Value of Heterogeneous Information in Stochastically Congestible Facilities

by

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Abstract

This thesis studies the effects of heterogeneous information on traffic equilibria and the resulting travel costs (both individual and social) when commuters make departure time choices to cross an unreliable bottleneck link. Increasing adoption and improved predictive abilities of Traveler Information Systems (TIS) enable commuters to plan their trips; however, there are inherent heterogeneities in information access and TIS accuracies, which can significantly affect commuters' choices and the equilibrium level of congestion. Our work addresses the open problem raised in Arnott et al. (1991) about the need to consider asymmetrically informed commuters in the bottleneck model of traffic congestion. We consider a Bayesian game with two heterogeneous commuter populations: one population is privately informed of the realized network state while the other only knows the public information about the distribution of states. We characterize the equilibrium of the game, in which each population chooses a departure rate function over time to minimize its expected cost based on its private belief about the state and the behavior of the other population. We provide a full equilibrium characterization for the complete range of values of link reliability, incident probability, and information penetration. This uncovers a rich structure of population strategies, which can broadly be categorized into two distinct regimes. Specifically, when information penetration is above a certain threshold, the populations' equilibrium strategies are non-unique, and the relative value of information (VoI) is 0, i.e. the two populations face the same cost. However, the aggregate departure rate function is unique and remains unchanged as more commuters gain access to information. On the other hand, when information penetration is below the threshold, equilibrium is unique, and VoI is positive and decreasing in information penetration. Importantly, we find that the lowest social cost is always achieved when a certain fraction of commuters are uninformed. The more unreliable the link, the higher the optimal information penetration that achieves this minimum. We define the Value of Heterogeneity (VoH) as the difference between the optimal social cost and the cost under complete information penetration, and find that it is significant (upto 20%) under practically relevant conditions.

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Chapter 1

Introduction

Over the last decade, there have been significant advancements in the technology and reach of Traveler Information Systems (TIS). In particular, smartphone navigation apps based on GPS and crowd-sourcing have gained widespread adoption among commuters. These services provide commuters with information about network conditions, including uncertain network state and likely congestion. Commuters may use different TISs due to marketing, costs, availability, etc., such as Google Maps/Waze Apple Maps, etc. The information provided by different TISs is generally not identical due to technological differences in traffic data collection and prediction between various providers. Furthermore, traditional means of traffic information such as traffic radio also continue to be used by some commuters. On the other hand, some commuters may not have access to any means of gaining information about traffic conditions, or may choose not to use such means. All these factors contribute to an inherently heterogeneous information structure among the population of commuters, with some having more accurate information about traffic conditions than others. Thus commuters maintain private beliefs about traffic conditions such as network state, demand, etc. Furthermore, commuters may include predictions of other commuters' decisions, and in particular other commuters' information, in their own decision making. From a practical viewpoint, if a commuter is aware that a significant number of other commuters have access to information about network state, she would be likely to consider how they will react based on their information, and make her decisions accordingly. Thus, to study the effect of heterogeneity of information, it is important to model commuters' beliefs about other commuters in addition to their own beliefs about traffic conditions.

The information structure (penetration, accuracy, timing of provision, etc.) has a significant effect on commuters' decisions and therefore their costs of commuting. Results obtained under symmetric information settings are not directly applicable to practical scenarios, since there is significant information heterogeneity between commuters in practice. Despite this, heterogeneous information structures have received little attention in the literature on traffic congestion. In this thesis, we address this gap in the literature and explore the effects of heterogeneous information. We aim to answer the following questions:

1. How does increasing penetration of accurate bottleneck state information affect commuters' time-of-travel decisions and costs?

2. In particular, what level of information penetration is socially optimal in minimizing total travel costs?

1.1 Our Contribution

To address the questions mentioned above, we consider a Bayesian game model which incorporates heterogeneous beliefs about both link state as well as other commuters' beliefs. The game is played on a single incident-prone bottleneck link which connects an origin node to a destination node. The state of the link determines its capacity. The link faces a fixed demand comprised of non-atomic commuters with identical preferences. Commuters have a single preferred arrival time at the destination, and incur a cost for arriving early or late, as well as a cost for time spent queuing at the bottleneck. Commuters choose their departure times to minimize the total travel costs they face. Each commuter is subscribed to one of two TISs; each TIS sends a noisy signal of the state to its subscribers. In the case of a single bottleneck which we consider, all commuters subscribed to a TIS receive an identical signal; thus, they can naturally be modeled as one population. This game is a contribution from a modeling perspective as the first attempt to incorporate an asymmetric information structure with Vickrey (1969)'s seminal bottleneck model of traffic congestion.

In equilibrium, each population chooses its departure rate function such that all its members face the minimum expected cost based on their beliefs. We characterize the equilibrium strategies (i.e. departure rate functions) under certain assumptions on the information structure. Firstly, we assume that, for a given state, the signals reported by the two TISs are independent. This is likely to be the case if the TISs make their predictions independently without sharing data or predictive techniques. Secondly, we assume that the accuracies of each TIS are common knowledge. Under this assumption, the populations' beliefs are derived from a common prior, which is the joint distribution of the state and signals received by the populations. Specifically, given its received signal, each population uses the common prior to update its belief about the state and the signals received by other populations via Bayes' rule. Finally, we assume that a one population is perfectly informed about the state while the other is not informed. This assumption may seem restrictive, but, as mentioned before, some commuters may not use TISs at all, making them in effect an "uninformed population". Similarly, those commuters who do use a TIS may have access to near-perfect information due to the exceedingly high accuracies of modern TISs. We make these assumptions because the equilibrium structure is already rich even under this specific case, and our analysis provides insights on the effect the asymmetric information structure has on equilibrium strategies and costs relative to symmetric information cases.

We now mention our main results on equilibrium characterization. We characterize the equilibrium strategies for the complete range of values of link reliability, incident probability, and information penetration. Link reliability refers to the extent of capacity loss in case of an incident, while information penetration refers to the fraction of commuters with access to information about the state. We first provide the equilibrium strategies for the boundary cases of the game in which one or more of the parameters takes an extreme value, reducing it to a symmetric information setting.

Two of our boundary cases recover Vickrey's original deterministic bottleneck model, while the third, where all commuters are uninformed, recovers an example considered but not exhaustively analyzed in Arnott et al. (1988). These boundary cases provide useful insights for solving the equilibrium for the general case of asymmetric information.

For the general case, we first show that all equilibria must satisfy certain necessary conditions , which help narrow the search for equilibrium strategies. We show that equilibrium strategies can be broadly divided into two qualitatively different regimes depending on whether the information penetration is above or below a certain threshold. When information penetration is above the threshold, the populations' equilibrium strategies are non-unique, but the aggregate equilibrium departure rate function is unique and remains unchanged as more commuters gain access to information. We call this regime R0. On the other hand, when information penetration is below the threshold, we show that the populations' strategies are unique and sensitive to changes in the informed fraction. We call this regime R0'. Both R0 and R0' can be refined further in terms of the specific qualitative features of the equilibrium strategies. In particular, deriving the equilibrium strategies in R0' is rather involved, and requires us to exploit several additional necessary conditions that equilibria in R0' must satisfy. After listing these conditions, we then identify each of the four qualitative aspects in which equilibria in R0' can be distinguished. These include, among others, the number of intervals of queuing in the non-incident state and whether or not some informed players depart late on in the incident state. We then use these results to find the equilibrium departure rate functions for each population, and finally complete the derivation of the equilibrium strategies by setting up a system of linear equations which gives a unique solution.

The equilibrium characterization allows us to analyze the travel costs faced by the commuters in equilibrium. We specifically consider the effect of increasing information penetration on both the individual costs faced by commuters in each population as well as the social cost. To study the individual costs, we define the value of information (VoI) as the difference between the equilibrium costs of the two populations.

We mention the three main properties of VoI below. Firstly, in our model, VoI is non-negative, i.e. informed commuters never face higher costs than uninformed commuters. This indicates that gaining perfectly accurate information about the realized state is never detrimental. Secondly, VoI is largest when few other commuters are informed, and decreases with increasing information penetration. This is because of two opposite effects: 1) as more commuters gain information and make better timing decisions, they face an increasing congestion externality that increases their individual cost, and 2) information has a positive externality on uninformed commuters for a broad range of parameter values (but not all); they benefit from the better decision making of the growing fraction of informed commuters even though they do not have access to information themselves. Thirdly, VoI is 0 beyond the aforementioned threshold of information penetration (i.e. in R0). This reflects information saturation: information penetration has reached such an extent that its complete effect is seen by all commuters, even those who are uninformed.

In addition to the individual value of information described above, we also examine the social value. In particular, we analyze how the average (social) cost faced by all commuters is affected by changes in information penetration. As some commuters gain information, the social cost decreases reflecting the benefits of their better decision making. However, after a certain fraction of commuters are informed, informing further commuters can lead to an increase in the average cost, i.e. the lowest average cost is achieved when the fraction of informed commuters is less than 1. This is a counterintuitive and crucial result. It indicates that even when the information provided is perfectly accurate, providing it to a certain fraction of commuters is socially preferable to providing it to all commuters. In fact, we find that the average cost can be significantly lower (by upto 20% under practically relevant conditions) when the optimal fraction of commuters are informed as opposed to when all commuters are informed. We call this the Value of Heterogeneity (VoH). Finally, we study how the optimal information penetration varies with the link reliability. We find that the more reliable the link is, the lower the optimal fraction of informed commuters becomes. Our results on welfare are distinct from the analysis of the bottleneck model under symmetric information, and involve careful analysis of how private beliefs influence the equilibrium strategies of the population.

1.2 Related Work

We now present an overview of the literature related to our work. Our contribution is situated at the intersection of two streams of literature on traffic congestion: the first is the bottleneck model of congestion whereas the second is the effect of heterogeneous information on commuters' equilibrium behavior and costs.

The classical bottleneck model of traffic congestion, proposed by Vickrey (1969), considers a fixed number of identical commuters who must cross a bottleneck link of fixed capacity. Commuters wish to arrive at the destination at the same time, and face a trade-off between departing and thus arriving inconveniently early (or late) or traveling at the peak time and facing a long queue. In addition to solving for the user equilibrium, Vickrey also solved for the socially optimal strategy and determined the time-varying toll which achieves it. The model has been extended in various directions, including elastic demand, stochastic capacity and multiple routes, among others. Small (2015) provides a recent review of the many extensions. The authors R. Arnott, A. de Palma and R. Lindsey have written several papers on extensions of the bottleneck model. Arnott et al. (1990) examines the equilibrium under a coarse (step-wise) toll and solves for optimal capacity under various tolling schemes. Arnott et al. (1993) consider elastic demand and solves for optimal capacity. Arnott et al. (1994) consider commuters who differ in their valuation of early (late) arrival cost and queuing cost. Arnott et al. (1988, 1991, 1999) introduce stochastic capacity to the bottleneck model. They derive the equilibrium strategy when both demand and capacity can vary, and commuters have (identical) noisy information about the capacity. The former two papers consider a network of two routes in parallel, while the latter considers a single route under more general assumptions on the cost functions. They find that improving the accuracy of information provided to commuters can exacerbate congestion and drive up costs unless the provided information is perfectly accurate. Our results are analogous to these observations in that increasing the fraction of informed commuters can exacerbate congestion. However, in our model, gaining access to perfect information never increases the cost for those who gain it. Finally, Zhang et al. (2010) extend the of idea stochastic capacity further by allowing for the capacity to change over time within a single instance of the commute.

One common assumption in the above-mentioned literature on the bottleneck model is that all commuters have access to identical information about the state of the link(s)[1]. This is partially because heterogeneous information is difficult to model; it creates a rich information structure which makes the derivation of the equilibrium strategies greatly involved. However, as described earlier, modeling heterogeneous information is important since commuters clearly have varying information in practice. Indeed, in their concluding remarks, authors have cited the homogeneity of information available to commuters as an assumption that should be relaxed in future work on the bottleneck model (Arnott et al. (1991)). Our work provides this relaxation by assuming that a fraction of commuters have access to information about the bottleneck state while the remaining commuters do not.

The other area of research this thesis contributes to is studying the effect of asymmetric information structures on traffic congestion. Mahmassani and Jayakrishnan (1991) use a dynamic simulation to identify several effects of information on traffic congestion. From their results, we would like to emphasize three particular effects: (i) there exists an optimal fraction (less than 1) of commuters with information that results in the minimum social cost, (ii) the cost for informed commuters increases as the information penetration increases, and (iii) even uninformed commuters enjoy a reduction in cost when others have access to information. Our equilibrium results are similar to these: we find that the first two effects exist universally while the third also exists for a broad range of parameter values. However, we find that under certain conditions, uninformed commuters may be worse of due to others having access to a TIS, an effect that is also reported by Levinson (2003) in his simulations. Similarly

[1] Arnott et al. (1991) do consider a single commuter with access to private information, but under the assumption of non-atomic commuters, this does not change the information structure.

to us, some recent papers use modeling rather than simulation to study the effects of heterogeneous information. Acemoglu et al. (2016) consider different commuters having knowledge about the existence of different routes in a network congestion game. The authors show that an "Informational Braess' Paradox" can occur, characterized by commuters having knowledge about additional routes being worse off than those who do not. Our model instead adopts a Bayesian framework to study the effects of asymmetric information. A similar information structure to ours is considered in Wu et al. (2017), who formulate a Bayesian congestion game where route costs are affected by a random network state. The authors characterize the equilibria of their game under both objective beliefs, which admit a common prior, and subjective beliefs, which do not. In comparison, our model assumes an objective belief structure. Their results are similar to ours in that they find that information heterogeneity is always socially beneficial. However, under their model, unlike ours, commuters with access to more accurate information may be worse off than others.

The above mentioned equilibrium analyses which incorporate heterogeneous information do not include departure time as a choice variable. This is a limitation as it ignores the temporal dynamics of congestion, reducing the model to a static one. To the best of our knowledge, our work is the first analytical study to incorporate both heterogeneous information and departure time choice into the same model of traffic congestion. Since our focus is not on route choice, we consider a single pair of origin and destination nodes connected by a single link. We hope that this serves as the basis for a more complete study of joint departure time and route choices under an asymmetric information structure.

Our work also draws from, and be considered part of, the wider literature on the effects of TISs (not specifically asymmetric information) on traffic congestion. Ben-Akiva et al. (1996) consider the merits of predictive vs instantaneous information. They find that using predictive information, as is the case in our model, rather than instantaneous information, reduces travel time only slightly. Ben-Akiva et al. (1991) propose a dynamic model to study the value of TISs, and note the importance of the fraction of informed commuters as a policy variable. Khattak et al. (1996) use a survey to explore how commuters react to information provided by TISs.

Finally, the value of information in a broader context (not applying specifically to traffic congestion) has also been studied extensively in the game theoretic literature. Several papers find that improving the quality of the information provided to one player can decrease the welfare of either individual players or society, or both, in equilibrium (see Hirshleifer (1971) and Haenfler (2002)). However, other studies find that under certain constraints on information structures, information is always welfare improving (see Neyman (1991) and Gossner and Mertens (2001)). Our results are along the lines of the latter group of studies. However, in contrast to the majority of the literature, which considers the value of information to be the benefit gained by unilaterally improving the accuracy of the information available to one population, we define the value of information VoI as the relative difference in the costs of the informed and uninformed commuters.

In summary, our work extends the literature on the bottleneck model of traffic congestion to include heterogeneously informed commuters. We provide a game theoretical model that can capture many of the experimental effects observed in Mahmassani and Jayakrishnan (1991). By considering travel time as the decision variable, our model complements recent work by Acemoglu et al. (2016) and Wu et al. (2017) who consider the effects of heterogeneous information in congestion games with route choice decisions.

This thesis is structured as follows. In Chapter 2, we introduce our Bayesian game model and define its' equilibrium concept. In Chapter 3, we characterize the equilibrium structure for certain boundary cases. These provide background for understanding the equilibrium structure of the general case, which we characterize in Chapter 4. In Chapter 5, we analyze the equilibrium costs and value of information. We conclude with a discussion of implications and future work in Chapter 6.

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Chapter 2

Model

Our model comprises of a bottleneck link which can either be in a nominal or incident state. There are two Traffic Information Services (TISs) which predict the state of the link with different levels of accuracy. This induces information heterogeneity in a fixed population of commuters. In Section 2.1, we describe the basic model followed by the information structure in Section 2.2. Next, we introduce our Bayesian game in Section 2.3 before finally defining the equilibrium concept in Section 2.4.

