoted as $\ell\in\bar{b}\cup\{i\}$. Since ℓ is replaced, we must have $\eta_r^\ell \le \eta_r^m$ for any $m\in h_r(\bar{b}\cup\{j\})$. If $\ell = i$, then $h_r(\bar{b}\cup\{i,j\}) = h_r(\bar{b}\cup\{j\})$. Hence, $\overline{V}_r(i|j,\bar{b}) = 0$, and we arrive at a contradiction with (C.11b). On the other hand, if $\ell\neq i$, then ℓ is a rider in group $\bar{b}$. This implies that $\ell\in\bar{b}$ should be replaced by j when j is added to the set $\{k\}\cup\bar{b}$, which contradicts the assumption of this case that $h_r\left(\bar{b}\cup\{j,k\}\right) = h_r\left(\bar{b}\cup\{k\}\right)$.

(B-3). $h_r\left(\bar{b}\cup\{i,j\}\right) = h_r\left(\bar{b}\cup\{i\}\right)$ and $h_r\left(\bar{b}\cup\{j,k\}\right) \neq h_r\left(\bar{b}\cup\{k\}\right)$. Analogous to case *B-2*, we know that $h_r\left(\bar{b}\cup\{j,k\}\right) = h_r\left(\bar{b}\cup\{j\}\right)$ and $\eta_r^j \ge \eta_r^k$. Moreover, since $h_r\left(\bar{b}\cup\{i,j\}\right) = h_r\left(\bar{b}\cup\{i\}\right)$, we must have $\eta_r^j \le \eta_r^i$. Therefore, $\overline{V}_r(\bar{b}\cup\{i,j\}) = \overline{V}_r(\bar{b}\cup\{i\})$, and $\overline{V}_r(i|j,\bar{b}) = \overline{V}_r(\bar{b}\cup\{i\}) - \overline{V}_r(\bar{b}\cup\{j\})$. Since $\eta_r^j \le \eta_r^i$ and $\eta_r^j \ge \eta_r^k$, we know that $\overline{V}_r(i|k,\bar{b}) = \overline{V}_r(\bar{b}\cup\{i\}) - \overline{V}_r(\bar{b}\cup\{k\}) \ge \overline{V}_r(\bar{b}\cup\{i\}) - \overline{V}_r(\bar{b}\cup\{j\}) = \overline{V}_r(i|j,\bar{b})$, which contradicts (C.11b).

(B-4). $h_r\left(\bar{b}\cup\{i,j\}\right) \neq h_r\left(\bar{b}\cup\{i\}\right)$ and $h_r\left(\bar{b}\cup\{j,k\}\right) \neq h_r\left(\bar{b}\cup\{k\}\right)$. In this case, if $h_r\left(\bar{b}\cup\{i,j\}\right) = h_r\left(\bar{b}\cup\{j\}\right)$, then $\overline{V}_r(i|j,\bar{b}) = \overline{V}_r(i,j,\bar{b}) - \overline{V}_r(j,\bar{b}) = \overline{V}_r(j,\bar{b}) - \overline{V}_r(j,\bar{b}) = 0$, which contradicts (C.11b). On the other hand, if $h_r\left(\bar{b}\cup\{i,j\}\right) \neq h_r\left(\bar{b}\cup\{j\}\right)$, then one rider $\ell\in\bar{b}$ must be replaced by j when j is added into the set $\bar{b}\cup\{i\}$, i.e. $h_r\left(\bar{b}\cup\{i,j\}\right) = h_r\left(\bar{b}\setminus\{\ell\}\cup\{i,j\}\right)$. Hence, $\eta_r^\ell \le \eta_r^i$ and $\eta_r^\ell \le \eta_r^j$. If $\eta_r^\ell \le \eta_r^k$, then under the assumption that $|h_r\left(\bar{b}\cup\{j,k\}\right)| = |h_r\left(\bar{b}\cup\{k\}\right)|$ and $h_r\left(\bar{b}\cup\{j,k\}\right) \neq h_r\left(\bar{b}\cup\{k\}\right)$, we must have $h_r\left(\bar{b}\cup\{j,k\}\right) = h_r\left(\bar{b}\setminus\{\ell\}\cup\{j,k\}\right)$. Then, we can check that $\overline{V}_r(j|i,b) = \overline{V}_r(j|k,b)$, which contradicts (C.11a).

On the other hand, if $\eta_r^\ell > \eta_r^k$, then $h_r\left(\bar{b}\cup\{j,k\}\right) = h_r\left(\bar{b}\cup\{j\}\right)$. In this case, $\overline{V}_r(i|j,\bar{b})$ is the change of trip value by replacing ℓ with i, and $\overline{V}_r(i|k,\bar{b})$ is the change of trip value by replacing k with i. Since $\eta_r^k < \eta_r^\ell$, we must have $\overline{V}_r(i|j,\bar{b}) < \overline{V}_r(i|k,\bar{b})$, which contradicts

(C.11b).

Case C: $h_r\left(\bar{b} \cup \{i,j\}\right) = h_r\left(\bar{b} \cup \{i\}\right) \cup \{j\}$ and $|h_r\left(\bar{b} \cup \{j,k\}\right)| = |h_r\left(\bar{b} \cup \{k\}\right)|$. We further consider the following sub-cases:

(C-1). $h_r\left(\bar{b} \cup \{j,k\}\right) = h_r\left(\bar{b} \cup \{k\}\right)$. In this case, $\eta_r^j \leq \eta_r^m$ for all $m \in h_r(\bar{b} \cup \{k\})$, and $\eta_r^j < \theta(|h_r(\bar{b}\cup\{k\})+1|) - \theta(|h_r(\bar{b}\cup\{k\})|)$. Since $h_r\left(\bar{b} \cup \{i,j\}\right) = h_r\left(\bar{b} \cup \{i\}\right) \cup \{j\}$, we know that $\eta_r^j \geq \theta(|h_r(\bar{b}\cup\{i\})+1|) - \theta(|h_r(\bar{b}\cup\{i\})|)$. Since carpool disutility is non-decreasing in rider group size, for η_r^j to satisfy both inequalities, we must have $|h_r(\bar{b}\cup\{i\})| < |h_r(\bar{b}\cup\{k\})|$. Then, we must have $h_r(\bar{b}\cup\{i\}) = h_r(\bar{b})$ and $h_r(\bar{b}\cup\{k\}) = h_r(\bar{b})\cup\{k\}$. Therefore, $\overline{V}_r(i,j,\bar{b}) = \overline{V}_r(j,\bar{b})$ and $\overline{V}_r(i,k,\bar{b}) = \overline{V}_r(k,\bar{b})$. Hence, $\overline{V}_r(i|j,\bar{b}) = \overline{V}_r(i|k,\bar{b}) = 0$, which contradicts (C.11b).

(C-2). $h_r\left(\bar{b} \cup \{j,k\}\right) \neq h_r\left(\bar{b} \cup \{k\}\right)$. Since $|h_r\left(\bar{b} \cup \{j,k\}\right)| = |h_r\left(\bar{b} \cup \{k\}\right)|$, j replaces a rider ℓ in $\bar{b}\cup\{k\}$, and $\eta_r^\ell \leq \ell_r^m$ for all $m \in \bar{b}\cup k$. If $\ell = k$, then $h_r\left(\bar{b} \cup \{j,k\}\right) = h_r\left(\bar{b} \cup \{j\}\right)$. Therefore, $\overline{V}_r(j|i,\bar{b}) = \eta_r^j - \big(\theta(|h_r(\bar{b} \cup \{i\})| + 1) - \theta(|h_r(\bar{b} \cup \{i\})|)\big)$ and $\overline{V}_r(j|k,\bar{b}) = \eta_r^j - \eta_r^k$. If $\eta_r^k \leq \theta(|h_r(\bar{b} \cup \{i\})| + 1) - \theta(|h_r(\bar{b} \cup \{i\})|)$, then (C.11a) is contradicted. Thus, $\eta_r^k > \theta(|h_r(\bar{b} \cup \{i\})| + 1) - \theta(|h_r(\bar{b} \cup \{i\})|)$. Since k is replaced by j when j is added to $\bar{b} \cup \{k\}$, we must have $\eta_r^k < \theta(|h_r(\bar{b} \cup \{j\})| + 1) - \theta(|h_r(\bar{b} \cup \{j\})|)$. For η_r^k to satisfy both inequalities, we must have $|h_r(\bar{b}\cup\{j\})| > |h_r(\bar{b}\cup\{i\})|$. Hence, $h_r(\bar{b}\cup\{j\}) = h_r(\bar{b})\cup\{j\}$ and $h_r(\bar{b}\cup\{i\}) = h_r(\bar{b})$. Then, $\overline{V}_r(i|j,\bar{b}) = \overline{V}_r(\bar{b} \cup \{i,j\}) - \overline{V}_r(\bar{b} \cup \{j\}) = 0$, which contradicts (C.11b).

On the other hand, if $\ell \in \bar{b}$, then we know from (C.10) that $\eta_r^\ell < \theta(|h_r\left(\bar{b} \cup \{k\}\right)| + 1) - \theta(|h_r\left(\bar{b} \cup \{k\}\right)|)$. Additionally, since $h_r\left(\bar{b} \cup \{i,j\}\right) = h_r\left(\bar{b} \cup \{i\}\right) \cup \{j\}$, we know from (C.9) that $\eta_r^\ell \geq \theta(|h_r\left(\bar{b} \cup \{i\}\right)| + 1) - \theta(|h_r\left(\bar{b} \cup \{i\}\right)|)$. If η_r^ℓ satisfies both inequalities, then we must have $|h_r\left(\bar{b} \cup \{i\}\right)| < |h_r\left(\bar{b} \cup \{k\}\right)|$. Therefore, $h_r\left(\bar{b} \cup \{i\}\right) = h_r(\bar{b})$. Then, $\overline{V}_r(i|j,\bar{b}) = 0$, which contradicts (C.11b).

Case D: $|h_r\left(\bar{b} \cup \{i,j\}\right)| = |h_r\left(\bar{b} \cup \{i\}\right)|$ and $h_r\left(\bar{b} \cup \{j,k\}\right) = h_r\left(\bar{b} \cup \{k\}\right) \cup \{j\}$. We further consider the following sub-cases:

(D-1). $h_r\left(\bar{b} \cup \{i,j\}\right) = h_r\left(\bar{b} \cup \{i\}\right)$. In this case, analogous to *(C-1)*, we know that $|h_r(\bar{b} \cup \{k\})| < |h_r(\bar{b} \cup \{i\})|$. Therefore, $h_r(\bar{b} \cup \{k\}) = h_r(\bar{b})$ and $h_r(\bar{b} \cup \{i\}) = h_r(\bar{b}) \cup \{i\}$. Therefore, $\eta_r^k < \eta_r^i$. Additionally, since $h_r\left(\bar{b} \cup \{i,j\}\right) = h_r\left(\bar{b} \cup \{i\}\right)$, $\eta_r^j < \eta_r^i$. Then, $\overline{V}_r(i|j,\bar{b}) = \overline{V}_r(i,\bar{b}) - \overline{V}_r(j,\bar{b})$ and $\overline{V}_r(i|k,\bar{b}) = \overline{V}_r(i,\bar{b}) - \overline{V}_r(\bar{b})$. Since $\overline{V}$ is monotonic, $\overline{V}_r(j,\bar{b}) \geq \overline{V}_r(\bar{b})$ so that $\overline{V}_r(i|j,\bar{b}) \leq \overline{V}_r(i|k,\bar{b})$, which contradicts (C.11b).

(D-2). $h_r\left(\bar{b} \cup \{i,j\}\right) \neq h_r\left(\bar{b} \cup \{i\}\right)$. Since $|h_r\left(\bar{b} \cup \{i,j\}\right)| = |h_r\left(\bar{b} \cup \{i\}\right)|$, j replaces

the rider $\ell \in \bar{b} \cup \{i\}$ such that $\eta_r^\ell \leq \eta_r^m$ for all $m \in h_r(\bar{b} \cup \{i\})$. If $\ell = i$, then analogous to case *C-2*, we know that if (C.11b) is satisfied, then $|h_r(\bar{b} \cup \{j\})| < |h_r(\bar{b} \cup \{k\})|$. Hence, $h_r(\bar{b} \cup \{j\}) = h_r(\bar{b})$ and $V(j|i, \bar{b}) = 0$, which contradicts (C.11a).

On the other hand, if $\ell \in \bar{b}$, then again analogous to case *C-2*, we know that $|h_r\left(\bar{b} \cup \{k\}\right)| < |h_r\left(\bar{b} \cup \{i\}\right)|$. Therefore, $h_r\left(\bar{b} \cup \{k\}\right) = h_r(\bar{b})$, and $h_r\left(\bar{b} \cup \{i\}\right) = h_r(\bar{b}) \cup \{i\}$. Then, $\overline{V}_r(j|i, \bar{b}) = \overline{V}_r(\bar{b} \setminus \{\ell\} \cup \{i, j\}) - \overline{V}_r(i, \bar{b})$, and $\overline{V}_r(j|k, \bar{b}) = \overline{V}_r(\bar{b} \cup \{j\}) - \overline{V}_r(\bar{b})$. Since $\ell \neq i$, $\overline{V}_r(i|j, \bar{b}) = \overline{V}_r(\bar{b} \setminus \{\ell\} \cup \{i, j\}) - \overline{V}_r(j, \bar{b}) = \eta_r^i - \eta_r^\ell$. Additionally, since $h_r(i, \bar{b}) = h_r(\bar{b}) \cup \{i\}$, $\overline{V}_r(i|k, \bar{b}) = \overline{V}_r(i, \bar{b}) - \overline{V}_r(\bar{b}) = \eta_r^i - (\theta(|h_r(\bar{b})| + 1) - \theta(|h_r(\bar{b})|))$. Since $h_r(\bar{b} \cup \{i\}) = h_r(\bar{b}) \cup \{i\}$ and $\ell \in \bar{b}$, we know from (C.9) that $\eta_r^\ell \geq \theta(|h_r(\bar{b})| + 1) - \theta(|h_r(\bar{b})|)$. Therefore, $\overline{V}_r(i|j, \bar{b}) \leq \overline{V}_r(i|k, \bar{b})$, which contradicts (C.11b).

From all above four cases, we can conclude that condition *(ii)* of Definition 5.5 is satisfied. We can thus conclude that $\overline{V}$ satisfies gross substitutes condition. $\square$

C.3 Proofs of Section 5.6

Proof of Lemma 5.9. We first show that for any optimal utility vector $u^* \in U^*$, there exists a vector λ^* such that (u^*, λ^*) is an optimal solution of (Dk^*). Since $u^* \in U^*$, there must exist a toll price vector τ^* such that (u^*, τ^*) is an optimal solution of (D). Consider $\lambda^* = (\lambda_r^*)_{r \in R^*}$ as follows:

$$\lambda_r^* = \sum_{e \in r} \tau_e^*, \quad \forall r \in R^*. \tag{C.12}$$

Since (u^*, τ^*) is feasible in (D), we can check that (u^*, λ^*) is also a feasible solution of (Dk^*). Moreover, since (x^*, u^*, τ^*) satisfies complementary slackness conditions with respect to (LP) and (D), (x^*, u^*, λ^*) also satisfies complementary slackness conditions with respect to (LPk^*) and (Dk^*). Therefore, (u^{m*}, λ^*) is an optimal solution of (Dk^*).