2.1 Bottleneck Model

We adopt the basic bottleneck model under unreliable capacity from Arnott et al. (1991). The model consists of an origin-destination pair connected by a single incident prone link. This link faces a fixed demand D comprised of risk-neutral non-atomic commuters. All commuters would prefer to reach the destination at a fixed time t^*. The demand is sufficiently large so that the commuters can not all cross simultaneously, making the link a *bottleneck* and causing a queue to form. Since the demand is fixed, each commuter's only decision is her *departure time* $t \in \mathbb{R}$, which is the time at which she departs from the origin (i.e., reaches the bottleneck link and join the queue, if any). Correspondingly the time at which a commuter crosses the bottleneck and arrives at the destination is referred to as her *arrival time*. The time interval of interest in our analysis is the interval in which all commuters cross the bottleneck, i.e. the interval between the earliest departure time and the latest arrival time. We refer to this interval as the *rush hour*.

The state of the link, denoted s, is is random and drawn from the set $S \triangleq \{n, a\}$, where n and a represent the nominal and abnormal (incident) states, respectively.[1] The state s is chosen from S by a fictitious player *Nature* according to an exogenous and fixed prior distribution θ. The prior distribution over S is given by $\theta(a) = p$, $\theta(n) = 1 - p$, where p denotes the probability of an incident. Once realized, the state is thereafter fixed throughout the game.

We define the *capacity* of the link as the maximum number of commuters that can traverse it per unit time. The capacity in a given state s is denoted c_s. We make two standard assumptions on the state-dependent capacities. First, we assume that $c_n \geq c_a$, representing the fact that the incident may reduce the capacity, but does not improve it. Second, we assume that $c_a > 0$, since assuming otherwise would imply that the commuters would not be able to traverse the bottleneck at all in the incident state. We define the ratio of the two capacities as $\rho \triangleq \frac{c_a}{c_n} \in (0, 1]$. Thus ρ is a measure of the reliability of the link; the higher the value of ρ, the smaller the capacity reduction due to an incident, and hence the more reliable the link is.

Each commuter's total individual travel cost is the sum of the disutility of time spent queuing (referred to as the *queuing cost*) and the disutility of arriving earlier or later than the preferred arrival time t^* (referred to as the *scheduling cost*). The cost of queuing is α per unit time, and the cost of arriving early (resp. late) is β (resp. γ) per unit time. These cost parameters are homogeneous across all commuters because of our assumption of identical preferences. We follow the standard assumptions that $\alpha > \beta$ and $\gamma > \beta$ (see Small (2015)). Also, without loss of generality, we set the the free-flow travel time on the bottleneck link to 0.

[1]This model can be extended to a setting where the set of states is of higher cardinality; however, we limit our attention to the case of a binary-valued state for the sake of simplicity.

2.2 Information Structure

We now introduce our information structure as determined by a set of two TISs that predict the link state with different accuracies. In many practical situations, TISs can be considered as independent sources of information about the underlying state, i.e., their technological processes (e.g., data collection and aggregation techniques) for predicting the state are distinct. This leads to differences in their accuracies. We assume that each commuter is subscribed exclusively to one TIS $i \in I \triangleq \{H, L\}$. We use H and L to denote *High accuracy* and *Low accuracy* TIS respectively, and refer to the respective commuter populations as *population* H and *population* L. The fraction of commuters in population H and L are respectively denoted as λ^H and λ^L, where $\lambda^H = \lambda$ and $\lambda^L = 1 - \lambda$. These TISs introduce information heterogeneity between the two populations due to the difference in their accuracies.

In our model, each TIS i sends a signal τ^i, which is its prediction of the state s, to its subscribed population. This signal is drawn from the set $\mathbf{\tau}^i \triangleq \{in, ia\}$, where in (resp. ia) represents the TIS i predicting that the state is n (resp. a). Thus Hn represents a signal sent by TIS H to its subscribed population H that it predicts the state is n, and similarly for Ha, Ln and La. We represent the *accuracy* of TIS i by a parameter $\eta^i \in [0.5, 1]$, which is the probability that it predicts the state correctly. For simplicity, we assume that the accuracy of a TIS's prediction does not change with the state, i.e., prediction accuracy of each TIS is identical across states. Since we consider that TIS H is more accurate than TIS L, the accuracy parameters satisfy $\eta^H > \eta^L$. For each TIS, the conditional probability of sending its subscribed population signal τ^i given that the state s can be written as:

$$\Pr(\tau^i = is|s) = \eta^i, \quad \Pr(\tau^i = is'|s) = 1 - \eta^i, \forall s \in S, \quad \forall i \in I, \qquad (2.1)$$

where s' denotes the complement of state s.

Our first assumption about the information structure is that the signals reported by the two TISs are independent of each other conditional on the state, i.e.:

Assumption 1. $Pr(\tau^H, \tau^L|s) = Pr(\tau^H|s)Pr(\tau^L|s), \quad \forall \tau^H \in \mathfrak{r}^H, \forall \tau^L \in \mathfrak{r}^L, \forall s \in S.$

That is, given the state, a population's knowledge of its own signal does not give it further information about the other population's signal. As mentioned earlier, this assumption is reasonable in situations where the TISs are independent entities who do not share data or predictive techniques.

Under Assumption 1, the joint distribution of the state s and the TIS signals τ^H,τ^L denoted $\pi(s, \tau^H, \tau^L)$, can be written as:

$$\pi(s, \tau^H, \tau^L) = \theta(s)\Pr(\tau^H|s)\Pr(\tau^L|s), \quad \forall \tau^H \in \mathfrak{r}^H, \forall \tau^L \in \mathfrak{r}^L, \forall s \in S. \tag{2.2}$$

We emphasize that information of each commuter population is incomplete, i.e. the populations do not know the realization of the state s, and the signal τ^i received by each population from its TIS is its *private information* about the state. After receiving its signal, in our model, the populations do not receive any further update on the state throughout the game.

Our second assumption is that the accuracies of the two TISs are common knowledge:

Assumption 2. *η^H and η^L are common knowledge.*[2]

Readers who wish to familiarize themselves with the notions of common knowledge and private information as they apply to games of incomplete information can refer to Fudenberg and Tirole (1991) or Osborne and Rubinstein (1994).

In addition to the TIS accuracies η^H and η^L, we follow the standard assumptions that the demand D, the set of states S, the prior distribution θ, the preferred arrival time t^*, the cost parameters α, β and γ, the capacities c_n and c_a (and hence their ratio ρ) and the information penetration fraction λ are all also common knowledge. Even under these assumptions, we are extending the previous settings of incomplete information in Arnott et al. (1991) (where they consider symmetric and imperfect information) to the case of asymmetric information.

[2]To say that x is common knowledge means that all parties know x, they all know that they know x, they all know that they all know that they know x, and so on ad infinitum.

2.3 Bayesian Game

In Bayesian games, the notion of *type* captures all the private information available to each population. Recall that, in our model, the private information of each population is the signal τ^i it receives from its TIS. Therefore we can view the type of population i as τ^i and the type space as $\boldsymbol{\tau}^i$. Thus the type sets for each population are $\boldsymbol{\tau}^H = \{Hn, Ha\}$ and $\boldsymbol{\tau}^L = \{Ln, La\}$ respectively. We define a generic type profile as $\tau \triangleq (\tau^H, \tau^L)$, and denote the set of type profiles as $\boldsymbol{\tau} \triangleq \boldsymbol{\tau}^H \times \boldsymbol{\tau}^L$.

Based on its type τ^i, each population generates an *interim* belief about the state and the other population's type, denoted $\mu^i(s, \tau^{-i}|\tau^i) \in \Delta(S \times \boldsymbol{\tau}^{-i})$. This belief is interim because the commuters will eventually learn the realized state once they traverse the bottleneck. Using this belief, each population then chooses a departure rate function $r^{\tau^i} : \mathbb{R} \to \mathbb{R}$, which is a mapping from continuous time to an instantaneous rate of departure at that time.[3] For a given time instant $t \in \mathbb{R}$, the departure rate $r^{\tau^i}(t)$ denotes the number of commuters of population i departing per unit time from the origin (when the signal they received is τ^i). Note that this departure rate is an aggregate quantity, i.e., it is a result of individual choices made by population i's (non-atomic) commuters. We denote the space of all such departure rate functions as $\mathcal{F}$. A *strategy* of population i, denoted σ^i, is a map from its type space $\boldsymbol{\tau}^i$ to $\mathcal{F}$. Thus $\sigma^i(\tau^i) = r^{\tau^i}$ means that, at any time $tinr$, population i's strategy is to depart from the origin according to the departure rate function $r^{\tau^i}(t)$. Following standard notation, we denote the support of a function $r : \mathbb{R} \to \mathbb{R}$, i.e., the subset of the domain where r is non-zero as follows:

$$\text{supp}(r) = \{t \in \mathbb{R} | r(t) \neq 0\}. \tag{2.3}$$

We define a strategy profile as a tuple $\sigma \triangleq (\sigma^H, \sigma^L)$ and say that a strategy profile

[3]The frame of reference (i.e. "zero" time) is irrelevant. Recall that we have set $t^* = 0$ without loss of generality.

is feasible if and only if it satisfies the following conditions:

$$\int_{\mathbb{R}} r^{\tau^i}(t) \ \mathrm{d}t = \lambda^i D, \qquad \forall \tau^i \in \tau^i, \forall i \in I, \tag{2.4}$$

$$r^{\tau^i}(t) \geq 0, \qquad \forall t \in \mathbb{R}, \forall \tau^i \in \tau^i, \forall i \in I. \tag{2.5}$$

The first condition ensures that the demand of each population is satisfied, while the second ensures that departure rate is always nonnegative. Let Σ^i denote the set of all feasible strategies for population i. The set of feasible strategy profiles is denoted $\Sigma \triangleq \Sigma^H \times \Sigma^L$.

We now introduce the following quantities for a given strategy profile $\sigma = (\sigma^H, \sigma^L) \in \Sigma$:

- The total departure rate induced by both populations as a function of time can be written as:

$$r^{\sigma(\tau)}(t) \triangleq r^{\tau^H}(t) + r^{\tau^L}(t) \quad \forall t \in \mathbb{R}, \forall \tau \in \tau, \tag{2.6}$$

  where r^{τ^H} and r^{τ^L} are the departure rates of population H and L respectively.

- The queuing time faced by a commuter who departs at time t in state s is given by:

$$q_s^{\sigma(\tau)}(t) = \int_{T_s(t)}^{t} \frac{r^{\sigma(\tau)}(z) - c_s}{c_s} \ \mathrm{d}z, \quad \forall s \in S, \forall t \in \mathbb{R}, \tag{2.7}$$

  where $T_s(t)$ is the most recent time before t when there was no queue in state s. Note that the evolutions of $T_s(t)$ and $q_s^{\sigma(\tau)}(t)$ with respect to time are related and can be understood by considering the boundary conditions $q_s^{\sigma(\tau)}(t_0) = 0$, $T_s(t_{0,s}) = t_{0,s}$, where $t_{0,s}$ is the departure time of the first commuter in state s.

- The total travel cost faced by a commuter who departs at time t in state s, denoted the *state-dependent* cost, is given by:

$$\begin{aligned} C_s^{\sigma(\tau)}(t) = \alpha q_s(t) + \beta \max(t^* - t - q_s(t), 0) \\ + \gamma \max(t + q_s(t) - t^*, 0), \quad \forall t \in \mathbb{R}, \forall s \in S. \end{aligned} \tag{2.8}$$

This cost is a sum of the queuing cost faced by the commuter (if any) and her scheduling cost, which is the cost of arriving before or after the preferred time t^*.

Henceforth, we suppress the dependence of these quantities on σ for ease of presentation.

We are now ready to define our game:

Definition 1.

$$\Gamma \triangleq (I, S, \boldsymbol{\tau}, \Sigma, \mathcal{C}, \mu),$$

where:

$I = \{H, L\}$ is the set of commuter populations

$S = \{n, a\}$ is the set of nature states

$\boldsymbol{\tau} = (\boldsymbol{\tau}^i)_{i\in I}$ is the set of type profiles

$\Sigma = (\Sigma^i)_{i\in I}$ is the set of feasible strategy profiles

$\mathcal{C} = \{C_s\}_{s\in S}$ is the set of state-dependent cost functions as defined in (2.8)

$\mu = (\mu^i)_{i\in I}$ is the set of beliefs of each population about the state of nature and the other population's type, conditioned on its own type.

The setting of the game Γ is summarized in Fig. 2-1.

2.4 Equilibrium Concept

In our model, the joint distribution $\pi(s, \tau^H, \tau^L)$ defined in (2.2) is the common prior of the game, since under Assumption 2 it can be computed by both populations. The players derive their interim beliefs μ^i from the common prior, and use these beliefs to calculate their expected costs of playing a strategy. This in turn leads to the selection of a strategy profile.

Specifically, the interim beliefs are derived from the common prior as follows:

$$\mu^i(s, \tau^{-i}|\tau^i) = \frac{\pi(s, \tau^H, \tau^L)}{\Pr(\tau^i)}, \quad \forall s \in S, \forall \tau \in \boldsymbol{\tau}, \forall i \in I, \tag{2.9}$$

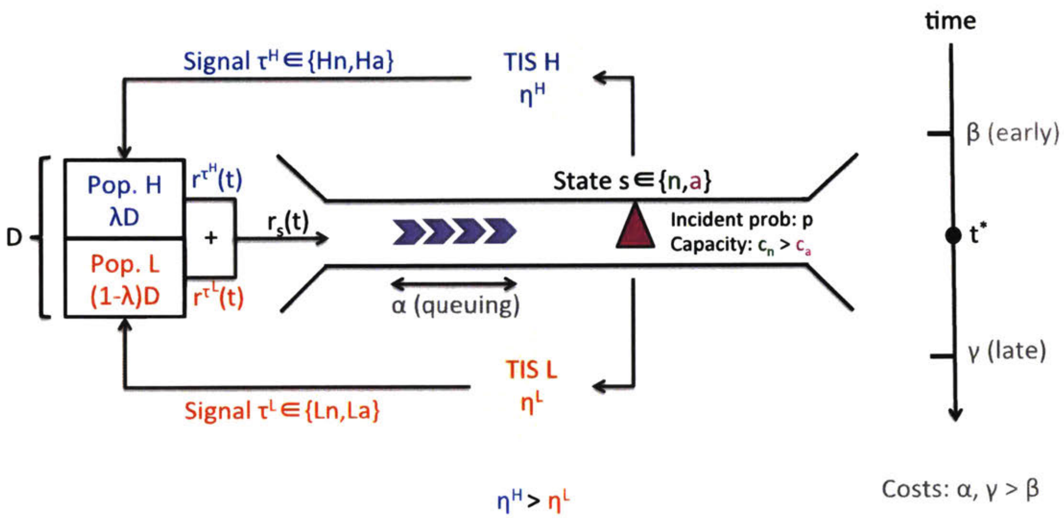


Figure 2-1: Bottleneck model with two asymmetrically informed commuter populations.

where the marginal distribution $\Pr(\tau^i)$ is given by:

$$\Pr(\tau^i) = \sum_{s \in S} \theta(s) \Pr(\tau^i | s), \quad \forall \tau^i \in \boldsymbol{\tau}^i, \forall i \in I. \tag{2.10}$$

Substituting (2.2) and (2.10) into (2.9), the interim belief of each type τ^i can be written as:

$$\mu^i(s, \tau^{-i} | \tau^i) = \frac{\theta(s)\Pr(\tau^i|s)\Pr(\tau^{-i}|s)}{\sum_{s \in S} \theta(s)\Pr(\tau^i|s)}, \quad \forall s \in S, \forall \tau^i \in \boldsymbol{\tau}^i, \forall i \in I, \tag{2.11}$$

where $\Pr(\tau^i|s)$ and $\Pr(\tau^{-i}|s)$ are the TIS accuracies defined in (2.1). Under the information structure of our game, the players of type Ln believe that the probability

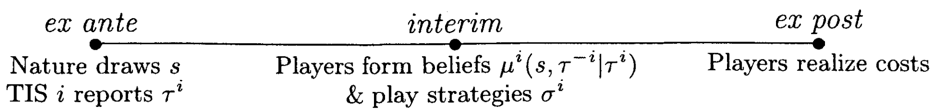


Figure 2-2: Timing of the game

of the state being n and the type of population H being Ha is:

$$\begin{aligned}\mu^L(n, Ha|Ln) &= \frac{\theta(n)\Pr(Ln|n)\Pr(Ha|n)}{\sum_{s\in S}\theta(s)\Pr(Ln|s)}\\ &= \frac{(1-p)\eta^L(1-\eta^H)}{(1-p)\eta^L + p(1-\eta^L)}.\end{aligned}$$

The beliefs of types La, Hn and Ha can be computed similarly.

Also note that population i's posterior belief about the state s conditional on its type τ^i can be derived from the belief μ^i as follows:

$$\Pr(s|\tau^i) = \frac{\mu^i(s, \tau^{-i}|\tau^i)}{\Pr(\tau^{-i}|s)}, \tag{2.12}$$

Given a type τ^i's interim beliefs $\mu^i(s, \tau^{-i}|\tau^i)$, the expected (ex ante) cost of playing strategy σ^i can be defined as a function of time as follows:

$$\mathbb{E}[C^{\tau^i}(t)] \triangleq \sum_{s\in S}\sum_{\tau^{-i}\in\boldsymbol{\tau}^{-i}} C_s(t)\mu^i(s, \tau^{-i}|\tau^i), \quad \forall t \in \mathbb{R}, \forall \tau^i \in \boldsymbol{\tau}^i, \forall i \in I, \tag{2.13}$$

where C_s is the state-dependent cost function from (2.8).