We next show that for any optimal solution (u^*, λ^*) of (Dk^*), we can find a toll price vector τ^* such that (u^*, τ^*) is an optimal solution of (D) (i.e. $u^* \in U^*$). We prove this argument by mathematical induction. To begin with, if the network only has a single edge $E = \{e\}$, then for any optimal solution (u^*, λ^*), we can check that (u^*, τ^*) where $\tau_e^* = \lambda_e^*$ is an optimal solution of (LP). We now prove that if this argument holds on two series-parallel networks G^1 and G^2, then it also holds on the network constructed by connecting G^1 and G^2 in parallel or in series. We prove the case of parallel connection and series connection separately as follows:

(Case 1). The network G is constructed by connecting G^1 and G^2 in parallel. In each network G^i $(i = 1, 2)$, we define E^i as the set of edges, R^i as the set of routes. We also define k^{i*} as the optimal route capacity vector computed from Alg. 1 in G^i, and $R^{i*} = \{R^i | k^{i*} > 0\}$ as the set of routes with positive capacity in k^{i*}. Since G^1 and G^2 are connected in parallel, we have $E^1 \cup E^2 = E$, $R^1 \cup R^2 = R$, $k^* = (k^{1*}, k^{2*})$, and $R^* = R^{1*} \cup R^{2*}$.

For each $i = 1, 2$, we consider the sub-problem, where riders organize trips on the sub-network G^i. For any (x^*, u^*, λ^*) on the original network G, we define the trip vector $x^{i*} = \left(x^*_{r^i}(b)\right)_{r^i \in R^i, b \in B}$ and the route toll price vector $\lambda^{i*} = \left(\tau^*_{r^i}\right)_{r^i \in R^i}$ for the subnetwork G^i. We can check that the vector x^{i*} is a feasible solution of (LPk^*) for the subproblem, where the route set R^* in the original problem (LPk^*) is replaced by R^{i*}, and k^* is replaced by k^{i*}, and the vector (u^*, λ^{i*}) is a feasible solution of (LPk^*). Additionally, since the original optimal solutions x^* and (u^*, λ^*) satisfy the complementary slackness conditions of constraints (LP.a)-(LP.b) and (D.a) for all $m \in M$ and all $r \in R^* = R^{1*} \cup R^{2*}$, we know that x^{i*} and (u^*, λ^{i*}) must also satisfy the complementary slackness conditions of these constraints in each subproblem. Therefore, x^{i*} is an optimal integer solution of (LPk^*) and (u^*, λ^{i*}) is an optimal solution of (Dk^*) in the subproblem P^i.

From our assumption of mathematical induction, there exists a toll price vector $\tau^{i*} = (\tau^*_e)_{e \in E^i}$ such that (u^*, τ^{i*}) is an optimal solution of (D) in each subproblem i with subnetwork G^i. Thus, (u^*, τ^{i*}) satisfies the feasibility constraints in (D) of each subproblem i, and x^{i*} and (u^*, τ^{i*}) satisfy the complementary slackness conditions with respect to constraints (LP.a) for each $m \in M$, (LP.b) for each $e \in E^i$, (D.a) for each $r^i \in R^i$. Consider the toll price vector $\tau^* = (\tau^{1*}, \tau^{2*})$. Since $R = R^1 \cup R^2$ and $E = E^1 \cup E^2$, (u^*, τ^*) must be feasible in (D) on the original network, and x^*, (u^*, τ^*) must satisfy the complementary slackness conditions with respect to constraints (LP.a) – (LP.b), and (D.a). Therefore, we can conclude that for any optimal solution (u^*, τ^*) of (Dk^*), there exists a toll price vector τ^* such that (u^*, τ^*) is an optimal solution of (D) in network G.

(Case 2). The network G is constructed by connecting G^1 and G^2 in series. Same as that in case 1, we define E^i as the set of edges in the subnetwork G^i ($i = 1, 2$), and R^i as the set of routes. Since G^1 and G^2 are connected in series, we have $E = E^1 \cup E^2$, and $R = R^1 \times R^2$.

We define a sub-trip (b, r^i) as the trip in the sub-network G^i where rider group b takes route $r^i \in R^i$. Analogous to the value of trip defined in (5.2), the value of each sub-trip $(b, r^i) \in B \times R^i$ is defined as:

$$V^i_{r^i}(b) = \sum_{m \in b} \left(\alpha^i_m - \beta^m t_{r^i}\right) - |b| \left(\pi(|b|) + \gamma(|b|)t_{r^i} + \sigma + \delta t_{r^i}\right), \quad \forall b \in B,\ \forall r^i \in R^i,\ \forall i = 1, 2, \tag{C.13}$$

where α^i_m is the value for rider m to travel from the origin to the destination of the subnetwork G^i. The value of α^i_m can be any number in $[0, \alpha^m]$ as long as $\alpha^1_m + \alpha^2_m = \alpha^m$. We can check that $V^1_{r^1}(b) + V^2_{r^2}(b) = V_{r^1 r^2}(b)$ is the value of the entire trip $(b, r^1 r^2)$ of the original network.

We denote the trip organization vector on G^i as $x^i = \left(x^i_{r^i}(b)\right)_{r^i \in R^i, b \in B}$, where $x^i_{r^i}(b) = 1$ if the sub-trip (b, r^i) is organized in G^i, and 0 otherwise. The optimal trip organization problem (LP) can be equivalently presented by (x^1, x^2) as follows:

$$\max_{x^1, x^2} \quad S(x^1, x^2) = \sum_{b \in B} \sum_{r^1 \in R^1} V^1_{r^1}(b) x^1_{r^1}(b) + \sum_{b \in B} \sum_{r^2 \in R^2} V^2_{r^2}(b) x^2_{r^2}(b)$$

$$s.t. \quad \sum_{r^i \in R^i} \sum_{b \ni m} x^i_{r^i}(b) \leq 1, \quad \forall m \in M, \quad \forall i = 1, 2 \tag{C.14a}$$

$$\sum_{r^i \ni e} \sum_{b \in B} x^i_{r^i}(b) \leq q_e, \quad \forall e \in E^i, \quad \forall i = 1, 2 \tag{C.14b}$$

$$\sum_{r^1 \in R^1} x^1_{r^1}(b) = \sum_{r^2 \in R^2} x^1_{r^2}(b), \quad \forall b \in B, \tag{C.14c}$$

$$x^i_{r^i}(b) \geq 0, \quad \forall b \in B, \quad \forall r^i \in R^i, \quad \forall i = 1, 2, \tag{C.14d}$$

where (C.14a) and (C.14b) are the constraints of x^i in the trip organization sub-problem on G^i. The constraint (C.14c) ensures that any rider group that takes a route in G^1 (resp. G^2) must also takes a route in G^2 (resp. G^1) to complete a trip in the original network G.

We denote k^{i*} as the optimal capacity vector of sub-network G^i computed from Alg. 1. Since Alg. 1 allocates capacity on routes in increasing order of their travel time, and the total travel time of each route is $t_{r^1r^2} = t_{r^1} + t_{r^2}$, we know that $k^{1*}_{r^1} = \sum_{r^2\in R^2} k^*_{r^1r^2}$ for all $r^1 \in R^1$ and $k^{2*}_{r^2} = \sum_{r^1\in R^1} k^*_{r^1r^2}$ for all $r^2 \in R^2$. Analogous to the proof of Lemma 5.1, any optimal integer solution of the following linear program is an optimal solution of (C.14):

$$\max_{x^1,x^2} \quad S(x^1,x^2) = \sum_{b\in B}\sum_{r^1\in R^1} V^1_{r^1}(b)x^1_{r^1}(b) + \sum_{b\in B}\sum_{r^2\in R^2} V^2_{r^2}(b)x^2_{r^2}(b)$$

$$s.t. \quad \sum_{r^i\in R^i}\sum_{b\ni m} x^i_{r^i}(b) \leq 1, \quad \forall m \in M, \quad \forall i = 1,2, \tag{C.15a}$$

$$\sum_{b\in B} x^i_{r^i}(b) \leq k^{i*}_{r^i}, \quad \forall r^i \in R^i, \quad \forall i = 1,2, \tag{C.15b}$$

$$\sum_{r^1\in R^1} x^1_{r^1}(b) = \sum_{r^2\in R^2} x^1_{r^2}(b), \quad \forall b \in B, \tag{C.15c}$$

$$x^i_{r^i}(b) \geq 0, \quad \forall b \in B, \quad \forall r^i \in R^i, \quad \forall i = 1,2. \tag{C.15d}$$

We note that a trip (b, r^1r^2) is organized if and only if both $x^1_{r^1}(b) = 1$ and $x^2_{r^2}(b) = 1$. Thus, any (x^1, x^2) is feasible in (C.14) (resp. (C.15)) if and only if there exists a feasible x in (LP) (resp. (LPk^*)) such that $x^1_{r^1}(b) = \sum_{r^2\in R^2} x_{r^1r^2}(b)$ and $x^2_{r^2}(b) = \sum_{r^1\in R^1} x_{r^1r^2}(b)$. Moreover, the value of the objective function $S\left(x^1, x^2\right)$ equals to $S(x)$ with the corresponding x:

$$\begin{aligned}
&S(x^1,x^2)\\
\overset{\text{(C.13)}}{=}&\sum_{b\in B}\sum_{r^1\in R^1} x^1_{r^1}(b)\left(\sum_{m\in b}\alpha^1_m\right) - \sum_{b\in B}\sum_{r^1\in R^1}\left(\sum_{m\in b}\beta^m t_{r^1} + |b|\left(\pi(|b|) + \gamma(|b|)t_{r^1} + \sigma + \delta t_{r^1}\right)\right) x^1_{r^1}(b)\\
&+\sum_{b\in B}\sum_{r^2\in R^2} x^2_{r^2}(b)\left(\sum_{m\in b}\alpha^2_m\right) - \sum_{b\in B}\sum_{r^2\in R^2}\left(\sum_{m\in b}\beta^m t_{r^2} + |b|\left(\pi(|b|) + \gamma(|b|)t_{r^2} + \sigma + \delta t_{r^2}\right)\right) x^2_{r^2}(b)\\
\overset{\text{(C.14c)}}{=}&\sum_{b\in B}\sum_{r\in R} x_r(b)\left(\sum_{m\in b}\alpha^m\right) - \sum_{b\in B}\sum_{r^1\in R^1}\left(\sum_{m\in b}\beta^m t_{r^1} + |b|\left(\pi(|b|) + \gamma(|b|)t_{r^1} + \sigma + \delta t_{r^1}\right)\right)\sum_{r^2\in R^2} x_{r^1r^2}(b)\\
&-\sum_{b\in B}\sum_{r^2\in R^2}\left(\sum_{m\in b}\beta^m t_{r^2} + |b|\left(\pi(|b|) + \gamma(|b|)t_{r^2} + \sigma + \delta t_{r^2}\right)\right)\sum_{r^1\in R^1} x_{r^1r^2}(b)\\
=&\sum_{b\in B}\sum_{r\in R} x_r(b)\left(\sum_{m\in b}\alpha^m - \beta^m t_r - |b|\left(\pi(|b|) + \gamma(|b|)t_r + \sigma + \delta t_r\right)\right) = S(x)
\end{aligned}$$

Therefore, given any optimal solution x^* of (LPk^*), (x^{1*}, x^{2*}), where $x^{1*}_{r^1}(b) = \sum_{r^2\in R^2} x^*_{r^1r^2}(b)$ and $x^{2*}_{r^2}(b) = \sum_{r^1\in R^1} x^*_{r^1r^2}(b)$, is an optimal integer solution of (C.15). Additionally, (x^{1*}, x^{2*})

is also an optimal solution of (C.14). Hence, the optimal values of (LP), (C.14), (LPk^*) and (C.15) are the same.

We introduce the dual variables $u^i = (u^i_m)_{m\in M, i=1,2}$ for constraints (C.14a), $\tau^i = (\tau^i_e)_{e\in E^i}$ for (C.14b) of each $i = 1, 2$, and $\chi = (\chi(b))_{b\in B}$ for (C.14c). Then, the dual program of (C.14) can be written as follows:

$$
\begin{aligned}
\min_{u^1,u^2,\tau^1,\tau^2,\chi} \quad & U = \sum_{m\in M} u^1_m + \sum_{m\in M} u^2_m + \sum_{e\in E^1} q_e\tau^1_e + \sum_{e\in E^2} q_e\tau^2_e \\
s.t. \quad & \sum_{m\in b} u^1_m + \sum_{e\in r^1} \tau^1_e + \chi(b) \geq V^1_{r^1}(b), \quad \forall b\in B, \quad \forall r^1 \in R^1, && \text{(C.16a)} \\
& \sum_{m\in b} u^2_m + \sum_{e\in r^2} \tau^2_e - \chi(b) \geq V^2_{r^2}(b), \quad \forall b\in B, \quad \forall r^2 \in R^2, && \text{(C.16b)} \\
& u^i_m,\ \tau^i_e \geq 0, \quad \forall m\in M, \quad \forall e\in E, \quad i = 1,2. && \text{(C.16c)}
\end{aligned}
$$

Similarly, we obtain the dual program of (C.15) with the same dual variables except for the route toll price vector $\lambda^i = \left(\lambda^i_{r^i}\right)_{r^i\in R^{i*}, i=1,2}$ for (C.15b):

$$
\begin{aligned}
\min_{u^1,u^2,\lambda^1,\lambda^2,\chi} \quad & U = \sum_{m\in M} u^1_m + \sum_{m\in M} u^2_m + \sum_{r^1\in R^{1*}} k^{1*}_{r^1}\lambda^1_{r^1} + \sum_{r^2\in R^{2*}} k^{2*}_{r^2}\lambda^2_{r^2} \\
s.t. \quad & \sum_{m\in b} u^1_m + \lambda^1_{r^1} + \chi(b) \geq V^1_{r^1}(b), \quad \forall b\in B, \quad \forall r^1 \in R^{1*}, && \text{(C.17a)} \\
& \sum_{m\in b} u^2_m + \lambda^2_{r^2} - \chi(b) \geq V^2_{r^2}(b), \quad \forall b\in B, \quad \forall r^2 \in R^{2*}, && \text{(C.17b)} \\
& u^i_m,\ \lambda^i_{r^i} \geq 0, \quad \forall m\in M, \quad \forall r^i \in R^{i*}, \quad i = 1,2. && \text{(C.17c)}
\end{aligned}
$$