The game is played as shown in Fig. 2-2. In the *ex ante* stage, Nature draws a realization of the state $s \in S$ according to the prior distribution θ. Each TIS i sends a signal τ^i to its subscribed population according to the conditional probability distribution given in (2.1). The received signal determines the population i's type τ^i. In the *interim* stage, each population forms an interim belief $\mu^i(s, \tau^{-i}|\tau^i)$, and both populations simultaneously choose a strategy σ^i based on the expected cost function $\mathbb{E}[C^{\tau^i}]$ of playing that strategy.[4] The populations play their respective strategies in the *interim* stage and realize their costs in the *ex post* stage.

[4]Alternatively, the interim stage an be viewed as a game in which the players are indexed by population types, i.e., the types becomes the players.

We now define an equilibrium of the game Γ:

Definition 2. A feasible strategy profile $\sigma^* = (\sigma^{H^*}, \sigma^{L^*}) \in \Sigma$, where $\sigma^{i^*}(\tau^i) = r^{\tau^{i^*}}$, is an equilibrium of the game Γ if $\forall \tau^i \in \mathfrak{T}^i, \forall i \in I$:

$$\mathbb{E}[C^{\tau^i}(t)] \leq \mathbb{E}[C^{\tau^i}(t')], \quad \forall t \in \text{supp}(r^{\tau^{i^*}}), \forall t' \in \mathbb{R}. \tag{2.14}$$

That is, in equilibrium, each population i chooses a strategy σ^{i^*}, which entails choosing a departure rate function $r^{\tau^{i^*}}$ for each realized type τ^i. Then, for each type τ^i, given the other populations' strategies, type τ^i's expected cost $\mathbb{E}[C^{\tau^i}]$ is equal over the support of $r^{\tau^{i^*}}$, and no greater than the expected cost at any point outside the support. In other words, in any equilibrium, all players of a given type face equal expected travel costs, and would not be able to decrease their costs if they unilaterally changed their departure rate at any time. Thus, no population has an incentive to deviate from σ^{i^*} when the other populations choose their equilibrium departure rates.

For simplicity, we restrict our subsequent analysis to the case where TIS H predicts the state perfectly while TIS L has no predictive ability. Thus, our third assumption on the information structure of the game is the following:

Assumption 3. *$\eta^H = 1$ and $\eta^L = 0.5$.*

With $\eta^H = 1$, population H always knows the true state. This implies that when the state is s, the only accessible type for population H is Hs. Under Assumptions 2 and 3, population H has complete information about all aspects of the game.

When $\eta^L = 0.5$, population L has no additional information about the state beyond the common knowledge. The beliefs of both types Ln and La are equivalent to the common prior. Thus they have identical strategies. Furthermore, population H has identical beliefs about both types, making them identical for the purpose of equilibrium characterization. Therefore, under Assumption 3, we combine Ln and La into a single equivalent type L that represents both subtypes.

Admittedly, Assumption 3 imposes further restrictions on the information structure of the game. However, it plays an important role in our analysis and enables us to completely characterize the equilibrium structure of the game. We believe that the general case of $0.5 < \eta^L < \eta^H < 1$ is significantly more complicated due to the manner in which each population type chooses an equilibrium strategy. We argue that Assumption 3 is of practical relevance in situations where a subset of commuters do not employ or trust TISs and so they can be viewed as an "uninformed population", i.e., having an information accuracy of 0.5. Similarly, since modern TISs use real-time traffic data to predict traffic conditions, their information accuracy can be viewed as almost perfect. Thus, assuming that the "informed population" has an information accuracy of 1 is also not unreasonable.

Henceforth, we will refer to populations H and L as *informed* and *uninformed* respectively. Furthermore, we can view λ to be the degree of *information penetration* amongst the commuters. This is important since one of our goals is to analyze the effects of changing the information penetration on the welfare of commuters.

For the remainder of the thesis, while referring to strategies, departure rate functions and costs in equilibrium we drop the * superscript unless stated otherwise. We now define the different costs which will be used in the equilibrium characterization (chapters 3 and 4) and welfare analysis (chapter 5). Again, these costs are all defined in equilibrium.

We define $\bar{C}_s^{\tau^i}$ as the *average of the costs faced by all type τ^i commuters in state s*, written as:

$$\bar{C}_s^{\tau^i} \triangleq \frac{1}{\lambda^i D} \int_{\mathbb{R}} C_s(t) r^{\tau^i}(t) \ \mathrm{d}t, \quad \forall s \in S, \forall \tau^i \in \boldsymbol{\tau}^i, \forall i \in I, \tag{2.15}$$

where the state-dependent cost function C_s is defined in (2.8).

Next, we define the *individual cost of type τ^i commuters*, which is the expectation of $\bar{C}_s^{\tau^i}$ over the set of states. It is given by:

$$\mathbb{E}[\bar{C}^{\tau^i}] \triangleq \sum_{s \in S} \bar{C}_s^{\tau^i} \Pr(s|\tau^i), \quad \forall \tau^i \in \boldsymbol{\tau}^i, \forall i \in I, \tag{2.16}$$

where $\Pr(s|\tau^i)$ is the posterior belief over states from (2.12). Under Assumption 3,

$\Pr(s|Hs) = 1$, and therefore $\mathbb{E}[\bar{C}^{Hs}] = \bar{C}_s^{Hs}$, $\forall s \in S$. Henceforth, we use the simpler notation $\bar{C}^{Hs}$ to denote $\mathbb{E}[\bar{C}^{Hs}]$ and $\bar{C}_s^{Hs}$.

For the same reason, $\mathbb{E}[C^{Hs}]$, the ex ante cost function of type Hs players from (2.13) is equal to the realized cost function of departing in state s i.e., $\mathbb{E}[C^{Hs}(t)] = C_s(t)$, $\forall t \in \mathbb{R}$, $\forall s \in S$. Therefore we simply use $C_s(t)$ to denote $\mathbb{E}[C^{Hs}(t)]$.

Also note that $\mathbb{E}[\bar{C}^{\tau^i}]$ can alternatively be derived from $\mathbb{E}[C^{\tau^i}]$ as follows:

$$\mathbb{E}[\bar{C}^{\tau^i}] = \int_{\mathbb{R}} \frac{\mathbb{E}[C^{\tau^i}(t)]r^{\tau^i}(t)}{\lambda^i D} \; \mathrm{d}t, \quad \forall \tau^i \in \boldsymbol{\tau}^i, \forall i \in I, \tag{2.17}$$

where $\mathbb{E}[C^{\tau^i}]$ is defined in (2.13). Note that eq. (2.16) and eq. (2.17) are equivalent since they are effectively an interchange of the order of expectation, which is permissible due to the non-negativity of all the quantities involved. Specifically, in the first derivation given in (2.16), $\bar{C}_s^{\tau^i}$ is first averaged over time, and then over states, while in the derivation given in (2.17), the order is reversed.

Next, we define the *individual cost of each population* i, which is the average of the individual costs faced by each of its types. It is given by:

$$\mathbb{E}[\bar{C}^i] \triangleq \sum_{\tau^i \in \boldsymbol{\tau}^i} \mathbb{E}[\bar{C}^{\tau^i}]\Pr(\tau^i), \quad \forall i \in I, \tag{2.18}$$

where $\Pr(\tau^i)$ is the marginal probability of population i having type τ^i from (2.10). This summation is degenerate for population L since it has only one type; therefore the individual cost of population L is simply given by (2.16).

Finally, we define the *social cost* $\mathbb{E}[\bar{C}]$, which is the average cost faced by any commuter in any state at any time. It can be calculated as an average of the individual costs of the populations:

$$\mathbb{E}[\bar{C}] \triangleq \sum_{i \in I} \mathbb{E}[\bar{C}^i]\lambda^i. \tag{2.19}$$

In the next two sections, we will characterize the equilibrium strategy profiles for the game Γ over the parameter space $\rho \times p \times \lambda \in (0,1] \times [0,1] \times [0,1]$, while holding the other parameters of the game constant.

Chapter 3

Boundary Cases

In this chapter, we discuss the equilibrium characterization for the boundary cases of the model described in Chapter 2 in which one or more of the parameters (ρ, p, λ) takes a limiting value. In these boundary cases, the game does not have asymmetric information. These cases are listed in Table 3.1, along with the parameter values that give rise to them. Section 3.1 describes the case where the capacity is deterministic and known, whereas Section 3.2 describes the case where the capacity is stochastic but its realization is known. Finally, Section 3.3 describes the case where the capacity is stochastic and unknown.[1] These boundary cases, which are all under symmetric information, are instructive for solving the general case which has asymmetric information structure (Chapter 4). For the fixed parameters, we use the values shown in Table 3.2 for all our figures and numerical analysis.

Boundary Case	Features	Parameter Values
Deterministic Case (classical Bottleneck model)	Capacity non-stochastic, known	$\rho = 1$ or $p \in \{0, 1\}$
Full Information	Capacity stochastic, realization known	$\rho \neq 1, p \neq \{0, 1\}, \lambda = 1$
Zero Information	Capacity stochastic, realization unknown	$\rho \neq 1, p \neq \{0, 1\}, \lambda = 0$

Table 3.1: List of Boundary Cases

[1]This is a special case of the incomplete (and symmetric) information game solved by Arnott et al. (1988).

Description	Symbol	Value	Units
Unit Cost of Travel Time	α	6.40	\$/hr
Unit Cost of Early Arrival	β	3.90	\$/hr
Unit Cost of Late Arrival	γ	15.21	\$/hr
Demand (Number of Commuters)	D	8000	veh.
Capacity in State n	c_n	4000	veh./hr

Table 3.2: Parameter values, taken from Table 1 of Arnott et al. (1991)

3.1 Deterministic Case

When $\rho = 1$, $p = 0$, or $p = 1$, the capacity becomes deterministic. When $p = 1$, the capacity is always c_a, whereas $p = 0$, or $p = 1$, it is c_n. We generically refer to the deterministic capacity as c. Since the prior distribution of the capacity is common knowledge, and is now degenerate, all commuters know the capacity c. The populations and types are all identical for the purposes of equilibrium analysis, and can be combined into one (homogeneous) population. Thus, we recover Vickrey's original Bottleneck Model in this case. A detailed analysis of this model can be found in Arnott et al. (1990). We only recap the result here, since it is instructive for our subsequent analysis.

The equilibrium strategy for this case is a piecewise constant departure rate function r whose support supp(r) is a contiguous interval around t^*. To describe this equilibrium strategy, we define the following time instants (in no particular order), which we call *epochs*:

- t_0: departure time of first commuter
- t_1: departure time of last commuter
- t_f: arrival time of last commuter
- $\tilde{t}$: departure time such that arrival is at t^*, referred to as *pivot time*
- $\hat{t}$: time of dissipation of queue

From Definition 2, we know that in equilibrium, the cost of departing at any time within supp(r) is equal. Taking the piecewise derivative of the cost function (2.8) within each interval in which it is continuous, and setting it to 0, we see that the equilibrium departure rate function r is given by:

$$r(t) = \begin{cases} \dfrac{\alpha c}{\alpha - \beta} & t \in [t_0, \tilde{t}], \\ \dfrac{\alpha c}{\alpha + \gamma} & t \in (\tilde{t}, t_1], \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

We refer to the expression $\dfrac{\alpha c}{\alpha - \beta}$ (resp. $\dfrac{\alpha c}{\alpha + \gamma}$) as r_e (resp. r_l), with the subscript representing early (resp. late) arrival. An intuitive explanation of this departure rate function is as follows: the queuing cost must build up until $\tilde{t}$ since it is offset by the decreasing scheduling cost; thus $r_e > c$. At $\tilde{t}$, the scheduling cost is 0 and the queue is at its largest. After $\tilde{t}$, the increasing scheduling cost is offset by a decreasing queue, thus $r_l < c$.

The queue begins to build up at t_0 and dissipates at t_1, so everyone except the first and last commuters faces a queue; thus $\hat{t} = t_1$. Furthermore, since the free flow travel time is assumed to be 0, $t_f = t_1$ as well. Recall from Chapter 2 that the rush hour is the interval from the earliest departure to the latest arrival, which is $[t_0, t_f]$ in our notation. Therefore the rush hour is precisely $\text{supp}(r)$ in this case.

We now show how the above-mentioned expressions for the epoch times are calculated. Since the bottleneck operates at full capacity throughout the rush hour, we can write:

$$t_1 - t_0 = \frac{D}{c}. \tag{3.2}$$

Furthermore, in equilibrium, all players face the same total cost, so equating the costs faced by the first and last players to depart gives:

$$\beta(t^* - t_0) = \gamma(t_1 - t^*). \tag{3.3}$$

Solving equations (3.2) and (3.3) gives:

$$\begin{aligned} t_0 &= t^* - \frac{\gamma D}{c(\beta+\gamma)}, \\ t_1 &= t^* + \frac{\beta D}{c(\beta+\gamma)}. \end{aligned} \tag{3.4}$$

Next, by equating the cost of the commuter who departs at the pivot time $\tilde{t}$ (and thus arrives at the preferred time t^*) with the first commuter departing at t_0, we get the equation:

$$\beta(t^* - t_0) = \alpha(t^* - \tilde{t}), \tag{3.5}$$

which gives the following expression for the pivot time:

$$\tilde{t} = t^* - \frac{\beta\gamma D}{c\alpha(\beta+\gamma)}. \tag{3.6}$$

Finally, the cost faced by any commuters in equilibrium, which we call the *deterministic cost* C, is given by:

$$C = \beta(t^* - t_0) = \frac{\beta\gamma D}{c(\beta+\gamma)}. \tag{3.7}$$

3.2 Full Information

Vickrey's model is also recovered in the second boundary case in Table 3.1: when $\rho \neq 1$, $p \neq \{0,1\}$, and $\lambda = 1$, the state is stochastic, but all commuters are perfectly informed of its realization. Thus, since all commuters are informed about the realized state, this case becomes an instance of the deterministic Bottleneck Model described in Section 3.1, with the realized capacity c_s. The epochs and departure rates are state-dependent and indexed by subscript s. The departure rate functions for this case (one for each state) are denoted $r_{s,1}$ where the 1 indicates the value of λ in this case. The definition of the rush hour is also revised to reflect that either state can be realized: it now refers to the interval from the earliest departure in either state to the latest arrival in either state. Under the condition $\rho < 1$, we can conclude from (3.4) that $t_{0,a} < t_{0,n}$ and $t_{1,n} < t_{1,a}$; i.e. the departures occur in a narrower interval in state n than in state a. Therefore, since $t_{1,a} = t_{f,a}$, the rush hour in this case is $[t_{0,a}, t_{1,a}]$.

This boundary case essentially amounts to averaging the deterministic cost faced in each state. Thus, the expected cost faced by any commuter in this case can be called the *full information cost.* This cost, denoted $\mathbb{E}[C_1]$, is given by:

$$\mathbb{E}[C_1] = p\frac{\beta\gamma D}{c_a(\beta+\gamma)} + (1-p)\frac{\beta\gamma D}{c_n(\beta+\gamma)} = \frac{(pc_n + (1-p)c_a)\beta\gamma D}{c_n c_a(\beta+\gamma)}. \tag{3.8}$$

It is worth noting here that when the deterministic capacity in Section 3.1 is c_n, then $\mathbb{E}[C_1] > C$. Thus the possibility of an incident increases the equilibrium cost relative to the case when there is no incident.

3.3 Zero Information

The third boundary case from Table 3.1 is in contrast to the two cases mentioned above; when $\rho \neq 1$, $p \neq \{0,1\}$, and $\lambda = 0$, the classical Bottleneck Model is not recovered. Instead, this case is a population game with a symmetric information structure where all commuters are uninformed, i.e. there is only one population (L). The equilibrium for this case can be obtained from Arnott et al. (1988) who deal with a general zero information game with an arbitrary joint prior distribution of capacity and demand. The authors consider this (boundary) case as a specific example to highlight certain properties of equilibrium costs; thus, their analysis of this case is not exhaustive in that they do not elaborate on all the qualitative features of the equilibrium strategy. It turns out that for our model these features provide useful insights that can be directly utilized in characterizing the equilibrium strategies for the game with heterogeneous information structure (Chapter 4). We now provide a complete characterization of equilibrium for the full range of parameters $\rho \times p \in$

$(0,1] \times [0,1]$ when all commuters are uninformed.[2] The equilibrium behavior is rich even though this is a boundary case; however we only mention the main results here for the sake of brevity. The reader is referred to Appendix A for complete details (including the proof of Theorem 1 mentioned below).