From strong duality, we know that the optimal value of (C.17) (resp. (Dk^*)) is the same as that of (C.15) (resp. (LPk^*)). Since the optimal values of (LPk^*) and (C.15) are identical, we know that the optimal values of (C.17) must be equal to that of (Dk^*). Additionally, we can check that for any feasible solution $(u^1, u^2, \lambda^1, \lambda^2, \chi)$ of (C.17) must correspond to a feasible solution (u, λ) of (Dk^*) such that $u^m = u^1_m + u^2_m$ and $\lambda_{r^1r^2} = \lambda^1_{r^1} + \lambda^2_{r^2}$. Then, for each (u^*, λ^*), we consider the optimal solution $(u^{1*}, u^{2*}, \lambda^{1*}, \lambda^{2*}, \chi^*)$ of (C.17), and define $\tilde{V}^1_{r^1}(b) = V^1_{r^1}(b) - \chi^*(b)$, $\tilde{V}^2_{r^2}(b) = V^2_{r^2}(b) + \chi^*(b)$ for each $r^1 \in R^1$, $r^2 \in R^2$ and $b \in B$. Then, for each $i = 1, 2$, (u^{i*}, λ^{i*}) is an optimal solution of the following linear program:

$$\begin{aligned}
\min_{u^i, \lambda^i} \quad & U^i = \sum_{m \in M} u_m^i + \sum_{r^i \in R^{i*}} k_{r^i}^{i*} \lambda_{r^i}^i \\
s.t. \quad & \sum_{m \in b} u_m^i + \lambda_{r^i}^i \geq \tilde{V}_{r^i}^i(b), \quad \forall b \in B, \quad \forall r^i \in R^{i*}, && \text{(C.18a)} \\
& u_m^i,\ \lambda_{r^i}^i \geq 0, \quad \forall m \in M, \quad \forall r^i \in R^{i*}. && \text{(C.18b)}
\end{aligned}$$

From the assumption of the mathematical induction, there exists toll price vector τ^{i*} such that (u^{i*}, τ^{i*}) is an optimal dual solution of the trip organization problem on the sub-network given $\tilde{V}$ value function for each $i = 1, 2$:

$$\begin{aligned}
\min_{u^i, \lambda^i} \quad & U = \sum_{m \in M} u_m^i + \sum_{e \in E^i} q_e \tau_e^i \\
s.t. \quad & \sum_{m \in b} u_m^i + \sum_{e \in r^i} \tau_e^i \geq \tilde{V}_{r^i}(b), \quad \forall b \in B, \quad \forall r^i \in R^i, && \text{(C.19)} \\
& u_m^i,\ \tau_e^i \geq 0, \quad \forall m \in M, \quad \forall e \in E^i.
\end{aligned}$$

Since the objective function (C.16) is the sum of the objective functions in (C.19) for $i = 1, 2$, and the constraints are the combination of the constraints in the two linear programs, we know that $(u^{1*}, u^{2*}, \tau^{1*}, \tau^{2*}, \chi^*)$ must be an optimal solution of (C.16). We consider the toll price vector $\tau^* = (\tau^{1*}, \tau^{2*})$. Since $(u^{1*}, u^{2*}, \tau^{1*}, \tau^{2*}, \chi^*)$ satisfies constraints (C.16a) and (C.16b) and $u^{m*} = u_m^{1*} + u_m^{2*}$ for all $m \in M$, (u^*, τ^*) is a feasible solution of (D) on the original network G. Furthermore, since (u^*, τ^*) achieves the same objective value as the optimal solution $(u^{1*}, u^{2*}, \tau^{1*}, \tau^{2*}, \chi^*)$ in (C.17), (u^*, τ^*) must be an optimal solution of (D) on the network G.

Finally, we conclude from cases 1 and 2 that in any series-parallel network, for any optimal solution (u^*, λ^*) of (Dk^*), there must exist a toll price vector τ^* such that (u^*, τ^*) is an optimal solution of (D). $\square$

Proof of Lemma 5.10. For any $u^* \in U^*$, we define $\lambda^* = (\lambda^*)_{r \in R}$ as follows:

$$\lambda_r^* = \max_{\bar{b} \in \bar{B}} \overline{V}_r(\bar{b}) - \sum_{m \in \bar{b}} u^{m*}, \quad \forall r \in R^*.$$

Analogous to the proof of Lemma 5.3, we can show that (y^*, u^*) is a Walrasian equilibrium if and only if (y^*, u^*, λ^*) satisfies the feasibility constraints of (LPk^*) and (Dk^*) and the complementary slackness conditions. Therefore, (u^*, λ^*) must be an optimal solution of (Dk^*) and $u^* \in U^*$. $\square$

Appendix D

Supplementary Material for Chapter 6

D.1 Proofs of Section 6.3

Proof of Lemma 6.1. We first show that the strategy in (6.5) is feasible. Since $\rho_{(1)} \leq 1$, and for any $i = 1, \ldots, m-1$, $\rho_{(i)} - \rho_{(i+1)} > 0$, $\sigma_d(s_d)$ is non-negative for any $s_d \in S_d$. Additionally,

$$\begin{aligned}
\sum_{s_d \in S_d} \sigma_d(s_d) &= \sigma_d(\emptyset) + \sum_{i=1}^{m-1} \sigma_d\left(\left\{e \in E | \rho_e \geq \rho_{(i)}\right\}\right) + \sigma_d\left(\left\{e \in E | \rho_e \geq \rho_{(m)}\right\}\right) \\
&= \left(1 - \rho_{(1)}\right) + \sum_{i=1}^{m-1} \left(\rho_{(i)} - \rho_{(i+1)}\right) + \rho_{(m)} \\
&= 1 - \rho_{(1)} + \rho_{(1)} - \rho_{(m)} + \rho_{(m)} \\
&= 1.
\end{aligned}$$

Thus, σ_d in (6.5) is a feasible strategy of the defender. Now we check that σ_d in (6.5) indeed induces ρ. Consider any $e \in E$ such that $\rho_e = 0$. Then, since $e \notin \left\{E | \rho_e \geq \rho_{(i)}\right\}$ for any $i = 1, \ldots, m$, and $e \notin \emptyset$, for any $s_d \ni e$, we must have $\sigma_d(s_d) = 0$. Thus, $\sum_{s_d \ni e} \sigma_d(s_d) = 0 = \rho_e$. Finally, for any $j = 1, \ldots, m$, consider any $e \in E$, where $\rho_e = \rho_{(j)}$:

$$\sum_{s_d \ni e} \sigma_d(s_d) = \sum_{i=j}^{m} \sigma_d\left(\left\{e \in E | \rho_e \geq \rho_{(i)}\right\}\right) = \rho_{(j)}.$$

Therefore, σ_d in (6.5) induces ρ. $\square$

Proof of Proposition 6.1. We prove the result by the principal of iterated dominance. We first show that any s_d such that $s_d \nsubseteq \bar{E}$ is strictly dominated by the strategy $s_d^{'} = s_d \cap \bar{E}$. Consider any pure strategy of the attacker, $s_a \in E$, the utilities of the defender with strategy s_d and $s_d^{'}$ are as follows:

$$u_d(s_d, s_a) = -C(s_d, s_a) - |s_d|p_d = -C(s_d, s_a) - (|s_d^{'}| + |s_d \setminus \bar{E}|)p_d,$$
$$u_d(s_d^{'}, s_a) = -C(s_d^{'}, s_a) - |s_d^{'}|p_d.$$

If $s_a \in \bar{E}$ or $s_a \notin s_d$ or $s_a = \emptyset$, then $C(s_d, s_a) = C(s_d^{'}, s_a)$, and thus $U_d(s_d, s_a) < U_d(s_d^{'}, s_a)$. If $s_a = e \in s_d \setminus \bar{E}$, then $e \notin \bar{E}$, and $C_e \leq C_\emptyset$. We have $C(s_d, s_a) = C_\emptyset \geq C_e = C(s_d^{'}, s_a)$, and thus $U_d(s_d^{'}, s_a) \geq -C(s_d, s_a) - |s_d^{'}|p_d > U_d(s_d, s_a)$. Therefore, any s_d such that $s_d \nsubseteq \bar{E}$ is a strictly dominated strategy. Hence, in Γ, any equilibrium strategy of the defender satisfies $\sigma_d^*(s_d) = 0$. From (6.4), we know that $\rho_e^* = 0$ for any $e \in E \setminus \bar{E}$.

We denote the set of defender's pure strategies that are not strictly dominated as $\bar{S}_d = \{s_d | s_d \subseteq \bar{E}\}$. Consider any $s_d \in \bar{S}_d$, we show that any $s_a \in E \setminus \bar{E}$ is strictly dominated by strategy $\emptyset$. The utility functions of the attacker with strategy s_a and $\emptyset$ are as follows:

$$u_a(s_d, s_a) = C(s_d, s_a) - p_a,$$
$$u_a(s_d, \emptyset) = C(s_d, \emptyset).$$

Since $s_d \subseteq \bar{E}$ and $s_a \in E \setminus \bar{E}$, $s_a \notin s_d$, thus $C(s_d, s_a) = C_{s_a} \leq C_\emptyset$. However, $C(s_d, \emptyset) = C_\emptyset$ and $p_a > 0$. Therefore, $U_a(s_d, \emptyset) > U_a(s_d, s_a)$. Hence, any $s_a \in E \setminus \bar{E}$ is strictly dominated. Hence, in equilibrium, the probability of the attacker choosing facility $e \in E \setminus \bar{E}$ is 0 in Γ.

We can analogously argue that in $\widetilde{\Gamma}$, $\tilde{\rho}_e^* = 0$ and $\widetilde{\sigma}_a^*(e, \tilde{\rho}) = 0$ for any $e \in E \setminus \bar{E}$. $\square$

D.2 Proofs of Section 6.4

Proof of Lemma 6.2. The utility functions of the attacker with strategy σ_a in Γ^0 and Γ are related as follows:

$$U_a^0(\sigma_d, \sigma_a) = U_a(\sigma_d, \sigma_a) + \mathbb{E}_{\sigma_d}\left[|s_d|\right] \cdot p_d.$$

Thus, for a given σ_d, any σ_a that maximizes $U_a^0(\sigma_d, \sigma_a)$ also maximizes $U_a(\sigma_d, \sigma_a)$. So the set of best response strategies of the attacker in Γ^0 is identical to that in Γ. Analogously, given any σ_a, the set of best response strategies of the defender in Γ is identical to that in Γ^0. Thus, Γ^0 and Γ are strategically equivalent, i.e. they have the same set of equilibrium strategy profiles. Using the interchangeability property of equilibria in zero-sum games, we directly obtain that for any $\sigma_d^* \in \Sigma_d^*$ and any $\sigma_a^* \in \Sigma_a^*$, (σ_d^*, σ_a^*) is an equilibrium strategy profile. □

Proof of Proposition 6.2. From Lemma 6.2, the set of attacker's equilibrium strategies Σ_a^* is the optimal solution of the following maximin problem:

$$\begin{aligned}
\max_{\sigma_a} \quad & \min_{s_d \in S_d} \left\{ \sum_{e \in \bar{E}} \left(C(s_d, e) + |s_d| p_d - p_a\right) \cdot \sigma_a(e) + \left(C(s_d, \emptyset) + |s_d| p_d\right) \cdot \sigma_a(\emptyset) \right\} \\
s.t. \quad & \sum_{e \in \bar{E}} \sigma_a(e) + \sigma_a(\emptyset) = 1, && \text{(D.1a)} \\
& \sigma_a(\emptyset) \geq 0, \quad \sigma_a(e) \geq 0, \quad \forall e \in \bar{E}. && \text{(D.1b)}
\end{aligned}$$

Given any $s_d \in S_d$, we can express the objective fucntion in (D.1) as follows:

$$\begin{aligned}
& \sum_{e \in \bar{E}} \left(C(s_d, e) + |s_d| p_d - p_a\right) \cdot \sigma_a(e) + \left(C(s_d, \emptyset) + |s_d| p_d\right) \cdot \sigma_a(\emptyset) \\
= & \sum_{e \in \bar{E}} \left(C(s_d, e) - p_a\right) \cdot \sigma_a(e) + C(s_d, \emptyset)\sigma_a(\emptyset) + |s_d| p_d \cdot \left(\sum_{e \in E} \sigma_a(e) + \sigma_a(\emptyset) \right) \\
\stackrel{\text{(D.1a)}}{=} & \sum_{e \in \bar{E}} \sigma_a(e) \cdot \left(C(s_d, e) - p_a\right) + |s_d| p_d + \sigma_a(\emptyset) \cdot C_\emptyset
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{e \in \bar{E}} \sigma_a(e) \cdot (C(s_d, e) - p_a) + p_d \cdot \left(\sum_{e \in \bar{E}} \mathbb{1}\{s_d \ni e\} \right) + \sigma_a(\emptyset) \cdot C_\emptyset \\
&= \sum_{e \in \bar{E}} \left(\sigma_a(e) \cdot (C(s_d, e) - p_a) + p_d \cdot \mathbb{1}\{s_d \ni e\} \right) + \sigma_a(\emptyset) \cdot C_\emptyset \\
&\overset{(6.1)}{=} \sum_{e \in s_d} \left(\sigma_a(e) \cdot (C_\emptyset - p_a) + p_d \right) + \sum_{e \in \bar{E} \setminus s_d} \sigma_a(e) \cdot (C_e - p_a) + \sigma_a(\emptyset) \cdot C_\emptyset.
\end{aligned}
$$

Therefore, we can write:

$$
\begin{aligned}
&\min_{s_d \in S_d} \left\{ \sum_{e \in \bar{E}} (C(s_d, e) + |s_d| p_d - p_a) \cdot \sigma_a(e) + (C(s_d, \emptyset) + |s_d| p_d) \cdot \sigma_a(\emptyset) \right\} \\
= &\min_{s_d \in S_d} \left\{ \sum_{e \in s_d} \left(\sigma_a(e) \cdot (C_\emptyset - p_a) + p_d \right) + \sum_{e \in \bar{E} \setminus s_d} \sigma_a(e) \cdot (C_e - p_a) + \sigma_a(\emptyset) \cdot C_\emptyset \right\} \\
= &\sum_{e \in \bar{E}} \min \left\{ \sigma_a(e) \cdot (C_\emptyset - p_a) + p_d,\ \sigma_a(e) \cdot (C_e - p_a) \right\} + \sigma_a(\emptyset) \cdot C_\emptyset \\
= &V(\sigma_a).
\end{aligned}
$$

Thus (D.1) is equivalent to (6.10), and Σ_a^* is the optimal solution set of (6.10)

By introducing an $|\bar{E}|$-dimensional variable $v = (v_e)_{e \in \bar{E}}$, (6.10) can be changed to a linear optimization program (6.11), and Σ_a^* is the optimal solution set of (6.11). □

Proof of Lemma 6.3. We first argue that the defender's best response is in (6.13). For edge $e \in E$ such that $\sigma_a(e) < \frac{p_d}{C_e - C_\emptyset}$, we have $(C_\emptyset - C_e)\, \sigma_a(e) + p_d > 0$. Since $\rho \in BR(\sigma_a)$ maximizes $U_d(\sigma_d, \sigma_a)$ as given in (6.6a), ρ_e must be 0. Additionally, Proposition 6.1 ensures that for any $e \in E \setminus \bar{E}$, ρ_e is 0.

Analogously, if $\sigma_a(e) > \frac{p_d}{C_e - C_\emptyset}$, then $(C_\emptyset - C_e)\, \sigma_a(e) + p_d < 0$, and the best response $\rho_e = 1$. Finally, if $\sigma_a(e) = \frac{p_d}{C_e - C_\emptyset}$, any $\rho_e \in [0, 1]$ can be a best response.