Since all commuters are type L (and this is common knowledge), their beliefs are equivalent to the common prior, which is the joint distribution π over the states. Furthermore, the set of types τ is degenerate, so π is reduced to the prior distribution θ. All commuters know the prior distribution θ, but not the realized state s. Therefore the resulting equilibrium departure rate function r is identical in both states, i.e. t_0 and t_1 are not state-dependent. However, the queuing time faced by commuters is state-dependent. Therefore the pivot time $\tilde{t}$, the last arrival time t_f and the queue dissipation time $\hat{t}$ are also state-dependent; we index these quantities by subscript s. Additionally, in this case, the last commuter to depart may face a queue, so the last departure time t_1, the queue dissipation time $\hat{t}_s$, and the last arrival time $t_{f,s}$ are not equal in general (in contrast to the first two boundary cases).

Similar to the Bottleneck Model, the departure rate function r is piecewise constant and its support is a contiguous interval around t^*. Depending on the parameters, r takes one of several qualitatively different forms, which we call *regimes*. These regimes can be distinguished in two aspects:

1. the existence and duration of the queue in state n, and

2. the queue faced by the last commuter in state a.

We discuss each of these distinctions below.

Firstly, whenever a queue forms in a state, it starts at t_0. In state a, a queue always forms and lasts beyond t^* in all regimes. In state n however, a queue may or may not form. This is the first distinction between regimes.[3] There are 3 possibilities in this regard:

[2]This model reduces to the deterministic case described in Section 3.1 for the extreme values $\rho = 1$ or $p \in \{0, 1\}$.

[3]While this distinction is not explicitly mentioned in Arnott et al. (1988), in can be inferred from one of the figures in the paper.

- R1: There is no queue ($\hat{t}_n = t_0$).

- R2: A queue forms but dissipates before t^* ($t_0 < \hat{t}_n < t^*$).

- R3: A queue forms and and does not dissipate by t^* ($\hat{t}_n \geq t^*$).

Since the last departure occurs at the same time in both states and the queue in state n (if it exists) dissipates before the queue dissipates in state a, the last arrival in state n is no later than the last arrival in state a, i.e. $t_{f,n} \leq t_{f,a}$.[4] Therefore, the rush hour in this case is $[t_0, t_{f,a}]$.

Secondly, regimes can be distinguished with regards to the queue faced in state a by the last commuter to depart. In fact, this can equivalently be viewed as the relative ordering between the last departure time t_1 and the queue dissipation time $\hat{t}_a$. There are two possibilities in this regard:

- RA: The last commuter does not face a queue ($t_1 = \hat{t}_a = t_{f,a}$).

- RB: The last commuter faces a non-zero queue ($t_1 < \hat{t}_a = t_{f,a}$).

Another equivalent interpretation of this distinction is whether or not there is a non-degenerate interval ($(t_1, \hat{t}_a]$) at the end of the rush hour in which there are no departures. This distinction is mentioned in Arnott et al. (1988).

We now mention our naming convention of regimes resulting from the two distinctions. The regime which is of type R1 with respect to the first distinction and type RA with respect to the second is referred to as R1A, and similarly for other regimes. Thus R1 refers to the set {R1A, R1B} and RA refers to the set {R1A, R2A, R3A}.

The following theorem states the existence and uniqueness of an equilibrium departure rate function for the full range of parameter values.

Theorem 1. *Given a* $(\rho, p) \in (0,1)^2$, *there exists a unique equilibrium departure rate function* r *over the domain* $[t_0, t_{f,a}]$. *Fig. 3-1 shows the partition of the* $\rho - p$ *parameter space into the above mentioned regimes.*

The boundaries between the regimes in terms of p *are as follows:*

[4] $\hat{t}_n < \hat{t}_a$ is shown in Proposition 1 in Appendix A.

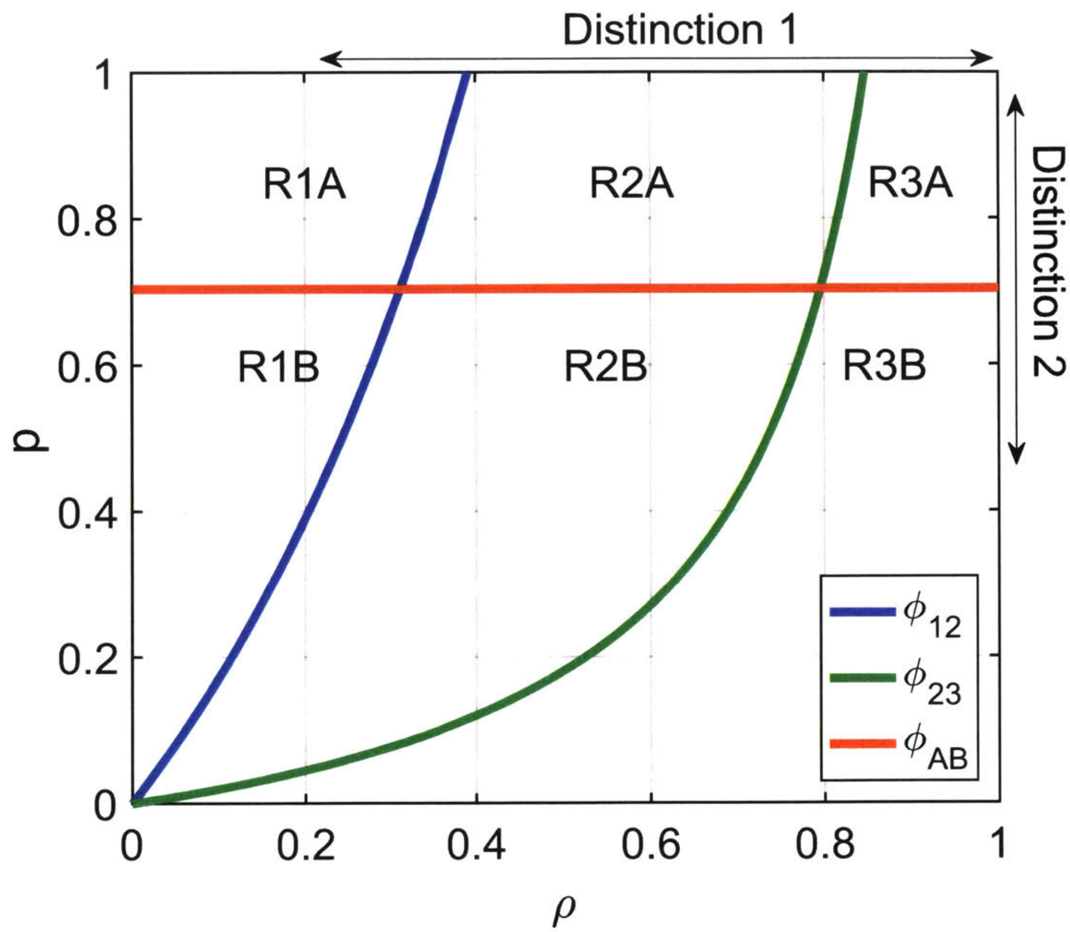


Figure 3-1: Equilibrium characterization under Zero Information

- *The threshold separating* R1 *and* R2 *is:*

$$\phi_{12}(\rho) = \frac{\beta\rho}{(\alpha - \beta)(1 - \rho)}. \tag{3.9}$$

- *The threshold separating* R2 *and* R3 *is:*

$$\phi_{23}(\rho) = \frac{\beta\rho}{(\alpha + \gamma)(1 - \rho)}. \tag{3.10}$$

- *The threshold separating* RA *and* RB *is:*

$$\phi_{AB} = \frac{\gamma}{\alpha + \gamma}. \tag{3.11}$$

Here we remark that ϕ_{23} and ϕ_{AB} are mentioned in Arnott et al. (1988) while ϕ_{12} is not.

An intuitive explanation of Fig. 3-1 is below. On one hand, as ρ increases with p fixed, the equilibrium shifts from R1 to R2 to R3. R1 exists for low values of ρ, where c_a is small relative to c_n, for example in the case of a severe incident, multiple lane closures, extreme weather, etc. The departures, which are based on minimizing expected individual travel cost, are spread out to avoid large queuing delays in state a. Thus, when the realized state is n, the departure profile is too conservative and the available capacity is underutilized, resulting in no queues. For intermediate values of ρ, such as in the case of moderate incidents, the departure rate function is moderately spread out such that a queue builds up in state n as well, but is short lived and dissipates before t^*. Finally, when ρ is close enough to 1, for example in the case of mildly inconvenient weather conditions etc., the departures are more concentrated around t^*. Here, the difference between the queue dissipation times $\tilde{t}_n$ and $\tilde{t}_a$ is small, and the queue lasts beyond t^* even in state n. Note that the actual values of c_n and c_a are not needed for the characterization of the regimes; the value of their ratio ρ is sufficient. The numerical value of c_n simply scales the length of the rush hour.

On the other hand, we can analyze the effect of increasing p while keeping ρ fixed, i.e. increasing the chance of an incident. As p becomes larger, the probability of long queuing delays for those commuters departing in the middle of the rush hour increases. Thus departing late in the rush hour to avoid long queues becomes more desirable. Departure times progressively shift later as p increases until t_1 becomes equal to $\hat{t}_a$ at $p = \phi_{AB}$, and remains equal thereafter, meaning that departures occur throughout the rush hour (i.e. RA). For $p < \phi_{AB}$, there exists a non-zero interval in the last part of the rush hour in which no departures take place, and so $t_1 < \hat{t}_a$ (i.e. RB).

The expected cost faced by commuters in this case is called the *zero information cost* $\mathbb{E}[\bar{C}_0]$. Since it is equal for all commuters in equilibrium, we can use the first commuter to depart as an example to calculate it. The first commuter departs at t_0 and does not face a queue in either state, thus $\mathbb{E}[\bar{C}_0]$ is given by:

$$\mathbb{E}[\bar{C}_0] = \beta(t^* - t_0), \tag{3.12}$$

where the method for determining t_0 is given in Appendix A. As shown in Arnott et al. (1988), $\mathbb{E}[\bar{C}_0]$ is always greater than $\mathbb{E}[C_1]$, indicating that full information is welfare improving over zero information.

Chapter 4

Equilibrium Characterization

In Chapter 3, we described the equilibrium strategies for the boundary cases of the Bayesian game Γ, in which one or more of the parameters (ρ, p, λ) took an extreme value. In this chapter, we provide complete equilibrium characterization for the game for any $\rho \times p \times \lambda \in (0,1] \times [0,1] \times [0,1]$. This is the asymmetric information game from Definition 1. In Section 4.1, we describe the necessary conditions for equilibrium strategies. In Section 4.2 and Section 4.3, we distinguish between the two main equilibrium regimes that govern our results on welfare analysis in Chapter 5. We also describe the refinement of each of these regimes into various subregimes based on certain qualitative distinctions within each regime. Finally, in Section 4.4 we describe how the parameter space is partitioned into the various subregimes. Before proceeding further, we recall the definitions of key quantities from Chapter 3 and extend them where necessary for the general case.

The definitions of the epochs $t_{0,s}$, $t_{1,s}$, $t_{f,s}$, $\tilde{t}_s$ and $\hat{t}_s$ are carried forward from Chapter 3. Furthermore, we define $t_0^{\tau^i}$ (resp. $t_1^{\tau^i}$) as the earliest (resp. latest) departure time of type τ^i commuters. Therefore, under Assumption 3, $t_{0,s}$ (resp. $t_{1,s}$) can be expressed as follows:

$$t_{0,s} = \min\{t_0^{Hs}, t_0^{L}\}, \quad t_{1,s} = \max\{t_1^{Hs}, t_1^{L}\}$$

Analogously, t_0 (resp. t_1) is the earliest (resp. latest) departure time of any commuter in any state, that is:

$$t_0 = \min\{t_{0,n}, t_{0,a}\} = \min\{t_0^{\tau^i};\ \tau^i \in \mathrm{T}^i, i \in I\},$$

$$t_1 = \max\{t_{1,n}, t_{1,a}\} = \max\{t_1^{\tau^i};\ \tau^i \in \mathrm{T}^i, i \in I\}$$

Table 4.1 summarizes the aforementioned epochs. The rush hour can now be defined as $[t_0, \max\{t_{f,n}, t_{f,a}\}]$, the interval from the earliest departure of any commuter in any state to the latest arrival of any commuter in any state.

Epoch	Definition
$t_0^{\tau^i}$	earliest departure time of τ^i commuters
$t_{0,s}$	earliest departure time of any commuter in state s
t_0	earliest departure time in any state
$t_1^{\tau^i}$	latest departure time of τ^i commuters
$t_{1,s}$	latest departure time of any commuter in state s
t_1	latest departure time in any state
$t_{f,s}$	latest arrival time in state s
$\tilde{t}_s$	pivot time in state s
$\hat{t}_s$	queue dissipation time in state s
$T_s(t)$	last time before t when $q_s(t) = 0$

Table 4.1: Definitions of epochs.

4.1 Necessary Conditions

The departure rates and costs of the each type must satisfy certain conditions in equilibrium. These conditions can be derived from Definition 2. As in the case of zero information (Section 3.3), the equilibrium strategy takes qualitatively different forms, called regimes, depending on the parameter values (ρ, p, λ). However all regimes that are admissible in equilibrium exhibit the common features described in Conditions 1 to 4. The first three conditions are to do with population H commuters, while the last is to do with state a.

Condition 1. The realized (ex post) costs of all type Hs commuters are equal:

$$C_s(t) \equiv \bar{C}^{Hs}, \ \forall t \in \text{supp}(r^{Hs}), \ \forall s \in S.$$

To arrive at this condition, first note from Definition 2 that the expected cost for all commuters of a given type must be equal, so $\mathbb{E}[C^{Hs}(t)] = const.$ for all $t \in \text{supp}(r^{Hs})$. Secondly, recall that under Assumption 3, because population H is fully informed, the expected cost function of type Hs commuters $\mathbb{E}[C^{Hs}]$ is simply the state-dependent cost function C_s defined in (2.8), i.e. $\mathbb{E}[C^{Hs}(t)] = C_s(t)$ for all $t \in \mathbb{R}$. Therefore $C_s(t) = const.$ for all $t \in \text{supp}(r^{Hs})$. Furthermore, by definition (see (2.17)), $\bar{C}^{Hs}$ is a weighted average of $C_s(t)$ over $\text{supp}(r^{Hs})$. Since $C_s(t)$ is constant over $\text{supp}(r^{Hs})$, $\bar{C}^{Hs}$ must equal that constant value i.e. $C_s(t) = \bar{C}^{Hs}$ $\forall t \in \text{supp}(r^{Hs})$.

In other words, the ex post costs incurred by type Hs commuters are exactly their ex ante costs, which must be equal for all type Hs commuters in equilibrium. Note that this condition does not hold for type L commuters: they face equal expected (ex ante) costs due to Definition 2, but the realized costs of type L commuters departing at different times need not be identical.

Condition 2. $\bar{C}^{Hs}$ is the lowest cost achievable at any time in state s:

$$\bar{C}^{Hs} = \inf_{t \in \mathbb{R}} \{C_s(t)\}, \ \forall s \in S.$$

Again, this condition follows from the fact that type Hs commuters are completely informed about the state (from Assumption 3). Therefore, if achieving a lower ex post cost were possible, they would adjust their departure rate to achieve it, contradicting the equilibrium condition. Note that this condition implies that no type L commuter has a lower ex post cost in any state s than type Hs commuters.

Condition 3. All type Hs commuters (except possibly the first and the last) face a queue:

$$q_s(t) > 0,\ \forall t \in \{\text{supp}(r^{Hs}) \backslash \{t_0^{Hn}, t_1^{Hn}\}\},\ \forall s \in S.$$

This condition follows from Condition 1, i.e. that type Hs commuters face equal ex post costs. For this to happen, the change in the scheduling cost over time must be offset by an equal and opposite change in the queuing cost.[1] Thus, for all departure times such that arrival is early, i.e. the set $\{t : \{t \leq \tilde{t}_s\} \bigcap \text{supp}(r^{Hs})\}$, the queue must continuously grow (possibly from 0 at t_0^{Hs}). Conversely, for all departure times such that arrival is late, i.e. the set $\{t : \{t > \tilde{t}_s\} \bigcap \text{supp}(r^{Hs})\}$, the queue must continuously decrease (possibly to 0 at t_1^{Hs}). This ensures that there is a non-zero queue throughout the interior of supp(r^{Hs}). Note that this condition does not assume (or require) that supp(r^{Hs}) is a single contiguous interval.

Condition 4. All commuters in state a (except possibly the first and the last) face a queue:

$$T_a(t) = t_{0,a}, \forall t \in [t_{0,a}, t_{f,a}), \quad \hat{t}_a = t_{f,a}$$

This condition must be proved separately for the two regimes described in Section 4.2 and Section 4.3. It is therefore proved in Appendix B. Note that this condition does not necessarily hold true for state n.