We next prove (6.14). We show that if a feasible σ_a violates (6.14a), i.e., there exists a facility, denoted $\bar{e} \in \bar{E}$ such that $\sigma_a(\bar{e}) > \frac{p_d}{C_{\bar{e}} - C_\emptyset}$, then σ_a cannot be an equilibrium strategy. There are two cases:

1. There exists another facility $\widehat{e} \in \bar{E}$ such that $\sigma_a(\widehat{e}) < \frac{p_d}{C_{\widehat{e}} - C_\emptyset}$. Consider an attacker's

strategy $\sigma_a^{'}$ defined as follows:

$$\sigma_a^{'}(e) = \sigma_a(e), \quad \forall e \in \bar{E} \setminus \{\bar{e}, \widehat{e}\}, \quad \sigma_a^{'}(\emptyset) = \sigma_a(\emptyset),$$
$$\sigma_a^{'}(\bar{e}) = \sigma_a(\bar{e}) - \epsilon,$$
$$\sigma_a^{'}(\widehat{e}) = \sigma_a(\widehat{e}) + \epsilon,$$

where ϵ is a sufficiently small positive number so that $\sigma_a^{'}(\bar{e}) > \frac{p_d}{C_{\bar{e}} - C_\emptyset}$ and $\sigma_a^{'}(\widehat{e}) < \frac{p_d}{C_{\widehat{e}} - C_\emptyset}$. We obtain:

$$V(\sigma_a^{'}) - V(\sigma_a) = \epsilon \left(C_{\widehat{e}} - C_\emptyset\right) > 0$$

The last inequality holds from (6.7a) and $\widehat{e} \in \bar{E}$. Therefore, σ_a cannot be an attacker's equilibrium strategy.

2. If there does not exist such $\bar{e}$ as defined in case (a), then for any $e \in \bar{E}$, we have $\sigma_a(e) \geq \frac{p_d}{C_e - C_\emptyset}$. Now consider $\sigma_a^{'}$ as follows:

$$\sigma_a^{'}(e) = \sigma_a(e), \quad \forall e \in E \setminus \{\bar{e}\},$$
$$\sigma_a^{'}(\bar{e}) = \sigma_a(\bar{e}) - \epsilon,$$
$$\sigma_a^{'}(\emptyset) = \sigma_a(\emptyset) + \epsilon,$$

where ϵ is a sufficiently small positive number so that $\sigma_a^{'}(\bar{e}) > \frac{p_d}{C_{\bar{e}} - C_\emptyset}$. We obtain:

$$V(\sigma_a^{'}) - V(\sigma_a) = \epsilon \left(C_\emptyset - (C_\emptyset - p_a)\right) = \epsilon p_a > 0.$$

Therefore, σ_a also cannot be an attacker's equilibrium strategy.

Thus, we can conclude from cases (a) and (b) that in equilibrium σ_a^* must satisfy (6.14a). Additionally, from Proposition 6.1, (6.14b) is also satisfied. □

Proof of Theorem 6.1. We first prove the attacker's equilibrium strategies in each regime. From Proposition 6.2 and Lemma 6.3, we know that σ_a^* maximizes $V(\sigma_a)$, which can be equivalently re-written as in (6.15). We analyze the attacker's equilibrium strategy set in each regime subsequently:

1. Type I regimes Λ^i:

   - $i = 0$:

     Since $p_a > C_{(1)} - C_\emptyset$, we must have $C_\emptyset > C_e - p_a$ for any $e \in \bar{E}$. There is no vulnerable facility, and thus $\sigma_a^*(\emptyset) = 1$.

   - $i = 1, \ldots, K$:

     Since p_d satisfies (6.19) or (6.20), we obtain:

$$\sum_{e \in \cup_{k=1}^{i} \bar{E}_{(k)}} \frac{p_d}{C_e - C_\emptyset} = \sum_{k=1}^{i} \frac{p_d \cdot E_{(k)}}{C_{(k)} - C_\emptyset} < 1 \tag{D.2}$$

     Therefore, the set of feasible attack strategies satisfying (6.24c)-(6.24d) is a non-empty set. We also know from Lemma 6.3 that σ_a^* satisfies (6.14a). Again from (6.19) or (6.20), for any $k = 1, \ldots, i$, we have $C_{(k)} - p_a > C_\emptyset$ and for any $k = i+1, \ldots, K$, we have $C_{(k)} - p_a < C_\emptyset$. Since $\{C_{(k)}\}_{k=1}^{K}$ satisfy (6.8), to maximize $V(\sigma_a)$ in (6.15), the optimal solution must satisfy (6.24c)-(6.24d).

2. Type II regimes Λ_j:

   - $j = 1$: From (6.21), we know that:

$$1 = \sum_{e \in \bar{E}_{(1)}} \sigma_a^*(e) < \frac{p_d E_{(1)}}{C_{(1)} - C_\emptyset}. \tag{D.3}$$

     Thus, the set of feasible attack strategies satisfying (6.25b)-(6.25c) is a non-empty set. Additionally, from Lemma 6.3, we know that σ_a^* satisfies (6.25b). Since $C_{(1)} > C_{(k)}$ for any $k = 2, \ldots, K$, and $C_{(1)} - p_a > C_\emptyset$. From (6.15) and (D.3), we know that in equilibrium the attacker targets facilities in $\bar{E}_{(1)}$ with probability 1. The set of strategies satisfying (6.25b)-(6.25c) maximizes (6.15), and thus is the set of attacker's equilibrium strategies.

- $j = 2, \ldots, K$: From (6.22), we know that:

$$0 < 1 - \sum_{k=1}^{j-1} \frac{p_d \cdot E_{(k)}}{C_{(k)} - C_\emptyset} < \frac{p_d \cdot E_{(j)}}{C_{(j)} - C_\emptyset}.$$

  Thus, the set of feasible attack strategies satisfying (6.26c)-(6.26e) is a non-empty set. From Lemma 6.3, we know that σ_a^* satifies (6.26d). Since $\{C_{(k)}\}_{k=1,\ldots,j}$ satisfies the ordering in (6.8), in order to maximize $V(\sigma_a)$ in (6.15), σ_a^* must also satisfy (6.26c) and (6.26e), and the remaining facilities are not targeted.

We next prove the defender's equilibrium security effort. By definition of Nash equilibrium, the probability vector ρ^* is induced by an equilibrium strategy if and only if it satisfies the following two conditions:

1. ρ^* is a best response to any $\sigma_a^* \in \Sigma_a^*$.

2. Any attacker's equilibrium strategy is a best response to ρ^*, i.e. the attacker has identical utilities for choosing any pure strategy in his equilibrium support set, and the utility is no less than that of any other pure strategies.

Note that in both conditions, we require ρ^* to be a best response to *any* attacker's equilibrium strategy. This is because given any $\sigma_a^* \in \Sigma_a^*$, (ρ^*, σ_a^*) is an equilibrium strategy profile (Lemma 6.2). We now check these conditions in each regime:

1. Type I regimes Λ^i:

- If $i = 0$:
  Since $\sigma_a^*(e) = 0$ for any $e \in E$. From Lemma 6.3, the best response of the defender is $\rho_e^* = 0$ for any $e \in E$.

- If $i = 1, \ldots, K$:
  From Lemma 6.3, we know that $\rho_e^* = 0$ for any $e \in E \setminus \left(\cup_{k=1}^{i} \bar{E}_{(k)}\right)$. Since $\sigma_a^*(\emptyset) > 0$, ρ_e^* must ensure that the attacker's utility of choosing any facility $e \in \cup_{k=1}^{i} \bar{E}_{(k)}$ is identical to that of choosing no attack $\emptyset$. Consider any $e \in \cup_{k=1}^{i} \bar{E}_{(k)}$:

$$U_a(\rho^*, e) = U_a(\rho^*, \emptyset),$$

$$\overset{(6.6b)}{\Rightarrow} \quad \rho_e^* \left(C_\emptyset - p_a\right) + \left(1 - \rho_e^*\right)\left(C_e - p_a\right) = C_\emptyset,$$

$$\Rightarrow \qquad \rho_e^* = \frac{C_e - p_a - C_\emptyset}{C_e - C_\emptyset}, \quad \forall e \in \cup_{k=1}^{i} \bar{E}_{(k)}.$$

For any $\bar{e} \in E \setminus \left(\cup_{k=1}^{i} \bar{E}_{(k)}\right)$, since $\rho_{\bar{e}}^* = 0$, the attacker receives utility $C_{\bar{e}} - p_a$ by targeting $\bar{e}$, which is lower than $C_\emptyset$. Therefore, ρ^* in (6.24a)-(6.24b) satisfies both conditions (1) and (2). ρ^* is the unique equilibrium strategy.

2. Type II regimes Λ_j:

- If $j = 0$:

  Consider an attacker's strategy σ_a such that:

$$\sigma_a(e) = \frac{1}{E_{(1)}}, \quad \forall e \in \bar{E}_{(1)},$$
$$\sigma_a(e) = 0, \quad \forall e \in E \setminus \bar{E}_{(1)}.$$

  Since p_d satisfies (6.21), we know that $\frac{1}{E_{(1)}} < \frac{p_d}{C_{(1)} - C_\emptyset}$. One can check that σ_a satisfies (6.25b)-(6.25c), and thus $\sigma_a \in \Sigma_a^*$. Therefore, we know from Lemma 6.3 that $\rho_e^* = 0$ for any $e \in E$.

- If $j = 1, \ldots, K$:

  Analogous to our discussion for $j = 0$, the following is an equilibrium strategy of the attacker:

$$\sigma_a^*(e) = \frac{p_d}{C_e - C_\emptyset}, \quad \forall e \in \cup_{k=1}^{j-1} \bar{E}_{(k)},$$
$$\sigma_a^*(e) = \frac{1}{E_{(j)}} \left(1 - \sum_{i=1}^{j-1} \frac{p_d E_{(k)}}{C_{(k)} - C_\emptyset}\right), \quad \forall e \in \bar{E}_{(j)},$$
$$\sigma_a^*(e) = 0, \quad \forall e \in E \setminus \left(\cup_{k=1}^{j} \bar{E}_{(k)}\right).$$

  From Lemma 6.3, we immediately obtain that $\rho_e^* = 0$ for any $e \in E \setminus \left(\cup_{k=1}^{j-1} \bar{E}_{(k)}\right)$.

  Furthermore, for any $e \in \cup_{k=1}^{j-1} \bar{E}_{(k)}$, the utility of the attacker in choosing e must be the same as the utility for choosing any facility in $\bar{E}_{(j)}$, which is $C_{(j)} - p_a$. Therefore, for any $e \in \cup_{k=1}^{j-1} \bar{E}_{(k)}$, ρ^* satisfies:

$$\begin{aligned} & & U_a(\rho^*, e) &= C_{(j)} - p_a, \\ &\overset{(6.6b)}{\Rightarrow} & \rho_e^* \left(C_\emptyset - p_a\right) + (1-\rho_e^*)\left(C_{(k)} - p_a\right) &= C_{(j)} - p_a, \\ &\Rightarrow & \rho_e^* &= \frac{C_{(k)} - C_{(j)}}{C_{(k)} - C_\emptyset}. \end{aligned}$$

Additionally, for any $e \in E \setminus \left(\cup_{k=1}^{j} \bar{E}_{(k)}\right)$, the utility for the attacker targeting e is $C_e - p_a$, which is smaller than $C_{(j)} - p_a$. Thus, both condition (1) and (2) are satisfied. ρ^* is the unique equilibrium security effort.

□

D.3 Proofs of Section 6.5

Proof of Lemma 6.4. For any non-vulnerable facility e, the best response strategy $\widetilde{\sigma}_a$ must be such that $\widetilde{\sigma}_a(e, \tilde{\rho}) = 0$ for any $\tilde{\rho}$.

Now consider any $e \in \{E | C_e - p_a > C_\emptyset\}$. If $\tilde{\rho}_e > \widehat{\rho}_e$, then we can write:

$$U_a(\tilde{\rho}, e) = \tilde{\rho}_e C_\emptyset + (1-\tilde{\rho}_e) C_e - p_a < C_\emptyset = U_a(\tilde{\rho}, \emptyset). \tag{D.4}$$

That is, the attacker's expected utility of targeting the facility e is less than the expected utility of no attack. Thus, in any attacker's best response, $\widetilde{\sigma}_a(e, \tilde{\rho}) = 0$ for any such facility e. Additionally, if $\tilde{\rho}_e = \widehat{\rho}_e$, then $U_a(e, \tilde{\rho}) = U_a(\emptyset, \tilde{\rho})$, i.e. the utility of targeting such facility is identical with the utility of choosing no attack, and is higher than that of any other pure strategies. Hence, the set of best response strategies of the attacker is $\Delta(\bar{E}^* \cup \{\emptyset\})$, where $\bar{E}^*$ is the set defined in (6.28).

Otherwise, if there exists a facility $e \in \{E | C_e - p_a > C_\emptyset\}$ such that $\tilde{\rho}_e < \widehat{\rho}_e$, then we obtain:

$$U_a(\tilde{\rho}, e) = \tilde{\rho}_e C_\emptyset + (1-\tilde{\rho}_e) C_e - p_a > C_\emptyset = U_a(\tilde{\rho}, \emptyset).$$

Thus, no attack cannot be chosen in any best response strategy, which implies that the attacker chooses to attack with probability 1. Finally, $\bar{E}^\diamond$ is the set of facilities which incur the highest expected utility for the attacker given $\tilde{\rho}$, thus $BR(\tilde{\rho}) = \Delta(\bar{E}^{\diamond})$. $\square$

Proof of Lemma 6.5. We first prove that the total attack probability is either 0 or 1 in any SPE. We discuss the following three cases separately:

- There exists at least one single facility $e \in \{\bar{E} | C_e - p_a > C_\emptyset\}$ such that $\tilde{\rho}^*_e < \widehat{\rho}_e$. Since $\widetilde{\sigma}^*_a(\tilde{\rho}^*) \in BR(\tilde{\rho}^*)$, from Lemma 6.4, we know that $\sum_{e \in \bar{E}} \widetilde{\sigma}^*_a(e, \tilde{\rho}^*) = 1$.

- For all $e \in \{\bar{E} | C_e - p_a > C_\emptyset\}$, $\tilde{\rho}^*_e > \widehat{\rho}_e$, i.e. the set $\bar{E}^*$ in (6.28) is empty. Since $\widetilde{\sigma}^*_a(\tilde{\rho}^*) \in BR(\tilde{\rho}^*)$, from Lemma 6.4, we know that no edge is targeted in SPE, i.e. $\sum_{e \in \bar{E}} \widetilde{\sigma}^*_a(e, \tilde{\rho}^*) = 0$.