In summary, Definition 2 imposes certain conditions on the costs and queues faced by commuters in equilibrium. This in turn imposes conditions on their admissible equilibrium strategies (departure rate functions). If a strategy fails to satisfy any of these conditions, it is not admissible in equilibrium. Thus the necessary conditions serve to reduce the search for equilibrium strategy profiles from the feasible set Σ to a smaller admissible set. This set admits two main qualitatively distinct equilibrium regimes, R0 and R0', which we describe in Section 4.2 and Section 4.3 respectively.

[1] The reader may wish to review the definitions of queuing cost and scheduling cost from Section 2.1.

4.2 Equilibrium In Regime R0

In this section, we describe the equilibrium when a *sufficiently large* fraction of commuters is informed. This regime, denoted R0, is closely related to the equilibrium under full information described in Section 3.2. Specifically, for any given (ρ, p), when the fraction of informed commuters (i.e. population H) is above a certain threshold, the aggregate equilibrium departure rate r_s in each state is identical to the corresponding equilibrium departure rate $r_{s,1}$ (see Section 3.2) in the case when *all* commuters are informed (i.e. under full information where $\lambda = 1$). Thus the individual costs $\mathbb{E}[\bar{C}^i]$ faced by both populations, and hence the social cost $\mathbb{E}[\bar{C}]$, are equal to the cost $\mathbb{E}[C_1]$ under full information. These properties of R0 are formalized in the theorem below:

Theorem 2. [Regime R0] *Consider the threshold information penetration:*

$$\lambda' \triangleq \begin{cases} \dfrac{\alpha(1-\rho)+\gamma}{\alpha+\gamma} & 0 < \rho \leq \dfrac{\beta}{\alpha}, \\ \dfrac{\alpha(1-\rho)(\beta(\alpha-\beta)+\gamma(\alpha+\gamma))}{(\alpha-\beta)(\alpha+\gamma)(\beta+\gamma)} & \dfrac{\beta}{\alpha} < \rho \leq 1. \end{cases} \tag{4.1}$$

For any $\rho \in (0,1]$*,* $p \in [0,1]$*,* $\lambda \in [\lambda', 1]$*, the following properties hold:*

$$r_s = r_{s,1}, \quad \forall s \in S \tag{4.2a}$$

$$\mathbb{E}[\bar{C}^i] = \mathbb{E}[C_1], \quad \forall i \in I \tag{4.2b}$$

$$\mathbb{E}[\bar{C}] = \mathbb{E}[C_1] \tag{4.2c}$$

Proof. This proof is structured as follows. We first show that when the fraction of type L commuters is sufficiently small (i.e. λ is greater than the threshold λ'), then the departures of type L commuters can fit within $r_{s,1}$ in both states. In other words, there exists a feasible r^L such that $r^L(t) \leq r_{s,1}(t)$, $\forall t \in \mathbb{R}$, $\forall s \in S$. We use this to derive the expression for λ'. We then show that this implies the properties given in (4.2).

First, recall that under Assumption 3, the departure rate of type L commuters is identical in both states. Next, note from Section 3.2 that departures are more spread out in state a than in state n, i.e. $(t_{1,n} - t_{0,n}) < (t_{1,a} - t_{0,a})$. Furthermore, when the departure rate in state n is non-zero, it is greater than the full information departure rate in state a, i.e. $r_{a,1}(t) < r_{n,1}(t)$, $\forall t \in \text{supp}(r_{n,1})$. Therefore, the maximum number of type L commuters that can fit in $r_{s,1}$ in both states is given by:

$$(1-\lambda')D = \int_{t_{0,n}}^{t_{1,n}} r_{a,1}(t)\ \mathrm{d}t \tag{4.3}$$

To solve this equation, we need to consider two cases separately:

- **Case 1:** $\tilde{t}_a \leq t_{0,n}$

  In this case, $r_{a,1}(t) \equiv r_{a,l}$ for all $t \in \text{supp}(r_{n,1})$. Therefore (4.3) becomes:

$$(1-\lambda')D = r_{a,l}(t_{1,n} - t_{0,n}).$$

  Substituting (3.1) and (3.2), this becomes:

$$(1-\lambda')D = \frac{\alpha c_a}{\alpha+\gamma}\frac{D}{c_n},$$

  giving:

$$\lambda' = \frac{\alpha(1-\rho)+\gamma}{\alpha+\gamma}$$

  Substituting the expressions for $\tilde{t}_a$ and $t_{0,n}$ given by (3.6) and (3.4) respectively, we can write:

$$t_{0,n} \leq \tilde{t}_a$$

$$\Rightarrow t^* - \frac{\gamma D}{c_n(\beta+\gamma)} \leq t^* - \frac{\beta\gamma D}{c_a \alpha(\beta+\gamma)}$$

$$\Rightarrow \rho \leq \frac{\beta}{\alpha}. \tag{4.4}$$

- **Case 2:** $\tilde{t}_a > t_{0,n}$

In this case, we can write:

$$r_{a,1}(t) = \begin{cases} r_{a,e} & t \in [t_{0,n}, \tilde{t}_a), \\ r_{a,l} & t \in [\tilde{t}_a, t_{1,n}], \\ 0 & \text{otherwise.} \end{cases}$$

Therefore in this case (4.3) becomes:

$$(1 - \lambda')D = r_{a,l}(t_{1,n} - \tilde{t}_a) + r_{a,e}(\tilde{t}_a - t_{0,n}).$$

Substituting (3.1), (3.4), and (3.6), this becomes:

$$\begin{aligned}(1 - \lambda')D =& \frac{\alpha c_a}{\alpha + \gamma}\left(t^* + \frac{\beta D}{c_n(\beta + \gamma)} - \left(t^* - \frac{\beta\gamma D}{c_a\alpha(\beta + \gamma)}\right)\right) \\ &+ \frac{\alpha c_a}{\alpha - \beta}\left(t^* - \frac{\beta\gamma D}{c_a\alpha(\beta + \gamma)} - \left(t^* - \frac{\gamma D}{c_n(\beta + \gamma)}\right)\right),\end{aligned}$$

giving:

$$\lambda' = \frac{\alpha(1 - \rho)(\beta(\alpha - \beta) + \gamma(\alpha + \gamma))}{(\alpha - \beta)(\alpha + \gamma)(\beta + \gamma)}$$

Combining these two cases results in the expression for λ' given in (4.1).

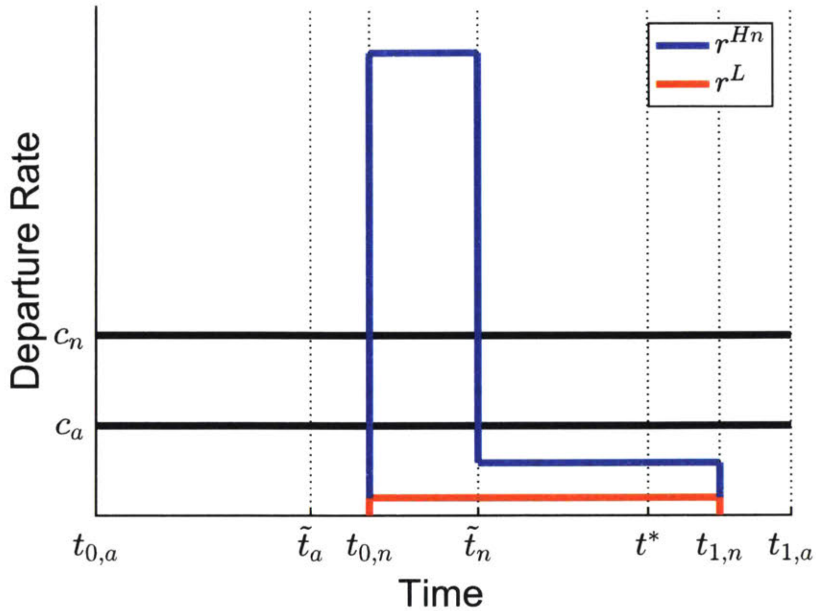
For $\lambda > \lambda'$, the strategy $r^{Hs}(t) = r_{s,1}(t) - r^L(t)$ for all $t \in \text{supp}(r_{s,1})$ is valid in equilibrium for population H. This results in aggregate departure functions r_s which are identical to $r_{s,1}$ (showing (4.2a)). Thus, type L commuters face the same cost as type Hs commuters in each state, and therefore $\mathbb{E}[\bar{C}^H]$ and $\mathbb{E}[\bar{C}^L]$ are also identical. Furthermore, $\mathbb{E}[\bar{C}^H]$ and $\mathbb{E}[\bar{C}^L]$, and hence the social cost $\mathbb{E}[\bar{C}]$, are equal to $\mathbb{E}[C_1]$ since the aggregate departures and arrivals are identical (showing (4.2b) and (4.2c)). □

This regime is called R0 since the relative value of information is 0. We will return to this observation in Section 5.1 while discussing the value of information.

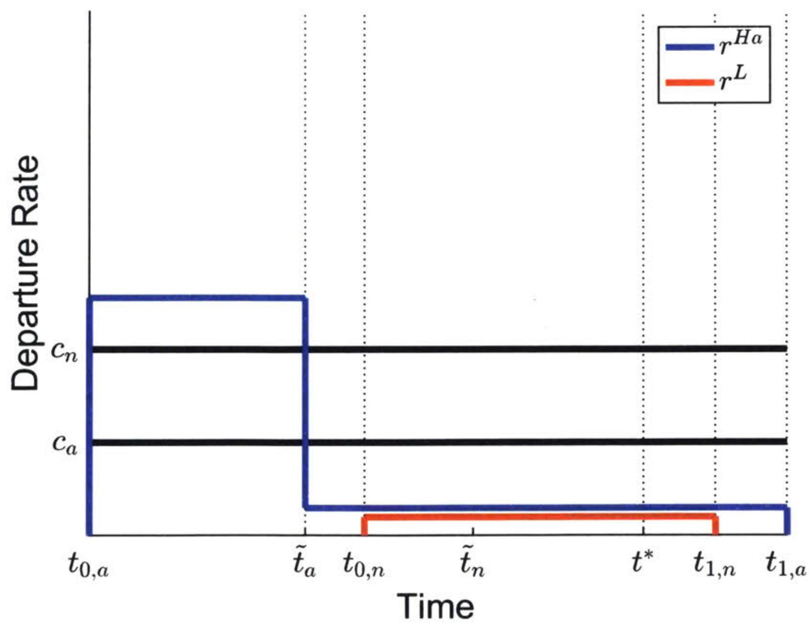
In R0, any set of feasible functions (r^{Hn}, r^{Ha}, r^{L}) which ensures that the aggregate departure rate functions r_s are identical to $r_{s,1}$ constitutes an equilibrium strategy profile. There are an infinite number of such strategy profiles, so the equilibrium is non-unique. Consequently, the type-dependent epochs $t_0^{\tau^i}$ and $t_1^{\tau^i}$ cannot be uniquely determined. However the aggregate departure rate functions r_s and thus the state-dependent epochs $t_{0,s}$, $\tilde{t}_s$ and $t_{1,s}$ are unique; thus we can say that the equilibrium in R0 is *essentially unique.* Fig. 4-1 (resp. Fig. 4-2) show one possible equilibrium strategy profile for the parameter values $\rho = 0.5$, $p = 0.5$, $\lambda = 0.9$ (resp. $\rho = 0.7$, $p = 0.5$, $\lambda = 0.7$).[2]

We refer to the two cases in the proof of Theorem 2 as *subregimes* of R0. For reasons we will see in Section 4.3.2, we call these subregimes R0⟨1,3⟩ and R0⟨3,3⟩ respectively.

[2]In Figs. 4-1, 4-2, 4-5 and 4-6, the height of the blue plot is the cumulative departure rate r_s. r^{Hs} is the difference in height between the blue and red plots.

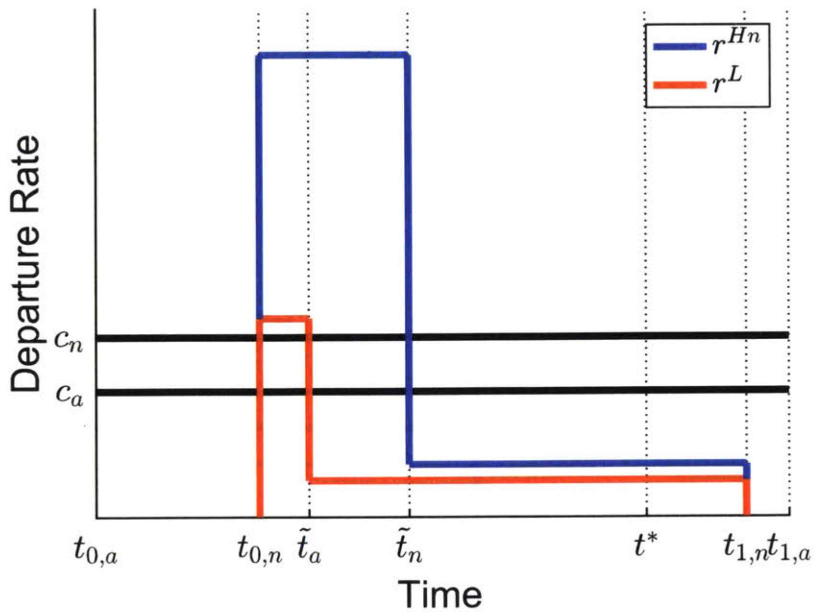


(a) state n

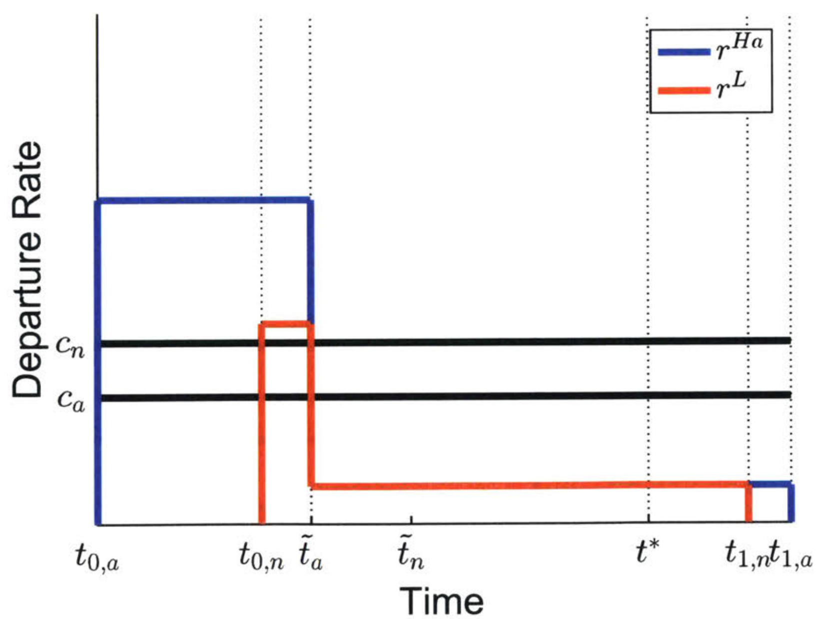


(b) state a

Figure 4-1: Population departure rates in R0$\langle 1,3 \rangle$ with $\rho = 0.5$, $p = 0.5$, $\lambda = 0.9$.



(a) state n



(b) state a

Figure 4-2: Population departure rates in R0⟨3,3⟩ with $\rho = 0.7$, $p = 0.5$, $\lambda = 0.7$.

4.3 Equilibrium In Regime R0'

Recall from Theorem 2 that R0 describes the equilibrium for $\lambda' \leq \lambda \leq 1$. For the remainder of the parameter space, where $0 \leq \lambda < \lambda'$, the equilibrium is described by a regime which we call R0'. This choice of notation reflects that the relative value of information in R0' is not 0, as discussed further in Chapter 5. Unlike in R0, where the equilibrium strategy profiles are non-unique for any given (ρ, p, λ), the equilibrium in R0' is unique:

Theorem 3. *For any $\rho \in (0, 1]$, $p \in [0, 1]$, $\lambda \in [0, \lambda')$ there exists a unique equilibrium strategy profile (r^{Hn}, r^{Ha}, r^{L}).*

The equilibrium strategy profiles in R0' are more complicated than in R0, and their derivation is more involved. The remainder of this section is dedicated to characterizing these equilibrium strategy profiles, which will ultimately prove Theorem 3. First, we provide some intuition for the equilibrium strategies below.