- For all $e \in \{\bar{E} | C_e - p_a > C_\emptyset\}$, $\tilde{\rho}^*_e \geq \widehat{\rho}_e$, and the set $\bar{E}^*$ in (6.28) is non-empty. For the sake of contradiction, we assume that in SPE, there exists a facility $e \in \bar{E}^*$ such that $\widetilde{\sigma}^*_a(e, \tilde{\rho}^*) > 0$, i.e. $\widetilde{\sigma}^*_a(\emptyset, \tilde{\rho}^*) < 1$. Then, we can write $U_d(\tilde{\rho}^*, \widetilde{\sigma}^*_a(\tilde{\rho}^*))$ as follows:

$$U_d(\tilde{\rho}^*, \widetilde{\sigma}^*_a(\tilde{\rho}^*)) = -C_\emptyset - (1 - \widetilde{\sigma}^*_a(\emptyset, \tilde{\rho}^*))p_a - \left(\sum_{e \in \bar{E}} \tilde{\rho}^*_e\right) p_d. \tag{D.5}$$

Now, consider $\tilde{\rho}'$ as follows:

$$\begin{aligned} \tilde{\rho}'_e &= \tilde{\rho}^*_e + \epsilon > \widehat{\rho}_e, \quad && \forall e \in \bar{E}^*, \\ \tilde{\rho}'_e &= \tilde{\rho}^*_e = 0, && \forall e \in E \setminus \bar{E}^*, \end{aligned}$$

where ϵ is a sufficiently small positive number. Given such a $\tilde{\rho}'$, we know from Lemma 6.4 that the unique best response is $\widetilde{\sigma}_a(\emptyset, \tilde{\rho}') = 1$. Therefore, the defender's utility is given by:

$$U_d(\tilde{\rho}', \widetilde{\sigma}_a(\tilde{\rho}')) = -C_\emptyset - \left(\sum_{e \in E} \tilde{\rho}'_e\right) p_d.$$

Additionally,

$$U_d(\tilde{\rho}', \widetilde{\sigma}_a(\tilde{\rho}')) - U_d(\tilde{\rho}^*, \widetilde{\sigma}_a(\tilde{\rho}^*)) = (1 - \widetilde{\sigma}_a(\emptyset, \tilde{\rho}^*))p_a - \epsilon p_d |\bar{E}^*|.$$

Since ϵ is sufficiently small and $\widetilde{\sigma}_a(\emptyset, \tilde{\rho}^*) < 1$, we obtain that $U_d(\tilde{\rho}', \widetilde{\sigma}_a(\tilde{\rho}')) > U_d(\tilde{\rho}^*, \widetilde{\sigma}_a(\tilde{\rho}^*))$. Therefore, $\tilde{\rho}^*$ cannot be a SPE. We can conclude that in this case, the attacker chooses not to attack with probability 1.

We next show that in any SPE, the defender's security effort on each vulnerable facility e is no higher than the threshold $\widehat{\rho}_e$ defined in (6.27). Assume for the sake of contradiction that there exists a facility $\bar{e} \in \{\bar{E} | C_e - p_a > C_\emptyset\}$ such that $\tilde{\rho}_{\bar{e}} > \widehat{\rho}_{\bar{e}}$. We discuss the following two cases separately:

- The set $\widehat{e} \in \{\bar{E} | C_e - p_a > C_\emptyset, \tilde{\rho}_e < \widehat{\rho}_e\}$ is non-empty. We know from Lemma 6.4 that $BR(\tilde{\rho}) = \Delta(\bar{E}^\diamond)$, where the set $\bar{E}^\diamond$ in (6.29) is the set of facilities which incur the highest utility for the attacker. Clearly, $\bar{E}^\diamond \subseteq \{\bar{E} | C_e - p_a > C_\emptyset, \tilde{\rho}_e < \widehat{\rho}_e\}$, and hence $\bar{e} \notin \bar{E}^\diamond$.

  We consider $\tilde{\rho}'$ such that $\tilde{\rho}'_{\bar{e}} = \tilde{\rho}_{\bar{e}} - \epsilon$, where ϵ is a sufficiently small positive number, and $\tilde{\rho}'_e = \tilde{\rho}_e$ for any other facilities. Then $\tilde{\rho}'_{\bar{e}} > \widehat{\rho}_{\bar{e}}$ still holds, and the set $\bar{E}^\diamond$ does not change. The attacker's best response strategy remains to be $BR(\tilde{\rho}') = \Delta(\bar{E}^\diamond)$. Hence, the utility of the defender given $\tilde{\rho}'$ increases by ϵp_d compared to that given $\tilde{\rho}$, because the expected usage cost $\mathbb{E}_\sigma[C]$ does not change, but the expected defense cost decreases by ϵp_d. Thus, such $\tilde{\rho}$ cannot be the defender's equilibrium effort.

- For all $e \in \{\bar{E} | C_e - p_a > C_\emptyset\}$, $\tilde{\rho}_e \geq \widehat{\rho}_e$. We have already argued that $\widetilde{\sigma}^*_a(\emptyset, \tilde{\rho}) = 1$ in this case. Since the defense cost $p_d > 0$, if there exists any e such that $\tilde{\rho}_e > \widehat{\rho}_e$, then by decreasing the security effort on e, the utility of the defender increases. Therefore, such $\tilde{\rho}$ cannot be an equilibrium strategy of the defender.

From both cases, we can conclude that for any $e \in \{\bar{E} | C_e - p_a > C_\emptyset\}$, $\tilde{\rho}^*_e \leq \widehat{\rho}_e$

Finally, any non-vulnerable facilities $e \in E \setminus \{E | C_e - p_a > C_\emptyset\}$ will not be targeted, hence we must have $\tilde{\rho}^*_e = 0$. □

Proof of Lemma 6.6. We first show that the threshold $\widetilde{p}_d(p_a)$ as given in (6.31) is a well-defined function of p_a. Given any $0 \leq p_a < C_{(1)} - C_\emptyset$, there is a unique $i \in \{1, \ldots, K\}$ such that $C_{(i+1)} - C_{\emptyset} \leq p_a < C_{(i)} - C_{\emptyset}$. Now, we need to show that there is a unique $j \in \{1, \ldots, i\}$ such that $\frac{\sum_{k=j+1}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}} \leq p_a < \frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}}$ (or $0 \leq p_a < \frac{E_{(i)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}}$ if $j = i$). Note that functions $\{p_d^{ij}\}_{j=1}^{i}$ are defined on the range $\left[0, \frac{\sum_{k=1}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}}\right]$. Since $\{C_{(k)}\}_{k=1}^{i}$ satisfies (6.8), we have:

$$\frac{\sum_{k=1}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}} \geq \frac{\sum_{k=1}^{i} E_{(k)}}{\frac{1}{C_{(i)}-C_{\emptyset}} \sum_{k=1}^{i} E_{(k)}} = C_{(i)} - C_{\emptyset}.$$

Hence, for any $C_{(i+1)} - C_{\emptyset} \leq p_a < C_{(i)} - C_{\emptyset}$, the value $\widetilde{p}_d(p_a)$ is defined as $p_d^{ij}(p_a)$ for a unique $j \in \{1, \ldots, i\}$. Therefore, we can conclude that for any $0 \leq p_a < C_{(1)} - C_{\emptyset}$, $\widetilde{p}_d(p_a)$ is a well-defined function.

We next show that $\widetilde{p}_d(p_a)$ is continuous and strictly increasing in p_a. Since for any $i = 1, \ldots, K$, and any $j = 1, \ldots, i$, the function $p_d^{ij}(p_a)$ is continuous and strictly increasing in p_a, $\widetilde{p}_d(p_a)$ must be piecewise continuous and strictly increasing in p_a. It remains to be shown that $\widetilde{p}_d(p_a)$ is continuous at $p_a \in \left\{C_{(i)} - C_{\emptyset}\right\}_{i=2}^{K} \cup \left\{\frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}}\right\}_{j=1,\ldots,i, i=1,\ldots K}$.

We now show that for any $i = 2, \ldots, K$, $\widetilde{p}_d(p_a)$ is continuous at $C_{(i)} - C_{\emptyset}$. Consider $p_a = C_{(i)} - C_{\emptyset} - \epsilon$ where ϵ is a sufficiently small positive number. There is a unique $\hat{j} \in \{1, \ldots, i\}$ such that $\widetilde{p}_d(p_a) = p_d^{i\hat{j}}(p_a)$. We want to argue that $\hat{j} \neq i$:

$$\begin{aligned} p_a \cdot \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}\right) &= \left(C_{(i)} - C_{\emptyset} - \epsilon\right) \cdot \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}\right) \\ &= E_{(i)} + \sum_{k=1}^{i-1} \frac{\left(C_{(i)} - C_{\emptyset}\right) E_{(k)}}{C_{(k)} - C_{\emptyset}} - \epsilon \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}\right) > E_{(i)}, \\ \Rightarrow \qquad p_a = C_{(i)} - C_{\emptyset} - \epsilon &> \frac{E_{(i)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}} \end{aligned}$$

Thus, $\hat{j} \in \{1, \ldots, i-1\}$, and from (6.31), $\frac{\sum_{k=\hat{j}+1}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}} \leq C_{(i)} - C_{\emptyset} - \epsilon < \frac{\sum_{k=\hat{j}}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}}$. Since ϵ is a sufficiently small positive number, we have:

$$\sum_{k=\hat{j}+1}^{i} E_{(k)} \leq \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}\right) \cdot \left(C_{(i)} - C_{\emptyset} - \epsilon\right)$$

$$
\begin{aligned}
&= E_{(i)} + \sum_{k=1}^{i-1} \frac{\left(C_{(i)} - C_{\emptyset}\right) E_{(k)}}{C_{(k)} - C_{\emptyset}} - \epsilon \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}} \right) \\
\Rightarrow \qquad \sum_{k=\hat{j}+1}^{i-1} E_{(k)} &\leq \sum_{k=1}^{i-1} \frac{\left(C_{(i)} - C_{\emptyset}\right) E_{(k)}}{C_{(k)} - C_{\emptyset}} + \epsilon \left(\sum_{k=1}^{i-1} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}} \right) \\
\Rightarrow \qquad \frac{\sum_{k=\hat{j}+1}^{i-1} E_{(k)}}{\sum_{k=1}^{i-1} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}} &\leq C_{(i)} - C_{\emptyset} + \epsilon.
\end{aligned}
$$

Analogously, we can check that $C_{(i)} - C_{\emptyset} + \epsilon < \frac{\sum_{k=\hat{j}}^{i-1} E_{(k)}}{\sum_{k=1}^{i-1} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}}$. Hence, from (6.31), when $p_a = C_{(i)} - C_{\emptyset} + \epsilon$, we have $\widetilde{p}_d(p_a) = p_d^{i-1\hat{j}}(p_a)$. Then,

$$
\begin{aligned}
\lim_{p_a \to \left(C_{(i)} - C_{\emptyset}\right)^-} \widetilde{p}_d(p_a) &= \lim_{\epsilon \to 0} p_d^{i\hat{j}}(C_{(i)} - C_{\emptyset} - \epsilon) \\
&\overset{(6.30)}{=} \frac{C_{(\hat{j})} - C_{\emptyset}}{\left(C_{(\hat{j})} - C_{\emptyset}\right) \cdot \left(\sum_{k=1}^{\hat{j}-1} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}} \right) + \sum_{k=\hat{j}}^{i-1} E_{(k)} - \sum_{k=1}^{i-1} \frac{p_a E_{(k)}}{C_{(k)} - C_{\emptyset}}} \\
&= \lim_{\epsilon \to 0} p_d^{i-1\hat{j}}(C_{(i)} - C_{\emptyset} + \epsilon) = \lim_{p_a \to \left(C_{(i)} - C_{\emptyset}\right)^+} \widetilde{p}_d(p_a).
\end{aligned}
$$

Thus, $\widetilde{p}_d(p_a)$ is continuous at $C_{(i)} - C_{\emptyset}$ for any $i = 2, \ldots, K$.

For any $i = 1, \ldots, K$, we next show that $\widetilde{p}_d(p_a)$ is continuous at $p_a = \frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}}$ for $j = 1, \ldots, i$:

$$
\begin{aligned}
\lim_{p_a \to \left(\frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}} \right)^-} \widetilde{p}_d(p_a) &= p_d^{ij} \left(\frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}} \right) = \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}} \right)^{-1} \\
&= p_d^{i(j-1)} \left(\frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}} \right) = \lim_{p_a \to \left(\frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_{\emptyset}}} \right)^+} \widetilde{p}_d(p_a).
\end{aligned}
$$

Hence, we can conclude that $\widetilde{p}_d(p_a)$ is continuous and strictly increasing in p_a.

Additionally, for any $i = 1, \ldots, K$, consider any p_a such that $C_{(i+1)} - C_{\emptyset} < p_a \leq C_{(i)} - C_{\emptyset}$

(or $0 < p_a \le C_{(K)} - C_\emptyset$ if $i = K$), then for any $j = 1, \ldots, i$, we have:

$$\begin{aligned}
p_d^{ij}(p_a) &\overset{(6.30)}{=} \frac{C_{(j)} - C_\emptyset}{\left(C_{(j)} - p_a - C_\emptyset\right) \cdot \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}\right) + \sum_{k=j}^{i} \frac{\left(C_{(k)} - p_a - C_\emptyset\right) E_{(k)}}{C_{(k)} - C_\emptyset}} \\
&> \frac{C_{(j)} - C_\emptyset}{\left(C_{(j)} - C_{(i+1)}\right) \cdot \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}\right) + \sum_{k=j}^{i} \frac{\left(C_{(k)} - C_{(i+1)}\right) E_{(k)}}{C_{(k)} - C_\emptyset}} \\
&= \frac{C_{(j)} - C_\emptyset}{\left(C_{(j)} - C_{(i+1)}\right) \cdot \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}\right)} \\
&\overset{(6.8)}{>} \left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}\right)^{-1} \\
&\overset{(6.17)}{=} \bar{p}_d(p_a).
\end{aligned}$$

Therefore, for any $0 < p_a < C_{(1)} - C_\emptyset$, we have:

$$\widetilde{p}_d(p_a) \overset{(6.31)}{\ge} \min_{j=1,\ldots,i} p_d^{ij}(p_a) > \bar{p}_d(p_a), \tag{D.6}$$

Finally, if $p_a = 0$, then we know that $\widetilde{p}_d(0) = p_d^{KK}(0)$. From (6.30), we can check that $p_d^{KK}(0) = \left(\sum_{k=1}^{K} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}\right)^{-1} = \bar{p}_d(0)$. If p_a approaches $C_{(1)} - C_\emptyset$, then $\widetilde{p}_d(p_a) = p_d^{11}(p_a)$, and we have:

$$\lim_{p_a \to C_{(1)} - C_\emptyset} \widetilde{p}_d(p_a) \overset{(6.30)}{=} \lim_{p_a \to C_{(1)} - C_\emptyset} \frac{C_{(1)} - C_\emptyset}{E_{(1)} - \frac{p_a E_{(1)}}{C_{(1)} - C_\emptyset}} = +\infty$$

□

We define the partition as:

$$\mathcal{P} \triangleq \left\{ \left\{\Lambda^i\right\}_{i=0}^{K}, \left\{\Lambda_j^i\right\}_{j=1,\ldots,i, i=1,\ldots,K,} \right\}, \tag{D.7}$$

where $\{\Lambda^i\}_{i=0}^{K}$ are type I regimes in the normal form game defined in (6.18)-(6.20), and Λ_j^i

is the set of (p_d, p_a), which satisfy:

$$p_d \in \begin{cases} \left(\left(\frac{E_{(1)}}{C_{(1)}-C_{\emptyset}}\right)^{-1}, +\infty\right), & \text{if } j = 1, \\ \left(\left(\sum_{k=1}^{j} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}\right)^{-1}, \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)}-C_{\emptyset}}\right)^{-1}\right), & \text{if } j = 2, \dots, K, \end{cases} \tag{D.8a}$$

$$p_a \in \begin{cases} \left(C_{(i+1)} - C_{\emptyset}, C_{(i)} - C_{\emptyset}\right), & \text{if } i = 1, \dots, K-1, \\ \left(0, C_{(K)} - C_{\emptyset}\right), & \text{if } i = K, \end{cases} \tag{D.8b}$$

We can check that sets in $\mathcal{P}$ are disjoint, and cover the whole space of (p_d, p_a). Lemma D.1 characterizes SPE in sets $\{\Lambda^i\}_{i=0}^{K}$, and Lemma D.2 characterizes SPE in sets $\left\{\Lambda^i_j\right\}_{i=1,j=1}^{i=K,j=i}$.

Lemma D.1. *In $\widetilde{\Gamma}$, for any (p_a, p_d) in the set Λ^i, where $i = 0, \dots, K$:*

- *If $i = 0$, then SPE is as given in* (6.37).

- *If $i = 1, \dots, K$: then SPE is as given in* (6.38).

Proof of Lemma D.1. If $i = 0$: The set of vulnerable facilities $\{\bar{E} | C_e - p_a > C_{\emptyset}\}$ is empty. Thus, $\widetilde{\sigma}^*_a(\emptyset, \tilde{\rho}) = 1$, and $\tilde{\rho}^*_e = 0$ for all $e \in E$.

For any $i = 1, \dots, K$: The set of vulnerable facilities is $\cup_{k=1}^{i} \bar{E}_{(k)}$. From Lemma 6.5, we have already known that for any $e \in \cup_{k=1}^{i} \bar{E}_{(k)}$, $\tilde{\rho}^*_e \leq \widehat{\rho}_e$. Assume for the sake of contradiction that there exists a facility $\bar{e} \in \cup_{k=1}^{i} \bar{E}_{(k)}$ such that $\tilde{\rho}_{\bar{e}} < \widehat{\rho}_{\bar{e}}$. From Lemma 6.4, we know that $\widetilde{\sigma}^*_a(\emptyset, \tilde{\rho}) = 0$, and $BR(\tilde{\rho}) = \Delta(\bar{E}^{\diamond})$, where $\bar{E}^{\diamond}$ is in (6.29). Clearly, $\bar{E}^{\diamond} \subseteq \cup_{k=1}^{i} \bar{E}_{(k)}$. We define λ as follows:

$$\lambda = \max_{e \in \cup_{k=1}^{i} \bar{E}_{(k)}} \{\tilde{\rho}_e C_{\emptyset} + (1 - \tilde{\rho}_e) C_e\} = \tilde{\rho}_e C_{\emptyset} + (1 - \tilde{\rho}_e) C_e, \quad \forall e \in \bar{E}^{\diamond}.$$

The utility of the defender can be written as:

$$U_d(\tilde{\rho}, \widetilde{\sigma}^*_a(\tilde{\rho})) = -\lambda - \left(\sum_{e \in E} \tilde{\rho}_e\right) \cdot p_d.$$

We now consider $\tilde{\rho}'$ as follows:

$$\tilde{\rho}'_e = \tilde{\rho}_e + \frac{\epsilon}{C_e - C_{\emptyset}}, \quad \forall e \in \bar{E}^{\diamond},$$

$$\tilde{\rho}'_e = \tilde{\rho}_e, \qquad \forall e \in E \setminus \bar{E}^\diamond,$$

where ϵ is a sufficiently small positive number. Under this deviation, we can check that the set $\bar{E}^\diamond$ does not change, but λ changes to $\lambda - \epsilon$. Therefore, the defender's utility can be written as:

$$\begin{aligned}
U_d(\tilde{\rho}', \widetilde{\sigma}_a(\tilde{\rho}')) &= -\lambda + \epsilon - \left(\sum_{e \in E} \tilde{\rho}'_e\right) \cdot p_d = -\lambda + \epsilon - \left(\sum_{e \in E} \tilde{\rho}_e\right) \cdot p_d - \sum_{e \in \bar{E}^\diamond} \frac{\epsilon p_d}{C_e - C_\emptyset} \\
=& U_d(\tilde{\rho}, \widetilde{\sigma}_a(\tilde{\rho})) + \epsilon \left(1 - \sum_{e \in \bar{E}^\diamond} \frac{p_d}{C_e - C_\emptyset}\right) \geq U_d(\tilde{\rho}, \widetilde{\sigma}_a(\tilde{\rho})) + \epsilon \left(1 - \sum_{e \in \cup_{k=1}^i \bar{E}_{(k)}} \frac{p_d}{C_e - C_\emptyset}\right) \\
=& U_d(\tilde{\rho}, \widetilde{\sigma}_a(\tilde{\rho})) + \epsilon \left(1 - \sum_{k=1}^{i} \frac{p_d E_{(k)}}{C_e - C_\emptyset}\right) \overset{(6.19)}{>} U_d(\tilde{\rho}, \widetilde{\sigma}_a(\tilde{\rho})).
\end{aligned}$$

Therefore, such $\tilde{\rho}$ cannot be an equilibrium strategy profile. We thus know that $\tilde{\rho}^*$ is as given in (6.38). The attacker's equilibrium strategy can be derived from Lemmas 6.4 and 6.5 directly. □

Lemma D.2. *For (p_a, p_d) in Λ^i_j, where $i = 1, \ldots, K$, and $j = 1, \ldots, i$, there are two cases of SPE:*
If $p_d > p_d^{ij}$, where p_d^{ij} is as given in (6.30)*:*

- *If $j = 1$, then SPE is as given in* (6.39)*.*

- *If $j = 2, \ldots, i$, then SPE is as given in* (6.40)*.*

If $p_d < p_d^{ij}$, then the SPE is as given in (6.38)*.*

Proof of Lemma D.2. Consider cost parameters in the set Λ^i_j defined in (D.7), where $i = 1, \ldots, K$ and $j = 1, \ldots, i$. The set of vulnerable facilities is $\cup_{k=1}^i \bar{E}_{(k)}$. From Lemma 6.5, we know that the defender can either secure all vulnerable facilities $e \in \cup_{k=1}^i \bar{E}_{(k)}$ with the threshold effort $\widehat{\rho}_e$ defined in (6.27), or leave at least one vulnerable facility secured less than the threshold effort. We discuss the two cases separately:

(1) If any $e \in \cup_{k=1}^i \bar{E}_{(k)}$ is secured with the threshold effort $\widehat{\rho}_e$, then from Lemma 6.5, we

know that the total probability of attack is 0. The defender's utility can be written as:

$$U_d(\widehat{\rho}, \widetilde{\sigma}_a^*(\widehat{\rho})) = -C_\emptyset - \left(\sum_{k=1}^{i} \frac{\left(C_{(k)} - p_a - C_\emptyset\right) \cdot E_{(k)}}{C_{(k)} - C_\emptyset} \right) \cdot p_d. \tag{D.9}$$

(2) If the set $\{E | C_e - p_a > C_\emptyset, \quad \tilde{\rho}_e < \widehat{\rho}_e\}$ is non-empty, then we define $\widetilde{P}$ as the set of feasible $\tilde{\rho}$ in this case. We denote $\tilde{\rho}^\dagger$ as the secure effort vector that incurs the highest utility for the defender among all $\tilde{\rho} \in \widetilde{P}$. Then, $\tilde{\rho}^\dagger$ can be written as:

$$\begin{aligned}
\tilde{\rho}^\dagger \in \underset{\tilde{\rho} \in \widetilde{P}}{\operatorname{argmax}}\ U_d(\tilde{\rho}, \widetilde{\sigma}_a^*(\tilde{\rho})) &= \underset{\tilde{\rho} \in \widetilde{P}}{\operatorname{argmax}} \left(-\mathbb{E}_{(\tilde{\rho}, \widetilde{\sigma}_a^*(\tilde{\rho}))}[C] - \left(\sum_{e \in E} \tilde{\rho}_e \right) \cdot p_d \right) \\
= \underset{\tilde{\rho} \in \widetilde{P}}{\operatorname{argmax}} \left(-\mathbb{E}_{(\tilde{\rho}, \widetilde{\sigma}_a^*(\tilde{\rho}))}[C] - \left(\sum_{e \in E} \tilde{\rho}_e \right) \cdot p_d + \left(\sum_{e \in E} \widetilde{\sigma}_a^*(e, \tilde{\rho}) \right) \cdot p_a - \left(\sum_{e \in E} \widetilde{\sigma}_a^*(e, \tilde{\rho}) \right) \cdot p_a \right)&.
\end{aligned} \tag{D.10}$$

We know from Lemma 6.4 that $\widetilde{\sigma}_a^*(\emptyset, \tilde{\rho}) = 0$. Therefore, $\sum_{e \in E} \widetilde{\sigma}_a^*(e, \tilde{\rho}) = 1$, and (D.10) can be re-expressed as:

$$\begin{aligned}
\tilde{\rho}^\dagger &\in \underset{\tilde{\rho} \in \widetilde{P}}{\operatorname{argmax}} \left(-\mathbb{E}_{(\tilde{\rho}, \widetilde{\sigma}_a^*(\tilde{\rho}))}[C] - \left(\sum_{e \in E} \tilde{\rho}_e \right) \cdot p_d + \left(\sum_{e \in E} \widetilde{\sigma}_a^*(e, \tilde{\rho}) \right) \cdot p_a - p_a \right) \\
&= \underset{\tilde{\rho} \in \widetilde{P}}{\operatorname{argmax}} \left(-\mathbb{E}_{(\tilde{\rho}, \widetilde{\sigma}_a^*(\tilde{\rho}))}[C] - \left(\sum_{e \in E} \tilde{\rho}_e \right) \cdot p_d + \left(\sum_{e \in E} \widetilde{\sigma}_a^*(e, \tilde{\rho}) \right) \cdot p_a \right).
\end{aligned}$$

Since in equilibrium, the attacker chooses the best response strategy, we have:

$$\mathbb{E}_{(\tilde{\rho}, \widetilde{\sigma}_a^*(\tilde{\rho}))}[C] - \left(\sum_{e \in E} \widetilde{\sigma}_a^*(e, \tilde{\rho}) \right) \cdot p_a = \max_{\widetilde{\sigma}_a \in \Delta(S_a)} \left(\mathbb{E}_{(\tilde{\rho}, \widetilde{\sigma}_a)}[C] - \left(\sum_{e \in E} \widetilde{\sigma}_a(e) \right) \cdot p_a \right). \tag{D.11}$$

Hence, $\tilde{\rho}^\dagger$ can be re-expressed as:

$$\begin{aligned}
\tilde{\rho}^\dagger &\overset{(D.11)}{=} \underset{\tilde{\rho} \in \widetilde{P}}{\operatorname{argmax}} \left(-\max_{\widetilde{\sigma}_a \in \Delta(S_a)} \left(\mathbb{E}_{(\tilde{\rho}, \widetilde{\sigma}_a)}[C] - \left(\sum_{e \in E} \widetilde{\sigma}_a(e) \right) \cdot p_a \right) - \left(\sum_{e \in E} \tilde{\rho}_e \right) \cdot p_d \right) \\
&= \underset{\tilde{\rho} \in \widetilde{P}}{\operatorname{argmax}} \left(-\max_{\widetilde{\sigma}_a \in \Delta(S_a)} \left(\mathbb{E}_{(\tilde{\rho}, \widetilde{\sigma}_a)}[C] - \left(\sum_{e \in E} \widetilde{\sigma}_a(e) \right) \cdot p_a + \left(\sum_{e \in E} \tilde{\rho}_e \right) \cdot p_d \right) \right)
\end{aligned}$$

$$= \underset{\tilde{\rho}\in\widetilde{P}}{\operatorname{argmax}} \min_{\widetilde{\sigma}_a\in\Delta(S_a)} \left(-\mathbb{E}_{(\tilde{\rho},\widetilde{\sigma}_a)}[C] + \left(\sum_{e\in E}\widetilde{\sigma}_a(e)\right)\cdot p_a - \left(\sum_{e\in E}\tilde{\rho}_e\right)\cdot p_d\right)$$
$$\overset{(6.9a)}{=} \underset{\tilde{\rho}\in\widetilde{P}}{\operatorname{argmax}} \min_{\widetilde{\sigma}_a\in\Delta(S_a)} U_d^0(\tilde{\rho},\widetilde{\sigma}_a).$$

Therefore, $\tilde{\rho}^\dagger$ is the defender's equilibrium strategy in the zero sum game, which is identical to the equilibrium strategy in the normal form game (recall Lemma 6.2). From Theorem 6.1, when p_a and p_d are in Λ_j^i, $\tilde{\rho}^\dagger$ is in (6.40) (or (6.39) if $j=1$). The defender's utility in this case is:

$$U_d(\tilde{\rho}^\dagger, \widetilde{\sigma}_a^*(\tilde{\rho}^\dagger)) = -C_{(j)} - \left(\sum_{k=1}^{j-1}\frac{\left(C_{(k)}-C_{(j)}\right)\cdot E_{(k)}}{C_{(k)}-C_\emptyset}\right)\cdot p_d. \tag{D.12}$$

Finally, by comparing U_d in (D.12) and (D.9), we can check that if $p_d > p_d^{ij}$, then $U_d(\tilde{\rho}^\dagger, \widetilde{\sigma}_a^*(\tilde{\rho}^\dagger)) > U_d(\widehat{\rho}, \widetilde{\sigma}_a^*(\widehat{\rho}))$. Thus, SPE is in (6.40) (or (6.39) if $j=1$). If $p_d < p_d^{ij}$, then $U_d(\tilde{\rho}^\dagger, \widetilde{\sigma}_a^*(\tilde{\rho}^\dagger)) < U_d(\widehat{\rho}, \widetilde{\sigma}_a^*(\widehat{\rho}))$, and SPE is in (6.38). □

Proof of Theorem 6.2. Type $\widetilde{\mathrm{I}}$ regimes $\widetilde{\Lambda}^i$:
If $i=0$: There is no vulnerable facility. Therefore, the attacker chooses not to attack with probability 1, and the defender does not secure any facility. SPE is as given in (6.37).

If $i=1,\dots,K$: Consider any $C_{(i+1)}-C_\emptyset < p_a < C_{(i)}-C_\emptyset$. From Lemma (6.6), we know that $\widetilde{p}_d(p_a) > \bar{p}_d(p_a)$, where $\widetilde{p}_d(p_a)$ is defined in (6.31) and $\bar{p}_d(p_a)$ is as defined in (6.16). From Lemma D.1, we know that SPE is as given in (6.38) for any $p_d < \bar{p}_d(p_a)$.