Since queuing times are short in state n, type Hn commuters travel in the middle of the rush hour, close to t^*, giving them a low scheduling cost as well as a low queuing cost. In state a, queues are longer in the middle of the rush hour, so type Ha commuters travel early to avoid long delays. For the same reason, they may also travel later on, when the queue has significantly subsided; they accept moderate scheduling costs in order to avoid larger queuing costs. Type L commuters must choose their departure rates taking both states into account, so they opt for a more conservative strategy, choosing a wider range of departure times centered around t^*. The bottom panel of Fig. 4-3 shows the departure intervals of each commuter type while the top panel shows the state-dependent cost functions C_s as well as the $\mathbb{E}[\bar{C}^{\tau^i}]$, the average costs faced by each commuter type. While this figure portrays an generic situation that is not the result of any exact equilibrium strategy, it offers some important insights. Several of the observations from Fig. 4-3 are formalized as necessary conditions in equilibrium in Section 4.3.1 below.

This section is structured as follows. First, in Section 4.3.1, we describe additional conditions apart from Conditions 1 to 4 that equilibria in R0' must satisfy. Next, in

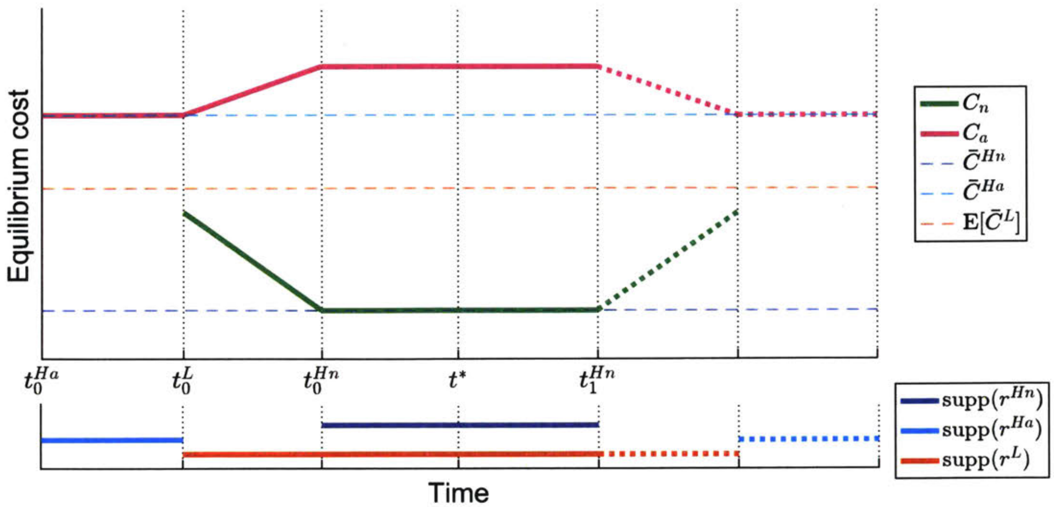


Figure 4-3: Equilibrium costs and departure times for generic equilibrium in R0'. The dotted lines represent intervals that may or may not have departures in equilibrium. The last two labels are missing for the same reason; the epochs they represent depend on the distinctions in Section 4.3.2.

Section 4.3.2, we describe the refinement of R0' into subregimes, and the distinctions between them. Then, in Section 4.3.3, we calculate the departure rates for these subregimes. Finally, in Section 4.3.4, we show how the epoch values are calculated, and complete the proof of Theorem 3.

4.3.1 Additional Necessary Conditions in R0'

In addition to Conditions 1 to 4, the equilibria in R0' satisfy a further set of conditions. These conditions impose a restrictive structure to the equilibrium strategies and play a significant role in the proof of Theorem 3.

Condition 5. Type Hn commuters depart in a contiguous interval around t^*:

$$\text{supp}(r^{Hn}) = [t_0^{Hn}, t_1^{Hn}], \text{ where } t_0^{Hn} < t^* < t_1^{Hn}.$$

Since queuing times are short in state n, the minimum cost is achieved by departing around t^* (as in the full information case). By Condition 2, type Hn commuters must achieve this cost. To satisfy Condition 1, their departures must be in a contiguous interval with the changing scheduling cost being offset by an equal change in queuing cost.

Condition 6. Type L commuters depart in a contiguous interval around t^* which contains the interval $\text{supp}(r^{Hn})$. Furthermore, type L commuters begin departing before type Hn commuters in state n:

$$\text{supp}(r^L) = [t_0^L, t_1^L], \quad t_0^L < t_0^{Hn}, \ t_1^L \geq t_1^{Hn}.$$

Under Assumption 3, Type L commuters have no information about the state, so they have the same departure rate in both states. When $\lambda < \lambda'$, not all type L commuters can depart concurrently in the interval $[t_0^{Hn}, t_1^{Hn}]$. Since this interval has a low cost in state n, some type L commuters do depart in it. The remainder depart before t_0^{Hn} or after t_1^{Hn}. With the relative values of β and γ we consider, the cost of departing after t_1^{Hn} can be either higher than or equal to the cost of departing before t_0^{Hn}), but not lower. Thus type Ha commuters must begin to depart before t_0^{Hn}.

Condition 7. Type Ha commuters do not depart concurrently with type L commuters:

$$\text{supp}(r^{Ha}) \bigcap \text{supp}(r^L) = \emptyset.$$

Due to Condition 1, all type Ha commuters must face equal cost, which is the lowest cost possible in state a. Any type L commuters departing concurrently with them must also face the same cost. In order to maintain the same expected cost as other type L commuters, these commuters would have to face equal cost in state n, which would be the highest cost faced in that state. However, type L commuters in state n either face the lowest cost in state n (in $\text{supp}(r^{Hn})$, or face time-varying cost (outside $\text{supp}(r^{Hn})$. Thus they cannot depart concurrently with type Ha commuters.

Condition 8. Type Ha commuters start departing before type L commuters, and depart continuously until type L commuters begin departing:

$$t_0^{Ha} < t_0^L, \ [t_0^{Ha}, t_0^L) \in \text{supp}(r^{Ha}).$$

From a combination of Condition 6 and Condition 7 , type Ha commuters can only depart before t_0^L or after t_1^L. From Condition 2, they must face the minimum cost achievable in state a. With the relative values of β and γ we consider, the cost of departing at the late end of the rush hour (after t_1^L) can be either higher than or equal to the cost at the beginning (before t_0^L), but not lower. Thus type Ha commuters must begin to depart before t_0^L. To satisfy Condition 1, this must be a contiguous interval with the changing scheduling cost being offset by an equal change in queuing cost. To satisfy Condition 2, this interval must be immediately preceding t_0^L, i.e. $[t_0^{Ha}, t_0^L)$.

Note that when it is possible to achieve the same cost later in the rush hour, some type Ha commuters depart in the late end of the rush hour as well. These different cases as described in Distinction 2.

Condition 9. The cost faced by all commuters departing in state a during $supp(r^{Hn})$ is equal, and is the highest cost faced by any commuters in state a:

$$C_a(t) = K \ \forall t \in [t_0^{Hn}, t_1^{Hn}], \quad K = \sup_{t \in \mathbb{R}} \{C_a(t)\}.$$

First note that, under Condition 6, type L commuters depart throughout this interval. Furthermore, under Condition 7, they are the only commuters departing in this interval in state a. Since $C_s(t)$ is equal $\forall t \in [t_0^{Hn}, t_1^{Hn}]$, therefore, to achieve an equal expected (ex ante) cost, $C_a(t)$ must also be equal in the same interval. Furthermore, this cost is the maximum incurred at any time in state a. This is because $C_n(t)$ is at its minimum in $[t_0^{Hn}, t_1^{Hn}]$. Therefore, $C_a(t)$ must be at its maximum in $[t_0^{Hn}, t_1^{Hn}]$ to ensure that, when averaged over states, the expected cost of departing in this interval can equal the expected cost faced by other type L commuters who depart outside the interval.

4.3.2 Refinement of R0' Into Subregimes

While all equilibrium strategies in R0' must satisfy Conditions 1 to 9, they can be distinguished further based on certain aspects which depend on the parameters $(\rho, p, \lambda) \in (0,1] \times [0,1] \times [0, \lambda')$. These distinctions refine R0' into several subregimes. Understanding these distinctions is necessary in order to determine the departure rate functions in Section 4.3.3 and Section 4.3.4.

In general, the subregimes of R0' are distinguishable in four aspects:

1. Existence and duration of the queue in state n during the interval $[t_0^L, t_0^{Hn}]$,
2. Contiguity of the departure interval(s) of type Ha players, i.e. supp(Ha),
3. Orderings of the pivot times $\tilde{t}_n$ and $\tilde{t}_a$ relative to the other epochs, and
4. the ordering of t_1^{Hn} and t_1^L, which are the last departure times of commuter types Hn and L respectively.

Before proceeding, we describe the naming convention that we use to denote the various subregimes. A single subregime is denoted Rx[y]$\langle$z$\rangle$, with x, y and z each representing a distinction, and $\langle\cdot\rangle$ being a tuple $\langle z_a, z_n\rangle$. The fourth distinction is nominally suppressed in our notation due to its lesser importance. As before, R1 refers to the set of all subregimes which are of type 1 with regards to the first distinction, and similarly for others. Furthermore, R$\cdot$[1]$\langle\cdot\rangle$ refers to the set of all subregimes which are of type 1 with regards to the second distinction, and similarly for other subregimes and other distinctions.

Distinction 1. Rx: Existence of Queue. The first distinction between subregimes is with regards to the existence and duration of a queue in state n during the interval $[t_0^L, t_0^{Hn}]$.

We find that there may be one or two disjoint intervals in state n in which a queue exists; see Fig. 4-4. In the former case, the queue dissipation time is $\hat{t}_n$ (by definition). In the latter case, we use $\hat{t}_n$ to refer to the dissipation time of the second queue and the additional notation $\hat{t}_{n'}$ to refer to the dissipation time of the first queue.

Recall from Condition 3 that there is necessarily a queue in state n during $\text{supp}(r^{Hn})$, i.e. during the interval $[t_0^{Hn}, t_1^{Hn}]$. However, there may or may not be a queue in the interval $[t_0^L, t_0^{Hn}]$, which immediately precedes $[t_0^{Hn}, t_1^{Hn}]$. Recall from Condition 6 that $t_0^L < t_0^{Hn}$, so this interval is non-degenerate (i.e. non-zero length), and from Condition 7 that only type L players depart in it. There are 3 possibilities with regards to the queue in this interval in state n:

R1: There is no queue during $[t_0^L, t_0^{Hn}]$.

In this case (Fig. 4-4a), $[t_0^{Hn}, t_1^{Hn}]$ is the only interval of queuing in state n.

R2: A queue begins to build at t_0^L and dissipates before t_0^{Hn}.

In this case (Fig. 4-4b), there are two intervals of queuing; the first is $[t_0^L, \hat{t}_{n'}]$, where $\hat{t}_{n'} < t_0^{Hn}$, whereas the second is $[t_0^{Hn}, t_1^{Hn}]$.

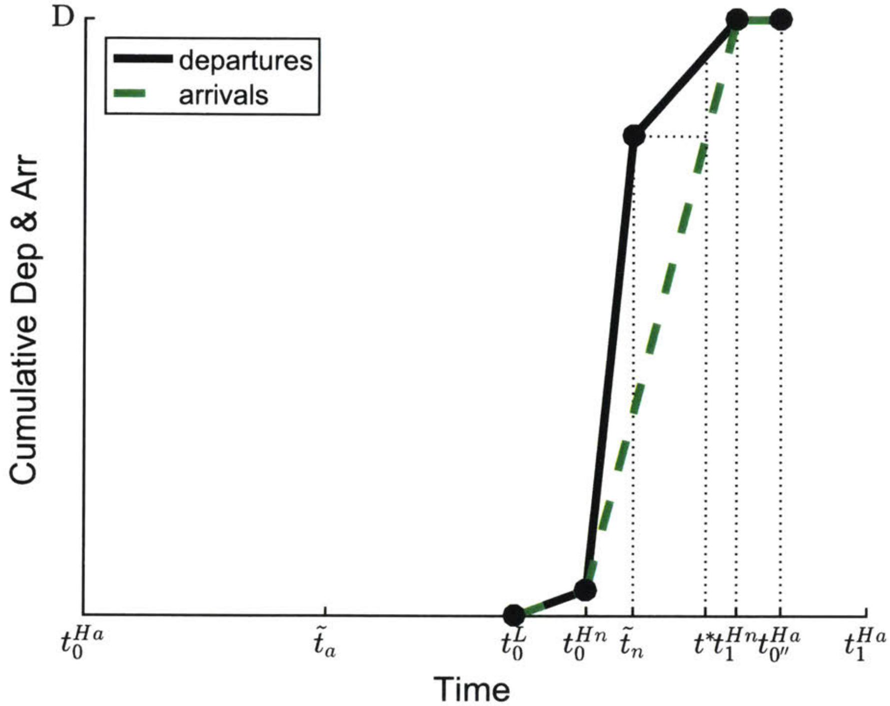
R3: A queue begins to build at t_0^L and does not dissipate by t_0^{Hn}.

In this case, (Fig. 4-4c), there is once again only one interval of queuing, which is $[t_0^L, t_1^{Hn}]$.

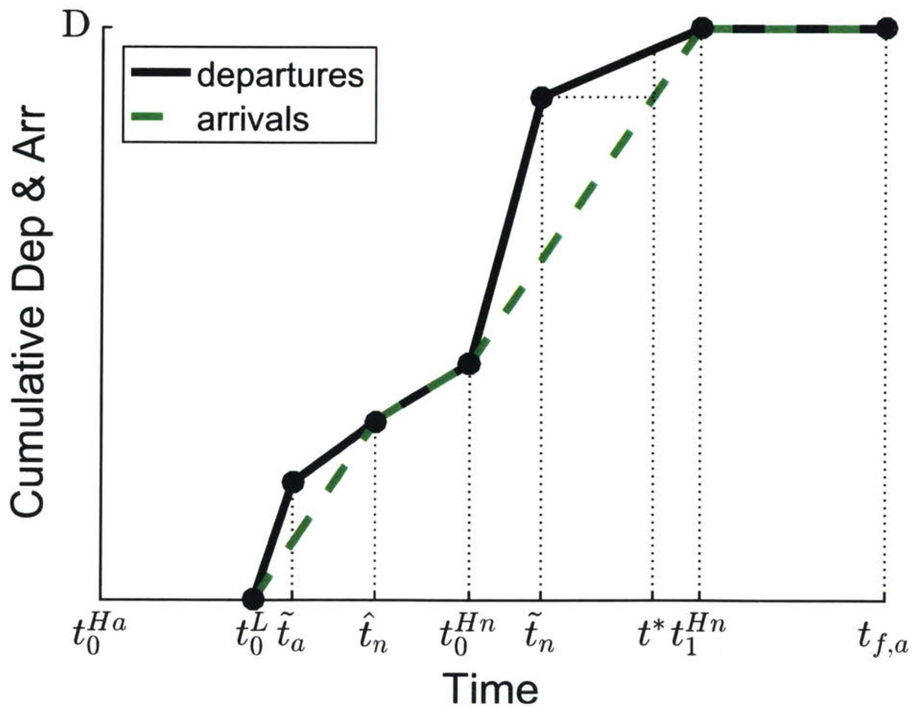
This distinction can also be seen in Figs. 4-5 and 4-6. In Fig. 4-5a, the departure rate is less than c_n throughout the interval $[t_0^L, t_0^{Hn}]$, so no queue forms (R1), whereas in Fig. 4-6a, it is initially greater than c_n, giving rise to a queue (which does not dissipate by t_0^{Hn}, i.e. R3).

This distinction is analogous to the first distinction in the zero information case (see Section 3.3), and there is a correspondence between these subregimes and the regimes R1-R3 of that case.