It remains to be shown that for any $\bar{p}_d(p_a) \le p_d < \widetilde{p}_d(p_a)$, SPE is also as given in (6.38). For any $C_{(i+1)}-C_\emptyset \le p_a < C_{(i)}-C_\emptyset$, there is a unique $\hat{\jmath}\in\{1,\dots,i\}$ such that $\frac{\sum_{k=\hat{\jmath}+1}^i E_{(k)}}{\sum_{k=1}^i \frac{E_{(k)}}{C_{(k)}-C_\emptyset}} \le p_a < \frac{\sum_{k=\hat{\jmath}}^i E_{(k)}}{\sum_{k=1}^i \frac{E_{(k)}}{C_{(k)}-C_\emptyset}}$, and from (6.31), we have:

$$\widetilde{p}_d(p_a) = p_d^{i\hat{\jmath}}(p_a) \ge p_d^{i\hat{\jmath}}\left(\frac{\sum_{k=\hat{\jmath}+1}^i E_{(k)}}{\sum_{k=1}^i \frac{E_{(k)}}{C_{(k)}-C_\emptyset}}\right) \overset{(6.30)}{=} \left(\sum_{k=1}^{\hat{\jmath}}\frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1},$$
$$\widetilde{p}_d(p_a) = p_d^{i\hat{\jmath}}(p_a) < p_d^{i\hat{\jmath}}\left(\frac{\sum_{k=\hat{\jmath}}^i E_{(k)}}{\sum_{k=1}^i \frac{E_{(k)}}{C_{(k)}-C_\emptyset}}\right) = \left(\sum_{k=1}^{\hat{\jmath}-1}\frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1}.$$

Consider any $j = \hat{j}+1, \ldots, i$, and any $\left(\sum_{k=1}^{j} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1} \leq p_d < \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1}$, the cost parameters (p_a, p_d) are in the set Λ_j^i as defined in (D.7). Additionally, from our definition of $\hat{j}$, we know that $p_a > \frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}}$. We now show that in Λ_j^i, $p_d < p_d^{ij}(p_a)$:

$$p_d^{ij}(p_a) \overset{(6.30)}{>} p_d^{ij}\left(\frac{\sum_{k=j}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}}\right) = \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1} \overset{(D.7)}{>} p_d.$$

Hence, from Lemma D.2, we know that for any $\left(\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1} \leq p_d \leq \left(\sum_{k=1}^{\hat{j}} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1}$, SPE is as given in (6.38). For any $\left(\sum_{k=1}^{\hat{j}} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1} < p_d < \widetilde{p}_d(p_a)$, the cost parameters (p_a, p_d) are in the set $\Lambda_i^{\hat{j}}$, and $p_d < \widetilde{p}_d(p_a) = p_d^{i\hat{j}}(p_a)$. Again from Lemma D.2, SPE is in (6.38).

Therefore, we can conclude that in regime $\widetilde{\Lambda}^i$, SPE is in (6.38).

Type $\widetilde{\text{II}}$ regimes $\widetilde{\Lambda}_j$, where $j = 1, \ldots, K$: Since $\widetilde{p}_d(p_a)$ is strictly increasing in p_a and $\lim_{p_a \to C_{(1)}-C_\emptyset} \widetilde{p}_d(p_a) = +\infty$, we know that for any $p_d > 0$, $p_a < \widetilde{p}_d^{-1}(p_d) < C_{(1)} - C_\emptyset$. Therefore, we can re-express $\widetilde{\Lambda}^1$ as follows:

$$\begin{aligned}
\widetilde{\Lambda}^1 &\overset{(6.35)}{=} \left\{ (p_a, p_d) \middle| p_a < \widetilde{p}_d^{-1}(p_d),\ p_d > \left(\frac{E_{(1)}}{C_{(1)}-C_\emptyset}\right)^{-1} \right\} \\
&= \left\{ (p_a, p_d) \middle| p_d > \widetilde{p}_d(p_a),\ p_d > \left(\frac{E_{(1)}}{C_{(1)}-C_\emptyset}\right)^{-1}, 0 \leq p_a \leq C_{(1)} - C_\emptyset \right\} \\
&\overset{(D.7)}{=} = \bigcup_{i=1}^{K} \left(\Lambda_j^i \bigcap \{(p_a, p_d) \,|p_d > \widetilde{p}_d(p_a)\}\right).
\end{aligned} \tag{D.13}$$

For any $j = 2, \ldots, K$, if $p_a > C_{(j)} - C_\emptyset$, then from Lemma 6.6, we have:

$$\widetilde{p}_d(p_a) > \bar{p}_d(p_a) \overset{(6.17)}{\geq} \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1}. \tag{D.14}$$

Therefore, for any $p_d < \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)}-C_\emptyset}\right)^{-1}$, we know that $p_a < \widetilde{p}_d^{-1}(p_d) < C_{(j)} - C_\emptyset$. Analogous to (D.13), we re-express the set $\widetilde{\Lambda}_j$ as follows:

$$\widetilde{\Lambda}_j \overset{(6.36)}{=} \left\{ (p_a, p_d) \middle| p_a < \widetilde{p}_d^{-1}(p_d),\ \left(\sum_{k=1}^{j} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1} \leq p_d < \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1} \right\}$$

$$\overset{(D.14)}{=} \left\{ (p_a, p_d) \middle| \begin{array}{l} p_d > \widetilde{p}_d(p_a),\ \left(\sum_{k=1}^{j} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1} \leq p_d < \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1}, \\ 0 \leq p_a \leq C_{(j)} - C_\emptyset \end{array} \right\}$$

$$\overset{(D.7)}{=} \bigcup_{i=j}^{K} \left(\Lambda_j^i \bigcap \{ (p_a, p_d) \, | p_d > \widetilde{p}_d(p_a) \} \right).$$

We next show that for any $j = 1, \ldots, K$, and any $i = j, \ldots, K$, the set $\Lambda_j^i \cap \{(p_a, p_d) \, | p_d > \widetilde{p}_d(p_a)\} \subseteq \Lambda_j^i \cap \{(p_a, p_d) \, | p_d > p_d^{ij}(p_a)\}$. Consider any cost parameters (p_a, p_d) in the set $\Lambda_j^i \cap \{(p_a, p_d) \, | p_d > \widetilde{p}_d(p_a)\}$, from (6.31), we can find $\hat{j}$ such that $\frac{\sum_{k=\hat{j}+1}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}} \leq p_a < \frac{\sum_{k=\hat{j}}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}}$, and $\widetilde{p}_d(p_a) = p_d^{i\hat{j}}(p_a)$. We discuss the following three cases separately:

- If $\hat{j} > j$, then we must have $p_a < \frac{\sum_{k=\hat{j}}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}} \leq \frac{\sum_{k=j+1}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}}$. Hence, from (6.30), $p_d^{ij}(p_a) < \left(\sum_{k=1}^{j} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1}$. From the definition of the set Λ_j^i in (D.7), we know that $p_d > p_d^{ij}(p_a)$ in this set, and thus $(p_a, p_d) \in \Lambda_j^i \cap \{(p_a, p_d) \, | p_d > p_d^{ij}(p_a)\}$.
- If $\hat{j} = j$, then we directly obtain that $(p_a, p_d) \in \Lambda_j^i \cap \{(p_a, p_d) \, | p_d > p_d^{ij}(p_a)\}$.
- If $\hat{j} < j$, then since $p_a \geq \frac{\sum_{k=\hat{j}+1}^{i} E_{(k)}}{\sum_{k=1}^{i} \frac{E_{(k)}}{C_{(k)} - C_\emptyset}}$, from (6.30), we have $\widetilde{p}_d(p_a) = p_d^{i\hat{j}}(p_a) \geq \left(\sum_{k=1}^{\hat{j}} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1} \geq \left(\sum_{k=1}^{j-1} \frac{E_{(k)}}{C_{(k)} - C_\emptyset} \right)^{-1}$. From the definition of the set Λ_j^i in (D.7), the set $\Lambda_j^i \cap \{(p_a, p_d) \, | p_d > \widetilde{p}_d(p_a)\}$ is empty, and thus can be omitted.

We can conclude from all three cases that $\Lambda_j^i \cap \{(p_a, p_d) \, | p_d > \widetilde{p}_d(p_a)\} \subseteq \Lambda_j^i \cap \{(p_a, p_d) \, | p_d > p_d^{ij}(p_a)\}$. Therefore, from Lemma D.2, SPE is in (6.40) (or (6.39) if $j = 1$) in the regime $\widetilde{\Lambda}_j$.

□

Appendix E

Supplementary Material for Chapter 7

E.1 Behavioral Justification

Our choice of the regression model (7.2) is motivated by the well-known discrete choice models (Ben-Akiva and Bierlaire [1999]) that are used to explain individual travelers' choices between the two modes – driving and taking public transit – based on the estimated utilities of the two modes. The utility of driving and taking public transit for individual travelers depends on their anticipated travel time and a mode-specific features that includes all other characteristics such as waiting time, walking distance, parking, fee, etc.

We note that travelers' anticipated travel time of each mode depends on their origins, destinations, and the selected path (subset of traffic or transit segments that connect from the origin to the destination). In most cases, travelers need to make the mode choice before they depart from the origin. Thus, travelers' anticipated travel time for a trip in time interval t is not necessarily the average travel time in t but instead is the time in a previous interval $t - j\delta$, where $\delta \geq 0$ is a unit non-negative time lag, and j depends on travelers' origins, destinations, and the delay of receiving travel time information.

Since we do not have individual-level data that records the origin, destination and selected path of travelers, we cannot estimate the utility of each mode for every individual. Instead, we consider a representative utility function that evaluates the average utility of each mode for all travelers. Since travelers' selected path may take any segment in the network, the representative utility depends on the travel time of all segments. Moreover, we assume that the maximum time lag for any traveler's decision of a trip in time interval t is $k\delta$. Then, we write the representative utility of driving (resp. transit) as $\phi_t^d\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$ (resp. $\phi_t^b\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$) plus a noise term with zero mean:

$$U_t^d = \phi_t^d\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right) + \zeta_t^d, \tag{E.1a}$$

$$U_t^b = \phi_t^b\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right) + \zeta_t^b, \tag{E.1b}$$

Given the representative utilities U_t^d and U_t^b, the probability of choosing the driving mode in the discrete choice model is given by:

$$\begin{aligned}\Pr(\text{Driving}) &= \frac{\exp\left\{U_t^d\right\}}{\exp\left\{U_t^d\right\} + \exp\left\{U_t^b\right\}} = \frac{1}{1 + \exp\left\{U_t^b - U_t^d\right\}} \\ &\overset{(\text{E.1})}{=} \frac{1}{1 + \exp\left\{\phi_t^b\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right) - \phi_t^d\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)\right\}},\end{aligned}$$

We take $\phi_t\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right) \triangleq \phi_t^b\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right) - \phi_t^d\left(x_{ti}, x_{(t-\delta)i}, \ldots, x_{(t-k\delta)i}\right)$. Therefore, as the number of travelers becomes large, the aggregate driving fraction evaluated by y_{ti} can be written as (7.2).

Bibliography

Daron Acemoglu and Asuman Ozdaglar. Competition and efficiency in congested markets. *Mathematics of Operations Research*, 32(1):1–31, 2007.

Daron Acemoglu, Munther A Dahleh, Ilan Lobel, and Asuman Ozdaglar. Bayesian learning in social networks. *The Review of Economic Studies*, 78(4):1201–1236, 2011.

Daron Acemoglu, Kostas Bimpikis, and Asuman Ozdaglar. Dynamics of information exchange in endogenous social networks. *Theoretical Economics*, 9(1):41–97, 2014.

Daron Acemoglu, Azarakhsh Malekian, and Asu Ozdaglar. Network security and contagion. *Journal of Economic Theory*, 166:536–585, 2016.

Daron Acemoglu, Ali Makhdoumi, Azarakhsh Malekian, and Asuman Ozdaglar. Fast and slow learning from reviews. Technical report, National Bureau of Economic Research, 2017.

David L Alderson, Gerald G Brown, W Matthew Carlyle, and R Kevin Wood. Solving defender-attacker-defender models for infrastructure defense. Technical report, Naval Postgraduate School Monterey CA Dept Of Operations Research, 2011.

David L Alderson, Gerald G Brown, W Matthew Carlyle, and R Kevin Wood. Assessing and improving the operational resilience of a large highway infrastructure system to worst-case losses. *Transportation Science*, 2017.

S Nageeb Ali. Herding with costly information. *Journal of Economic Theory*, 175:713–729, 2018.

Carlos Alós-Ferrer and Nick Netzer. The logit-response dynamics. *Games and Economic Behavior*, 68(2):413–427, 2010.

Tansu Alpcan and Tamer Baysar. *Network security: A decision and game-theoretic approach.* Cambridge University Press, 2010.

Saurabh Amin, Patrick Jaillet, and Manxi Wu. Efficient carpooling and toll pricing for autonomous transportation. *arXiv preprint arXiv:2102.09132*, 2021.

Robert Arnold, Vance C Smith, John Q Doan, Rodney N Barry, Jayme L Blakesley, Patrick T DeCorla-Souza, Mark F Muriello, Gummada N Murthy, Patty K Rubstello, Nick A Thompson, et al. Reducing congestion and funding transportation using road pricing in

Europe and Singapore. Technical report, United States. Federal Highway Administration, 2010.

Richard Arnott, Andre De Palma, and Robin Lindsey. Does providing information to drivers reduce traffic congestion? *Transportation Research Part A: General*, 25(5):309–318, 1991.

Dyana Bagby. Reporter analysis: Waze directions send traffic through quiet streets. *Reporter Newspapers*, September 2016.

Bianca Barragan. LA city council member wants to make Waze useless. *Curbed LA*, Apr 2015.

Tamer Basar and Geert Jan Olsder. *Dynamic noncooperative game theory*. SIAM, 1998.

Alan W Beggs. On the convergence of reinforcement learning. *Journal of Economic Theory*, 122(1):1–36, 2005.

Wolfgang W Bein, Peter Brucker, and Arie Tamir. Minimum cost flow algorithms for series-parallel networks. *Discrete Applied Mathematics*, 10(2):117–124, 1985.

MGH Bell, U Kanturska, J-D Schmöcker, and A Fonzone. Attacker–defender models and road network vulnerability. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 366(1872):1893–1906, 2008.

Moshe Ben-Akiva and Michel Bierlaire. Discrete choice methods and their applications to short term travel decisions. In *Handbook of Transportation Science*, pages 5–33. Springer, 1999.

Moshe Ben-Akiva, Andre de Palma, Kaysi Isam, and I. Kaysi. Dynamic network models and driver information systems. *Transportation Research Part A*, 25(5):251–266, 1991. ISSN 01912607.

Michel Benaim and Morris W Hirsch. Mixed equilibria and dynamical systems arising from fictitious play in perturbed games. *Games and Economic Behavior*, 29(1-2):36–72, 1999.

Dirk Bergemann and Stephen Morris. Bayes correlated equilibrium and the comparison of information structures in games. *Theoretical Economics*, 11(2):487–522, 2016.

Vicki Bier, Santiago Oliveros, and Larry Samuelson. Choosing what to protect: Strategic defensive allocation against an unknown attacker. *Journal of Public Economic Theory*, 9 (4):563–587, 2007.

Vicki M Bier and Kjell Hausken. Defending and attacking a network of two arcs subject to traffic congestion. *Reliability Engineering & System Safety*, 112:214–224, 2013.

Sushil Bikhchandani and John W Mamer. Competitive equilibrium in an exchange economy with indivisibilities. *Journal of Economic Theory*, 74(2):385–413, 1997.

David Blackwell. Equivalent comparisons of experiments. *The Annals of Mathematical Statistics*, pages 265–272, 1953.

Laura Bliss. Waze is worsening traffic in some LA neighborhoods: What should the city do? *CityLab*, Apr 2015.