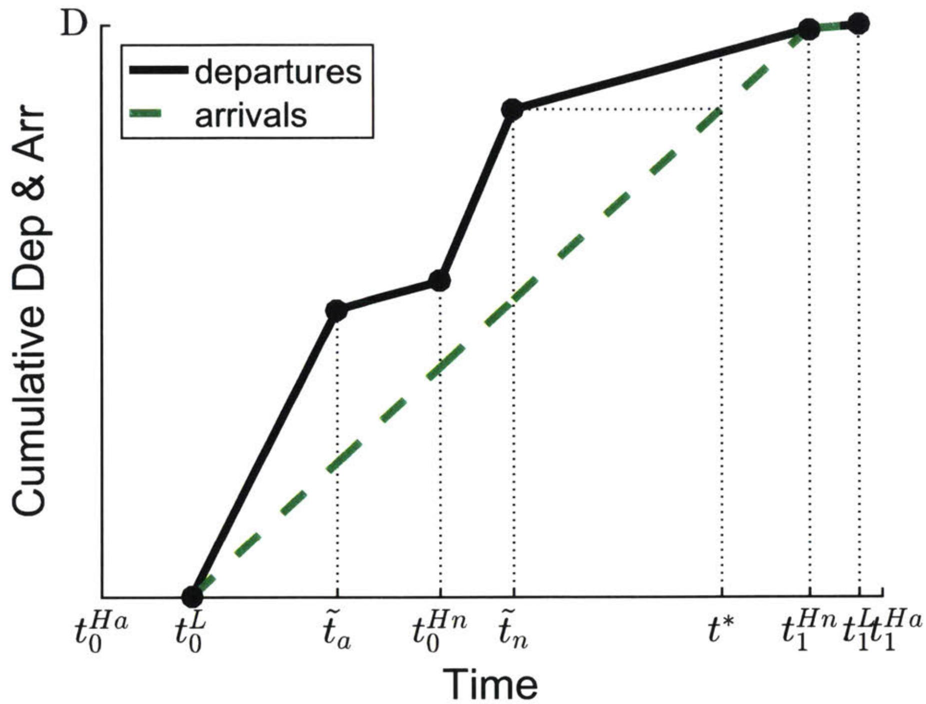
Distinction 2. R·[y]⟨·⟩: Contiguity of $\text{supp}(Ha)$. The second distinction between subregimes is with regards to the departure intervals of type Ha commuters, i.e. $\text{supp}(Ha)$. As noted in Condition 8, $\text{supp}(r^{Ha})$ always includes an interval $[t_0^{Ha}, t_0^L]$ at the beginning of the rush hour. When all type Ha commuters depart within this interval, the subregime is called R·[1]⟨·⟩. On the other hand, when some commuters depart in an additional interval at the end of the rush hour, the subregime is called



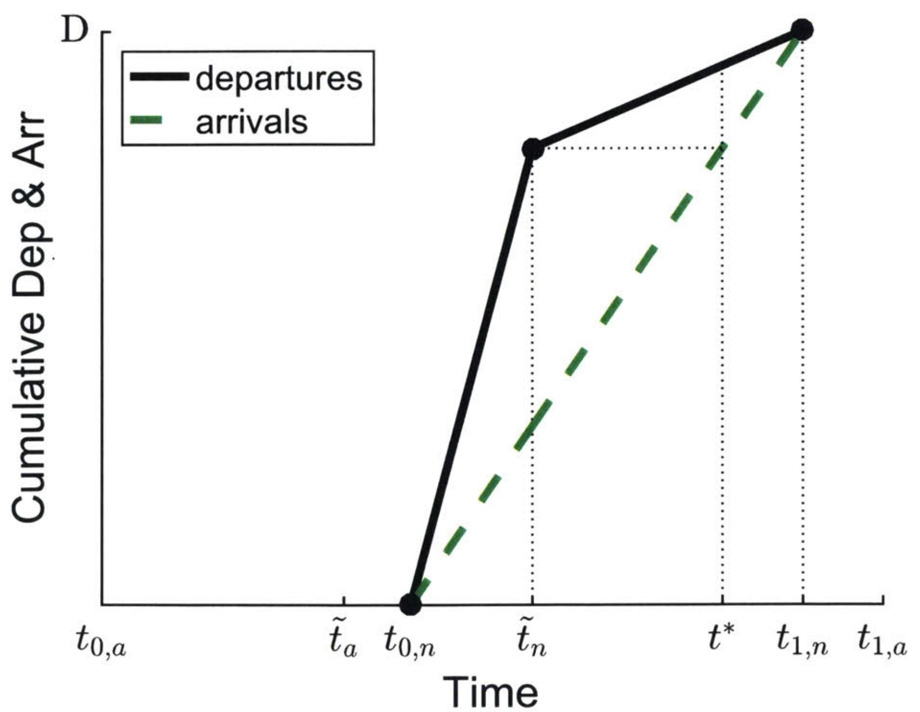
(a) R1[2]⟨1,3⟩ with $\rho = 0.2$, $p = 0.5$, $\lambda = 0.9$



(b) R2[1]⟨2,3⟩ with $\rho = 0.5$, $p = 0.25$, $\lambda = 0.5$



(c) R3[2]⟨2,3⟩ with $\rho = 0.8$, $p = 0.8$, $\lambda = 0.3$



(d) R0⟨1,3⟩ with $\rho = 0.5$, $p = 0.5$, $\lambda = 0.9$

Figure 4-4: Cumulative departures and arrivals in state n. R0 shown for comparison.

R·[2]⟨·⟩, reflecting the two intervals. We denote the first interval by $[t_{0'}^{Ha}, t_{1'}^{Ha}]$, and the second (if it exists) by $[t_{0''}^{Ha}, t_{1''}^{Ha}]$. Thus we can write $t_0^{Ha} = t_{0'}^{Ha}$ for both subregimes, and $t_1^{Ha} = t_{1'}^{Ha}$ (resp. $t_1^{Ha} = t_{1''}^{Ha}$) for R·[1]⟨·⟩ (resp. R·[2]⟨·⟩).

Fig. 4-5b and Fig. 4-6b show an example of each of these subregime types. Note that with the parameters $\beta < \gamma$, it is not possible in equilibrium for departures to occur only in the latter interval, since $C_a(t)$ is at least as low in the earlier interval. Also note that this distinction does not apply to R0 because for any given set of parameter values in R0, there are infinitely many possibilities for supp(r^{Ha}), and it can be composed of any number of distinct intervals.

Distinction 3. R·[·]⟨z⟩: Ordering of $\tilde{t}_n$ and $\tilde{t}_a$. The third distinction is with regards to the order of the pivot times $\tilde{t}_n$ and $\tilde{t}_a$ with respect to the other epochs.[3] This distinction is theoretic and holds little practical intuition or value, but is important in solving for the equilibrium strategies.

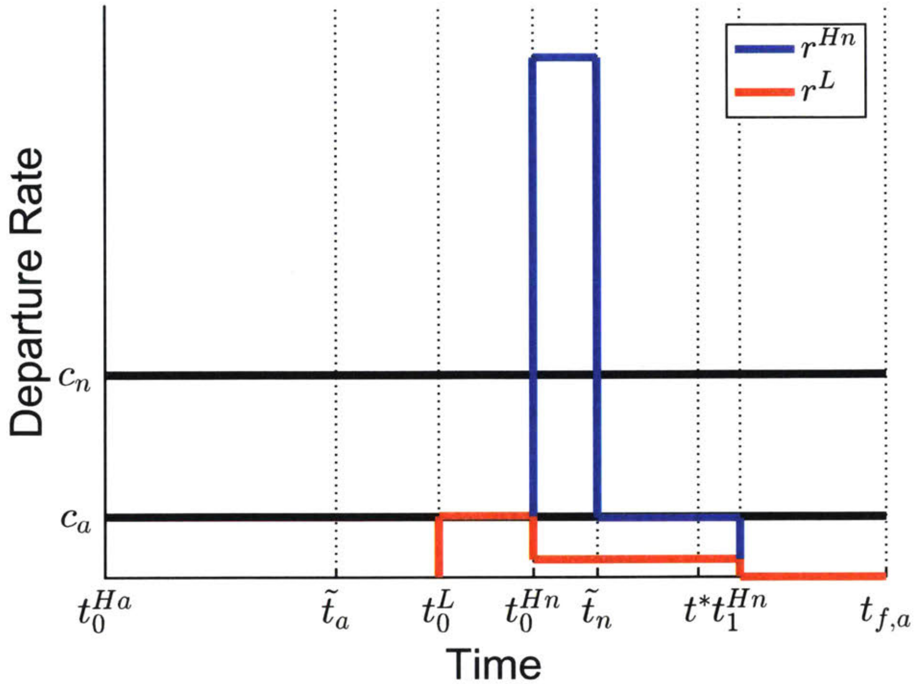
We know from a combination of Conditions 5, 6 and 8 that $t_0^{Ha} < t_0^{L} < t_0^{Hn} < t_1^{Hn}$. Thus, there are 3 intervals in which $\tilde{t}_a$ can be located:

1. $[t_0^{Ha}, t_0^{L})$
2. $[t_0^{L}, t_0^{Hn})$
3. $[t_0^{Hn}, t_1^{Hn})$

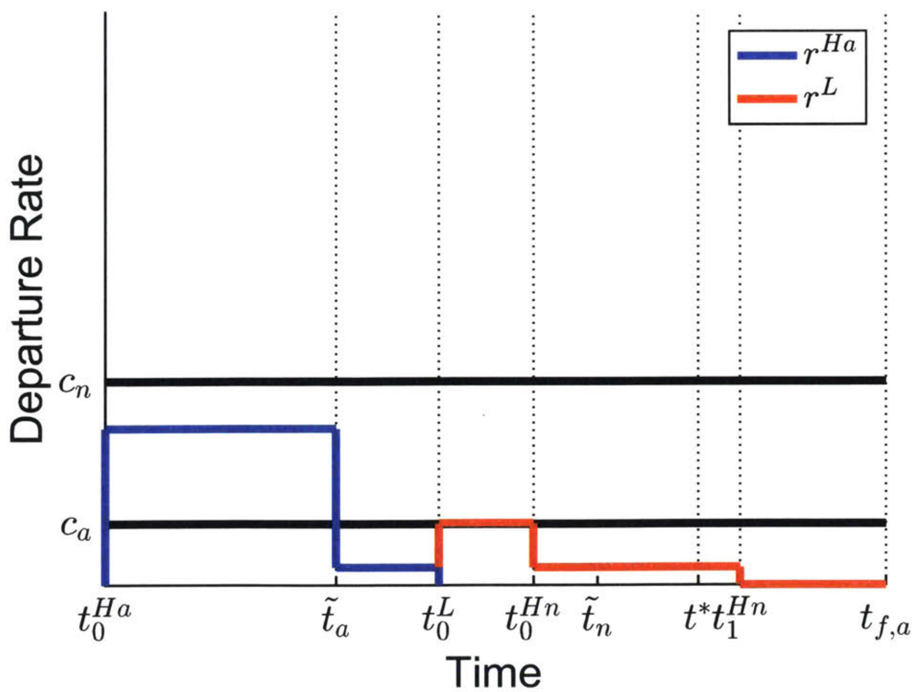
On the other hand, $\tilde{t}_n$ can only be in the latter two intervals. This is because t_0^L is the earliest departure time of any commuter in state n, and some arrivals must necessarily be early, so $t_0^L < \tilde{t}_n$. Further, we note that $\tilde{t}_a < \tilde{t}_n$. Let ⟨z⟩=$\langle z_a, z_n\rangle$ refer to the intervals in which $\tilde{t}_a$ and $\tilde{t}_n$ are respectively located, as enumerated above. Then, there are 5 possible values for ⟨z⟩, which are listed below:

- $\langle z_a, z_n\rangle$=⟨1,2⟩
- $\langle z_a, z_n\rangle$=⟨1,3⟩

[3] Recall that "pivot time" refers to the time of departure such that the arrival time is exactly at t^*.

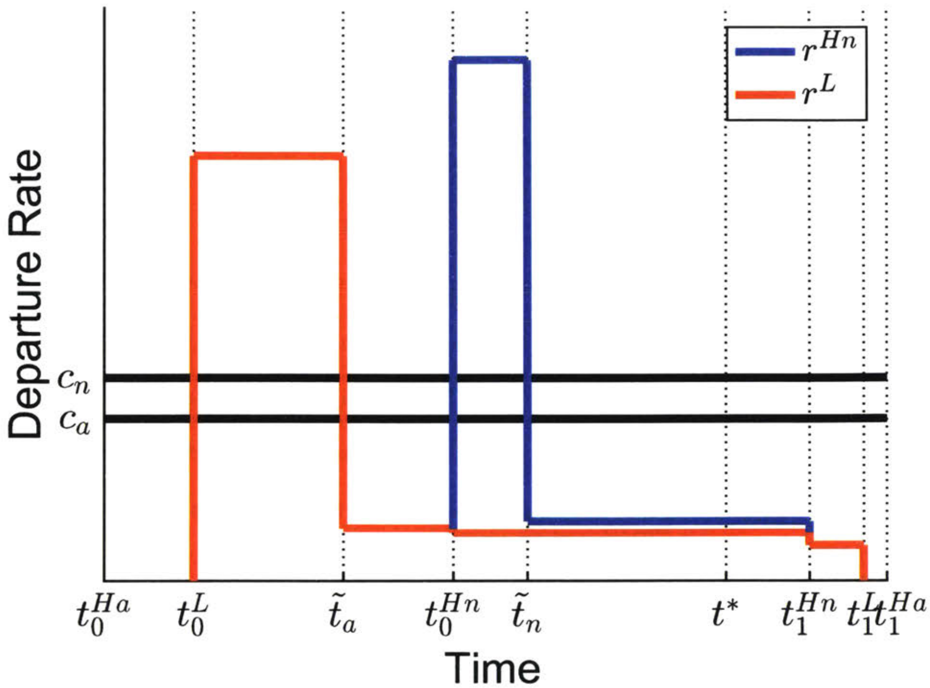


(a) state n

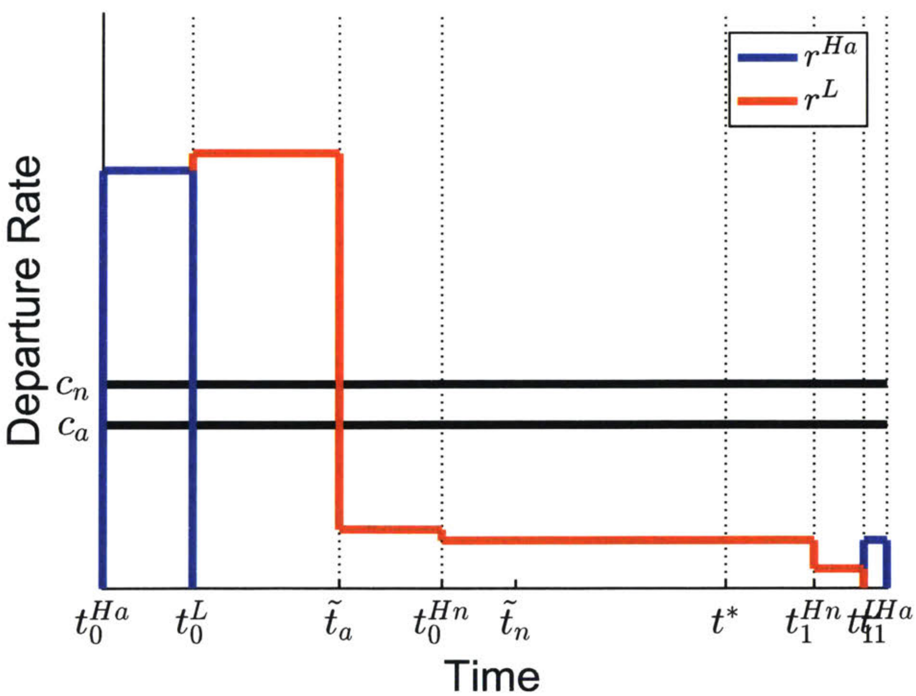


(b) state a

Figure 4-5: Population departure rates in R1[1]⟨1,3⟩ with $\rho = 0.3$, $p = 0.2$, $\lambda = 0.8$.



(a) state n



(b) state a

Figure 4-6: Population departure rates in R3[2]⟨2,3⟩ with $\rho = 0.8$, $p = 0.8$, $\lambda = 0.3$

- $\langle z_a, z_n \rangle = \langle 2{,}2 \rangle$

- $\langle z_a, z_n \rangle = \langle 1{,}3 \rangle$

- $\langle z_a, z_n \rangle = \langle 2{,}3 \rangle$

Fig. 4-4 shows examples of R·[·]⟨1,3⟩ (Fig. 4-4a) and R·[·]⟨2,3⟩ (Figs. 4-4b and 4-4c). These two subregimes account for the majority of the parameter space covered by R0', with the other three existing only for small ranges of parameter values. Where needed, we will use the notation R·[·]⟨·,3⟩, for example, to refer to subregimes with $\tilde{t}_n \in [t_0^{Hn}, t_1^{Hn})$ without any restriction on $\tilde{t}_a$, and similarly for others.

We note here the reason for choosing to refer to the subregimes of R0 as R0⟨1,3⟩ and R0⟨3,3⟩. Although the type-dependent epochs $t_0^{\tau^i}$ and $t_1^{\tau^i}$ are non-unique in R0, there is an analogous interpretation with respect to the state-dependent epochs $t_0^{\tau^i}$. We refer to the subregime in which $\tilde{t}_a \in [t_{0,a}, t_{0,n})$ as R0⟨1,3⟩ and the subregime in which $\tilde{t}_a \in [t_{0,n}, t_{1,n})$ as R0⟨3,3⟩. An example of the former is shown in Fig. 4-4d.

Distinction 4. Ordering of t_1^{Hn} and t_1^L. The final distinction between subregimes is with regards to the ordering of t_1^{Hn} and t_1^L. This distinction is analogous to the RA-RB distinction under zero information, so we use these labels to refer to it where needed. It is suppressed in the notation since it is of less importance than the other distinctions, both in solving for the equilibrium strategies and in welfare analysis.