Lawrence E Blume et al. The statistical mechanics of strategic interaction. *Games and Economic Behavior*, 5(3):387–424, 1993.

Lionel Bonaventure. US researchers expose Waze security lapse that allows hackers to track users. *i24NEWS*, Apr 2016.

Aron Brenner*, Manxi Wu*, and Saurabh Amin. Aggregate demand prediction in transportation networks. Working paper, 2021.

Gerald Brown, Matthew Carlyle, Javier Salmerón, and Kevin Wood. Defending critical infrastructure. *Interfaces*, 36(6):530–544, 2006.

Roberto Cominetti, Emerson Melo, and Sylvain Sorin. A payoff-based learning procedure and its application to traffic games. *Games and Economic Behavior*, 70(1):71–83, 2010.

José R Correa and Nicolás E Stier-Moses. Wardrop equilibria. *Wiley encyclopedia of operations research and management science*, 2011.

José R Correa, Andreas S Schulz, and Nicolás E Stier-Moses. Fast, fair, and efficient flows in networks. *Operations Research*, 55(2):215–225, 2007.

Mathieu Dahan and Saurabh Amin. Network flow routing under strategic link disruptions. In *Communication, Control, and Computing (Allerton), 2015 53rd Annual Allerton Conference on*, pages 353–360. IEEE, 2015.

G Dantzig and Delbert Ray Fulkerson. On the max flow min cut theorem of networks. *Linear Inequalities and Related Systems*, 38:225–231, 2003.

Sanmay Das, Emir Kamenica, and Renee Mirka. Reducing congestion through information design. In *2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1279–1284. IEEE, 2017.

Constantinos Daskalakis, Alan Deckelbaum, and Anthony Kim. Near-optimal no-regret algorithms for zero-sum games. In *Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 235–254. SIAM, 2011.

André de Palma and Robin Lindsey. Traffic congestion pricing methodologies and technologies. *Transportation Research Part C: Emerging Technologies*, 19(6):1377–1399, 2011.

Sven De Vries and Rakesh V Vohra. Combinatorial auctions: A survey. *INFORMS Journal on Computing*, 15(3):284–309, 2003.

Darrell Duffie, Semyon Malamud, and Gustavo Manso. Information percolation with equilibrium search dynamics. *Econometrica*, 77(5):1513–1574, 2009.

Marcin Dziubiński and Sanjeev Goyal. Network design and defence. *Games and Economic Behavior*, 79:30–43, 2013.

Marcin Dziubiński and Sanjeev Goyal. How do you defend a network? *Theoretical Economics*, 12(1):331–376, 2017.

David Easley and Nicholas M Kiefer. Controlling a stochastic process with unknown parameters. *Econometrica: Journal of the Econometric Society*, pages 1045–1064, 1988.

Anthony V Fiacco. Sensitivity and stability in NLP: Continuity and differential stability. *Encyclopedia of Optimization*, pages 3467–3471, 2009.

Anthony V Fiacco and Jerzy Kyparisis. Convexity and concavity properties of the optimal value function in parametric nonlinear programming. *Journal of Optimization Theory and Applications*, 48(1):95–126, 1986.

Lisa W. Foderaro. Navigation apps are turning quiet neighborhoods into traffic nightmares. *The New York Times*, Dec 2017.

Dean Foster and Hobart Peyton Young. Regret testing: Learning to play Nash equilibrium without knowing you have an opponent. *Theoretical Economics*, 1(3):341–367, 2006.

Jerome Friedman, Trevor Hastie, Robert Tibshirani, et al. *The Elements of Statistical Learning*, volume 1. Springer Series in Statistics New York, 2001.

Drew Fudenberg and David M Kreps. Learning mixed equilibria. *Games and Economic Behavior*, 5(3):320–367, 1993.

Drew Fudenberg and David K Levine. Self-confirming equilibrium. *Econometrica: Journal of the Econometric Society*, pages 523–545, 1993.

Drew Fudenberg, Fudenberg Drew, David K Levine, and David K Levine. *The Theory of Learning in Games*, volume 2. MIT Press, 1998.

Joseph Geha. Fremont: City takes steps to keep commuters on freeways and off its streets. *East Bay Times*, Nov 2016.

Branden Ghena, William Beyer, Allen Hillaker, Jonathan Pevarnek, and J Alex Halderman. Green lights forever: Analyzing the security of traffic infrastructure. *WOOT*, 14:7–7, 2014.

Faruk Gul and Ennio Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87(1):95–124, 1999.

Robert Hackett. Hackers are threatening to release 30GB of data from San Francisco 'Muni' hack. *Fortune*, Nov 2016.

Ross Haenfler. Social value of public information. *The American Economic Review*, 92(5): 1521–1534, 2002.

Frank H Hahn. Exercises in conjectural equilibria. In *Topics in Disequilibrium Economics*, pages 64–80. Springer, 1978.

Sergiu Hart and Andreu Mas-Colell. Regret-based continuous-time dynamics. *Games and Economic Behavior*, 45(2):375–394, 2003.

Andrew Hawkins. Uber and Lyft finally admit they're making traffic congestion worse in cities. *The Verge*, 2019.

Jack Hirshleifer. The private and social value of information and the reward to inventive activity. *The American Economic Review*, 61(4):561–574, 1971.

Josef Hofbauer and William H Sandholm. On the global convergence of stochastic fictitious play. *Econometrica*, 70(6):2265–2294, 2002.

Josef Hofbauer and William H Sandholm. Stable games and their dynamics. *Journal of Economic Theory*, 144(4):1665–1693, 2009.

Josef Hofbauer and Sylvain Sorin. Best response dynamics for continuous zero-sum games. *Discrete and Continuous Dynamical Systems Series B*, 6(1):215, 2006.

Ed Hopkins. Two competing models of how people learn in games. *Econometrica*, 70(6): 2141–2166, 2002.

Suzanne Jacobs. Traffic light study reveals serious hacking risk. *MIT Technology Review*, Sep 2014.

Ali Jadbabaie, Pooya Molavi, and Alireza Tahbaz-Salehi. Information heterogeneity and the speed of learning in social networks. Technical report, Columbia Business School Research Paper, 2013.

Li Jin, Mladen Cicic, Karl H Johansson, and Saurabh Amin. Analysis and design of vehicle platooning operations on mixed-traffic highways. *IEEE Transactions on Automatic Control*, 2020.

Ehud Kalai and Ehud Lehrer. Subjective equilibrium in repeated games. *Econometrica: Journal of the Econometric Society*, pages 1231–1240, 1993.

Ehud Kalai and Ehud Lehrer. Subjective games and equilibria. *Games and Economic Behavior*, 8(1):123–163, 1995.

Emir Kamenica. Bayesian persuasion and information design. *Annual Review of Economics*, 11, 2018.

Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.

Alexander S Kelso Jr and Vincent P Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica: Journal of the Econometric Society*, pages 1483–1504, 1982.

Anton Kolotilin, Tymofiy Mylovanov, Andriy Zapechelnyuk, and Ming Li. Persuasion of a privately informed receiver. *Econometrica*, 85(6):1949–1964, 2017.

Elias Koutsoupias and Christos Papadimitriou. Worst-case equilibria. In Christoph Meinel and Sophie Tison, editors, *STACS 99*, volume 1563 of *Lecture Notes in Computer Science*, pages 404–413. Springer Berlin Heidelberg, 1999.

Ehud Lehrer and Dinah Rosenberg. What restrictions do Bayesian games impose on the value of information? *Journal of Mathematical Economics*, 42(3):343–357, 2006.

Renato Paes Leme. Gross substitutability: An algorithmic survey. *Games and Economic Behavior*, 106:294–316, 2017.

Zhen Lian and Garrett van Ryzin. Autonomous vehicle market design. *Available at SSRN*, 2020.

Joan Lowy. Report: Automakers fail to fully protect against hacking. *AP News*, Feb 2015.

Jason R Marden and Jeff S Shamma. Revisiting log-linear learning: Asynchrony, completeness and payoff-based implementation. *Games and Economic Behavior*, 75(2):788–808, 2012.

Jason R Marden, Gürdal Arslan, and Jeff S Shamma. Regret based dynamics: convergence in weakly acyclic games. In *Proceedings of the 6th International Joint Conference on Autonomous Agents and Multiagent Systems*, page 42. ACM, 2007.

Jason R Marden, H Peyton Young, Gürdal Arslan, and Jeff S Shamma. Payoff-based dynamics for multiplayer weakly acyclic games. *SIAM Journal on Control and Optimization*, 48(1):373–396, 2009.

Laurent Mathevet, Jacopo Perego, and Ina Taneva. On information design in games. *Unpublished paper, Department of Economics, New York University.[1115]*, 2017.

Akihiko Matsui. Best response dynamics and socially stable strategies. *Journal of Economic Theory*, 57(2):343–362, 1992.

Emily Meigs, Francesca Parise, Asuman Ozdaglar, and Daron Acemoglu. Optimal dynamic information provision in traffic routing. *arXiv preprint arXiv:2001.03232*, 2020.

Katie Mettler. Somebody keeps hacking these dallas road signs with messages about Donald Trump, Bernie Sanders and Harambe the gorilla. *The Washington Post*, Jun 2016.

Igal Milchtaich. Network topology and the efficiency of equilibrium. *Games and Economic Behavior*, 2(57):321–346, 2006.

Paul Milgrom and John Roberts. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica: Journal of the Econometric Society*, pages 1255–1277, 1990.

Wendy W Moe and Peter S Fader. Dynamic conversion behavior at e-commerce sites. *Management Science*, 50(3):326–335, 2004.

Dov Monderer and Lloyd S Shapley. Fictitious play property for games with identical interests. *Journal of Economic Theory*, 68(1):258–265, 1996a.

Dov Monderer and Lloyd S Shapley. Potential games. *Games and Economic Behavior*, 14 (1):124–143, 1996b.

Abraham Neyman. The positive value of information. *Games and Economic Behavior*, 3(3): 350–355, 1991.

Evdokia Nikolova and Nicolás E Stier-Moses. A mean-risk model for the traffic assignment problem with stochastic travel times. *Operations Research*, 62(2):366–382, 2014.

Michael Ostrovsky and Michael Schwarz. Carpooling and the economics of self-driving cars. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 581–582, 2019.

Nancy Owana. Not fare: Hacker app resets subway card for free rides. *Phys.org - News and Articles on Science and Technology*, Sept 2012.

Jonathan Petit and Steven E Shladover. Potential cyberattacks on automated vehicles. *IEEE Transactions on Intelligent Transportation Systems*, 16(2):546–556, 2015.

Arthur C Pigou. The economics of welfare, 1920. *McMillan&Co., London*, 1932.

Robert Powell. Defending against terrorist attacks with limited resources. *American Political Science Review*, 101(3):527–541, 2007.

Hans Reijnierse, Anita van Gellekom, and Jos AM Potters. Verifying gross substitutability. *Economic Theory*, 20(4):767–776, 2002.

Jack Reilly, Sébastien Martin, Mathias Payer, et al. On cybersecurity of freeway control systems: Analysis of coordinated ramp metering attacks 2. In *Transportation Research Board 94th Annual Meeting*, 2015.

RT Rockafellar. Directional differentiability of the optimal value function in a nonlinear programming problem. In *Sensitivity, Stability and Parametric Analysis*, pages 213–226. Springer, 1984.

Linda Rosencrance. Hacker hits Toronto transit message system, jabs prime minister. *Computer World*, May 2006.

Robert W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2(1):65–67, dec 1973. ISSN 0020-7276.

Robert W Rosenthal. Correlated equilibria in some classes of two-person games. *International Journal of Game Theory*, 3(3):119–128, 1974.

Michael Rothschild. A two-armed bandit theory of market pricing. *Journal of Economic Theory*, 9(2):185–202, 1974.

Tim Roughgarden and Éva Tardos. Bounding the inefficiency of equilibria in nonatomic congestion games. *Games and Economic Behavior*, 47(2):389–403, 2004.

Larry Samuelson and Jianbo Zhang. Evolutionary stability in asymmetric games. *Journal of Economic Theory*, 57(2):363–391, 1992.

William H Sandholm. Potential games with continuous player sets. *Journal of Economic Theory*, 97(1):81–108, 2001.

William H Sandholm. Local stability under evolutionary game dynamics. *Theoretical Economics*, 5(1):27–50, 2010.

Marc Santora. Cuomo calls Manhattan traffic plan an idea 'whose time has come. *The New York Times*, 2017.

Lloyd Shapley. Some topics in two-person games. *Advances in Game Theory*, 52:1–29, 1964.

Auyon Siddiq and Terry Taylor. Ride-hailing platforms: Competition and autonomous vehicles. *Available at SSRN 3426988*, 2019.

Alexander Skabardonis, Pravin Varaiya, and Karl F Petty. Measuring recurrent and nonrecurrent traffic congestion. *Transportation Research Record*, 1856(1):118–124, 2003.

J Maynard Smith and George R Price. The logic of animal conflict. *Nature*, 246(5427): 15–18, 1973.

Hamidreza Tavafoghi and Demosthenis Teneketzis. Informational incentives for congestion games. In *2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1285–1292. IEEE, 2017.

Peter D Taylor and Leo B Jonker. Evolutionary stable strategies and game dynamics. *Mathematical Siosciences*, 40(1-2):145–156, 1978.

William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, 16(1):8–37, 1961.

Bernhard Von Stengel and Shmuel Zamir. Leadership with commitment to mixed strategies. Technical report, Citeseer, 2004.

Gerd Wachsmuth. On LICQ and the uniqueness of Lagrange multipliers. *Operations Research Letters*, 41(1):78–80, 2013.

Manxi Wu and Saurabh Amin. Securing infrastructure facilities: When does proactive defense help? *Dynamic Games and Applications*, pages 1–42, 2018.

Manxi Wu and Saurabh Amin. Information design for regulating traffic flows under uncertain network state. In *2019 57th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 671–678. IEEE, 2019a.

Manxi Wu and Saurabh Amin. Learning an unknown network state in routing games. *IFAC-PapersOnLine*, 52(20):345–350, 2019b.

Manxi Wu, Jeffrey Liu, and Saurabh Amin. Informational aspects in a class of Bayesian congestion games. In *2017 American Control Conference (ACC)*, pages 3650–3657. IEEE, 2017.

Manxi Wu, Saurabh Amin, and Asuman Ozdaglar. Bayesian learning with adaptive load allocation strategies. *2020 2nd Learning for Dynamics and Control (L4DC) Conference*, 2020a.

Manxi Wu, Saurabh Amin, and Asuman Ozdaglar. Multi-agent Bayesian learning with adaptive strategies: Convergence and stability. *arXiv preprint arXiv:2010.09128*, 2020b.

Manxi Wu, Saurabh Amin, and Asuman E Ozdaglar. Value of information in Bayesian routing games. *Operations Research*, 2020c.

Elleen Yu. Singapore readies satellite road toll system for 2021 rollout. *ZD mining*, 2020.

Kim Zetter. Hackers can mess with traffic lights to jam roads and reroute cars. *Wired*, Jun 2017.

Shanjiang Zhu, David Levinson, Henry X Liu, and Kathleen Harder. The traffic and behavioral effects of the I-35W Mississippi River Bridge collapse. *Transportation Research Part A: Policy and Practice*, 44(10):771–784, 2010.