This distinction inherits the "whether or not there is a non-degenerate interval without departures in the rush hour" interpretation of the RA-RB distinction under zero information, but not the "whether or not the last commuter in state a faces a queue" interpretation. This is because this distinction is concerned with the interval immediately after t_1^{Hn} in state a, which may not be the last interval in the rush hour (in R·[2]⟨·⟩, $[t_{0''}^{Ha}, t_{1''}^{Ha}]$ is the last interval in state a). Specifically, the interval of interest is $(t_1^{Hn}, t_{f,a}]$ in R·[1]⟨·⟩ and $(t_1^{Hn}, t_{0''}^{Ha})$ in R·[2]⟨·⟩.

Recall from Condition 6 that $t_1^L \geq t_1^{Hn}$. When $p \leq \phi_{AB}$, then $t_1^L = t_1^{Hn}$ (RB), since the departure rate $r^L(t) = r_a(t)$ in the interval of interest is 0. Thus there is a non-degenerate interval in state a in which there are no departures, but arrivals are still occurring. On the other hand, when $p > \phi_{AB}$, then $t_1^L > t_1^{Hn}$ (RA) since the $r^L(t)$ is positive in the interval of interest. In this case, there are departures throughout the rush hour in state a.

An example of this distinction can be seen in Figs. 4-5 and 4-6: the subregime in Fig. 4-5 is RB, since no departures occur in the interval immediately after t_1^{Hn}. On the other hand, the subregime in Figs. 4-6 and 4-6 is RA, since it does have departures in that interval.

Note that in R0, the type-dependent epochs $t_1^{\tau^i}$ are non-unique, so this distinction does not apply.

4.3.3 Equilibrium Departure Rates in R0'

In this section, we derive the departure rate functions r^{τ^i}. For each commuter type τ^i, the equilibrium departure rate $r^{\tau^i}(t)$ depends on the *experience* of the commuter departing at time t under the equilibrium strategy profile. For each state s, a commuter departing at a given time will either arrive early ($t \leq \tilde{t}_s$) or late ($t > \tilde{t}_s$), and either face a queue or not. Thus for a given state, a commuter faces one of the following experiences:

- **eq**: early arrival (e) and facing queue (q)
- **ez**: early arrival (e) and no queue (z)
- **lq**: late arrival (l) and facing queue (q)
- **lz**: late arrival (l) and no queue (z)

Over the two-state set S, a commuter's experience can be represented as tuples of the above mentioned types, representing the conditions he would face if he departed at the same time t in both states. For example, (ez,lq) represents experiencing early arrival and no queue in state n and late arrival and a queue in state a. Thus there are $4 \times 4 = 16$ types of experiences, not all of which can exist in equilibrium. Furthermore, we use $(\cdot$,lq), for example, to refer to experiencing late arrival and a queue in state a without any restriction on the experience in state n.

Theorem 4 lists the values of the departure rate functions r^{τ^i}.

Theorem 4. *The departure rate functions r^{Hn}, r^{Ha} and r^L are given in Table 4.2, Table 4.3 and Table 4.4 respectively.*

Interval	Experience	$r^{Hn}(t)$
$[t_0^{Hn}, \tilde{t}_a]$	(eq,eq)	$\frac{\alpha(c_n - c_a)}{\alpha - \beta}$
$[\tilde{t}_a, \tilde{t}_n]$	(eq,lq)	$\frac{\alpha c_n}{\alpha - \beta} - \frac{\alpha c_a}{\alpha + \gamma}$
$[\tilde{t}_n, t_1^{Hn}]$	(lq,lq)	$\frac{\alpha(c_n - c_a)}{\alpha + \gamma}$

Table 4.2: r^{Hn}

Interval	Experience	$r^{Ha}(t)$
$[t_{0'}^{Ha}, \min\{t_{1'}^{Ha}, \tilde{t}_a\}]$	$(\cdot,\text{eq})$	$\frac{\alpha c_a}{\alpha - \beta}$
$[\tilde{t}_a, t_{1'}^{Ha}]$, $[t_{0''}^{Ha}, t_{1''}^{Ha}]$	$(\cdot,\text{lq})$	$\frac{\alpha c_a}{\alpha + \gamma}$

Table 4.3: r^{Ha}

Theorem 4 is proved in Appendix C, giving us complete expressions for r^{Hn}, r^{Ha} and r^L. Using these, we can determine the values of the epochs using a system of linear equations, as shown next.

4.3.4 Equilibrium Departure Times in R0'

In this section, we show how the departure rates calculated in Section 4.3.3 can be used to determine the values of the epochs using a system of linear equations. We then argue that this system of linear equations always has a unique solution, proving the uniqueness of equilibrium claimed in Theorem 3. The equations linking the epochs are described below:

Interval	Experience	$r^L(t)$
$[t_0^L, \tilde{t}_a]$	(ez,eq)	$\frac{c_a(p\alpha+(1-p)\beta)}{p(\alpha-\beta)}$
$[\max\{t_0^L, \tilde{t}_a, \hat{t}_n\}, t_0^{Hn}]$	(ez,lq)	$\frac{c_a(p\alpha+(1-p)\beta)}{p(\alpha+\gamma)}$
$[t_1^{Hn}, \min\{\hat{t}_a, t_{0''}^{Ha}\}]$	(lz,lq)	$\max\{\frac{c_a(p\alpha-(1-p)\gamma)}{p(\alpha+\gamma)}, 0\}$
$[t_0^L, \min\{\tilde{t}_a, t_0^{Hn}\}]$	(eq,eq)	$\frac{\alpha}{(\alpha-\beta)\frac{p}{c_a}+(\alpha-\beta)\frac{1-p}{c_n}}$
$[\max\{t_0^L, \tilde{t}_a\}$, $\min\{\hat{t}_n, \tilde{t}_n, t_0^{Hn}\}]$	(eq,lq)	$\frac{\alpha}{(\alpha+\gamma)\frac{p}{c_a}+(\alpha-\beta)\frac{1-p}{c_n}}$
$[\tilde{t}_n, t_0^{Hn}]$	(lq,lq)	$\frac{\alpha}{(\alpha+\gamma)\frac{p}{c_a}+(\alpha+\gamma)\frac{1-p}{c_n}}$
$[t_0^{Hn}, \tilde{t}_a]$	(eq,eq), $t \in \text{supp}(r^{Hn})$	$\frac{c_a\alpha}{\alpha-\beta}$
$[\max\{t_0^{Hn}, \tilde{t}_a\}, t_1^{Hn}]$	(eq,lq) & (lq,lq), $t \in \text{supp}(r^{Hn})$	$\frac{c_a\alpha}{\alpha+\gamma}$

Table 4.4: r^L ($t \notin \text{supp}(r^{Hn})$ unless stated otherwise)

- **Feasibility Conditions (3 eqs.):** These are "conservation of commuters" type equations for each of the three commuter types. They follow directly from the first feasibility condition (2.4) for strategy profiles, and ensure that the integral of $r^{\tau^i}(t)$ over time equals the number of commuters of type τ^i:

$$\int_{t_0^{\tau^i}}^{t_1^{\tau^i}} r^{\tau^i}(t)\ \ \mathrm{d}t = \lambda^i D, \quad \forall \tau^i \in \tau^i, i \in I \tag{4.5}$$

These 3 equations would appear to have 6 unknowns, but there is a further restriction: combining Conditions 7 and 8, we note that $t_{1'}^{Ha} = t_0^L$. For R·[1]⟨·⟩, $t_1^{Ha} = t_{1'}^{Ha}$, so these equations have 5 degrees of freedom. For R·[2]⟨·⟩, there are two additional unknowns: $t_{0''}^{Ha}$ and $t_{1''}^{Ha}$. These are discussed below under "Equal Cost In Both Intervals of supp(r^{Ha})". Additionally, t_1^L is also always equal to one of the other epochs, but for different subregimes the epoch it is equal to is different. These different cases are enumerated later, towards the end of this section.

- **Pivot Time Definitions (2 eqs.):** These equations ensure that the definition of the pivot time $\tilde{t}_s$ is satisfied for each state, i.e. that a commuter departing at $\tilde{t}_s$ in state s arrives at t^*. This requires the queuing time at $\tilde{t}_s$ in state s to be $t^* - \tilde{t}_s$:

$$q_s(\tilde{t}_s) = \int_{T_s(\tilde{t}_s)}^{\tilde{t}_s} \frac{r_s(t) - c_s}{c_s} \, \mathrm{d}t = t^* - \tilde{t}_s, \quad \forall s \in S \tag{4.6}$$

These 2 equations appear to add 4 degrees of freedom to the system. However, both $T_n(\tilde{t}_n)$ and $T_a(\tilde{t}_a)$ happen to be equal to one of the other epochs, so these equations add two degrees of freedom. For $T_a(\tilde{t}_a)$, it follows from Condition 4 that it is equal to the first departure time in state a, which, from Condition 8, is t_0^{Ha}. The argument for $T_n(\tilde{t}_n)$ is more involved. First, recall from Distinction 1 that $\tilde{t}_n < t_1^{Hn}$. Then note that, as proved in Section 4.4, $\tilde{t}_n > t_0^{Hn}$ in R1 and R2. Combining these arguments with Condition 3, we conclude that $T_n(\tilde{t}_n) = t_0^{Hn}$ in both R1 and R2. In R3, the queue lasts from t_0^L to t_1^{Hn}, and $\tilde{t}_n > t_0^L$ as shown in Distinction 1. Thus we conclude that $T_n(\tilde{t}_n) = t_0^L$ in R3.

- **Queue Dissipation Time Definitions (2 eqs.):** Similar to the previous equations, these equations ensure that the definition of the queue dissipation time $\hat{t}_s$ is satisfied for each state:

$$q_s(\hat{t}_s) = \int_{T_s(\hat{t}_s)}^{\hat{t}_s} \frac{r_s(t) - c_s}{c_s} \, dt = 0, \quad \forall s \in S \tag{4.7}$$

These equations add only degree of freedom to the system since they introduce three additional correlations between the epochs:

1. First, recall from Distinction 1 that there is only one interval of queuing in R1 and R3 in state n. Therefore $T_n(\hat{t}_n)$ is equal to the time the queue begins, which is t_0^{Hn} in R1 and t_0^L in R3. In R2, recall that $\hat{t}_n$ refers to the dissipation time of the second queue, which starts at t_0^{Hn} (the dissipation

time of the first queue, $\hat{t}_{n'}$, requires an additional equation, described below under "Additional Queue Dissipation Time"). Therefore, in R2, $T_n(\hat{t}_n) = t_0^{Hn}$. Note that in all cases (R1-R3), $T_n(\hat{t}_n) = T_n(\tilde{t}_n)$.

2. Next, the argument for $T_a(\hat{t}_a)$ is identical to that for $T_a(\tilde{t}_a)$ above, giving $T_a(\hat{t}_a) = t_0^{Ha}$.
3. Next, recall from Distinction 1 that $\hat{t}_n = t_1^{Hn}$.

These 7 equations thus form a linear system with exactly 7 unknowns, once the equality of t_1^L to one of the other epochs is considered. When the subregime is not R2 or R·[2]⟨·⟩ (or both), there are no further epochs, and the system yields a unique solution. In R2 and R·[2]⟨·⟩, there are additional epochs, described below:

- R2 - **Additional Queue Dissipation Time:** Recall from Distinction 1 that there are two intervals of queuing in state n in R2. In this case, $\hat{t}_n$ refers to the dissipation time of the second queue, and has been considered previously in (4.7). The dissipation time of the first queue, $\hat{t}_{n'}$, requires another equation analogous to (4.7):

$$\int_{T(\hat{t}_{n'})}^{\hat{t}_{n'}} \frac{r_n(t) - c_n}{c_n} \, dt = 0 \tag{4.8}$$

  Recall from Distinction 1 that the first queue in R2 begins at t_0^L, so that $T(\hat{t}_{n'}) = t_0^L$. Thus this equation adds one additional degree of freedom to the system, which ensures that the solution is still unique.

- R·[2]⟨·⟩ - **Equal Cost In Both Intervals Of** $\text{supp}(r^{Ha})$**:** Recall that in R·[2]⟨·⟩, type Ha commuters depart in two separate intervals, adding the epochs $t_{0''}^{Ha}$ and $t_{1''}^{Ha}$. This case introduces an additional equation: it follows from Condition 1 that $C_a(t)$ must be equal for all $t \in [t_{0'}^{Ha}, t_{1'}^{Ha}] \bigcup [t_{0''}^{Ha}, t_{1''}^{Ha}]$. The departure rate r^{Ha} given in Table 4.3 ensures that $C_a(t)$ is constant *within* each of the two intervals, so it suffices to ensure that $C_a(t') = C_a(t'')$ for some $t' \in [t_{0'}^{Ha}, t_{1'}^{Ha}]$, $t'' \in [t_{0''}^{Ha}, t_{1''}^{Ha}]$. Choosing $t' = t_{0'}^{Ha}$ and $t'' = t_{1''}^{Ha}$ for ease, we get the following equation:

$$\beta(t^* - t_{0'}^{Ha}) = \gamma(t_{1''}^{Ha} - t^*) \tag{4.9}$$

However, from Condition 4, $\hat{t}_a = t_1^{Ha}$ in R·[2]⟨·⟩, so that there is only one degree of freedom added to the system.

Finally, we enumerate the different epochs t_1^L is equal to in different subregimes. In RB, $t_1^L = t_1^{Hn}$, as described in Distinction 4. In RA, either $t_1^L = \hat{t}_a$ (in R·[1]⟨·⟩, by Condition 4) or $t_1^L = t_{0''}^{Ha}$ (in R·[2]⟨·⟩).

Thus in all cases we get a perfectly determined system of equations, which we solve computationally to get the (unique) numerical values of the epochs. This completes the proof of Theorem 3.

It is important to note that it is only in R0 that there are analytical thresholds based on which we can determine which subregime the equilibrium falls in. In R0', the correct regime for a given set of parameter values (ρ, p, λ) can be determined computationally by brute force: the linear system is solved for each subregime, and the one which results in the correctly ordered epochs is the correct equilibrium strategy profile.

4.4 Equilibrium Characterization

In this section, we tie together the equilibrium characterization in two ways. Firstly, we enumerate the subregimes that exist in equilibrium. Secondly, we comment on the intuition behind why the equilibrium strategy changes from one subregime to another as the parameter values change.

Given the distinctions described in Section 4.3.2, with the A/B distinction excluded, it would appear there are $3 \times 2 \times 5 = 30$ possible subregimes in R0'. However, not all combinations of the various distinctions are possible in equilibrium:

- First, we note that subregimes R·[·]⟨·,2⟩ are not possible in R1 and R2. The queue length in an interval increases when $r_s(t) > c_s$. This can only happen when arrival is early, i.e. for $t \leq \tilde{t}_s$. In R1 and R2, there is no queue at t_0^{Hn} in state n, so by Condition 3, a queue must start building up at t_0^{Hn}. Therefore $\tilde{t}_n$ must fall after t_0^{Hn}, which means that the subregime is R·[·]⟨·,3⟩.

Thus there are 6 possible subregimes in R1, of which R1[1]⟨3,3⟩ and R1[2]⟨3,3⟩ do not exist in equilibrium under the values of the fixed parameters we consider.

- In R2, there is a further restriction. In state n, for the queue to build up starting at t_0^L and then to dissipate at $\hat{t}_{n'} < t_0^{Hn}$, there must be two different values of $r_n(t) = r^L(t)$ in the interval $[t_0^L, \hat{t}_{n'}]$, the first being greater than c_n and the second being smaller. Since we have already shown above that $\tilde{t}_n > t_0^{Hn}$ in R2, the only remaining possibility is that $\tilde{t}_a \in [t_0^L, \hat{t}_{n'}]$. This means that only subregimes R·[·]⟨2,3⟩ are possible in R2. Thus the only subregimes in R2 are R2[1]⟨2,3⟩ and R2[2]⟨2,3⟩.

- Finally, in R3, $\langle \tilde{t}_a, \tilde{t}_n \rangle$ can take all 5 of its possible values. Therefore there are 10 subregimes possible in R3. Among these subregimes, R3[2]⟨1,2⟩ does not exist in equilibrium under the values of the fixed parameters we consider.

Including the two subregimes that comprise R0, this gives a total of 17 subregimes which exist in equilibrium, which are depicted in Fig. 4-7.

The partitioning of the parameter space into the different subregimes, which is quite complex, is shown in Fig. 4-8. Some of the distinctions between the subregimes are peculiarities of this particular model and are not expected to be robust in real scenarios. However, others do have intuition and are discussed below.

Firstly, just like in the zero information case, equilibrium shifts from R1 to R2 to R3 as ρ increases. This is because departures and arrivals in state n become more compact and resemble those in state a more and more as ρ increases, leading to queuing occurring for longer periods of time. R0 can be considered the regime in which information is distributed widely enough (i.e. λ is sufficiently high) that even uninformed commuters (population L) enjoy the full benefit of it, and face equal costs as the informed commuters (population H). Thus this regime exists for high values of λ, i.e. above the threshold λ^*.

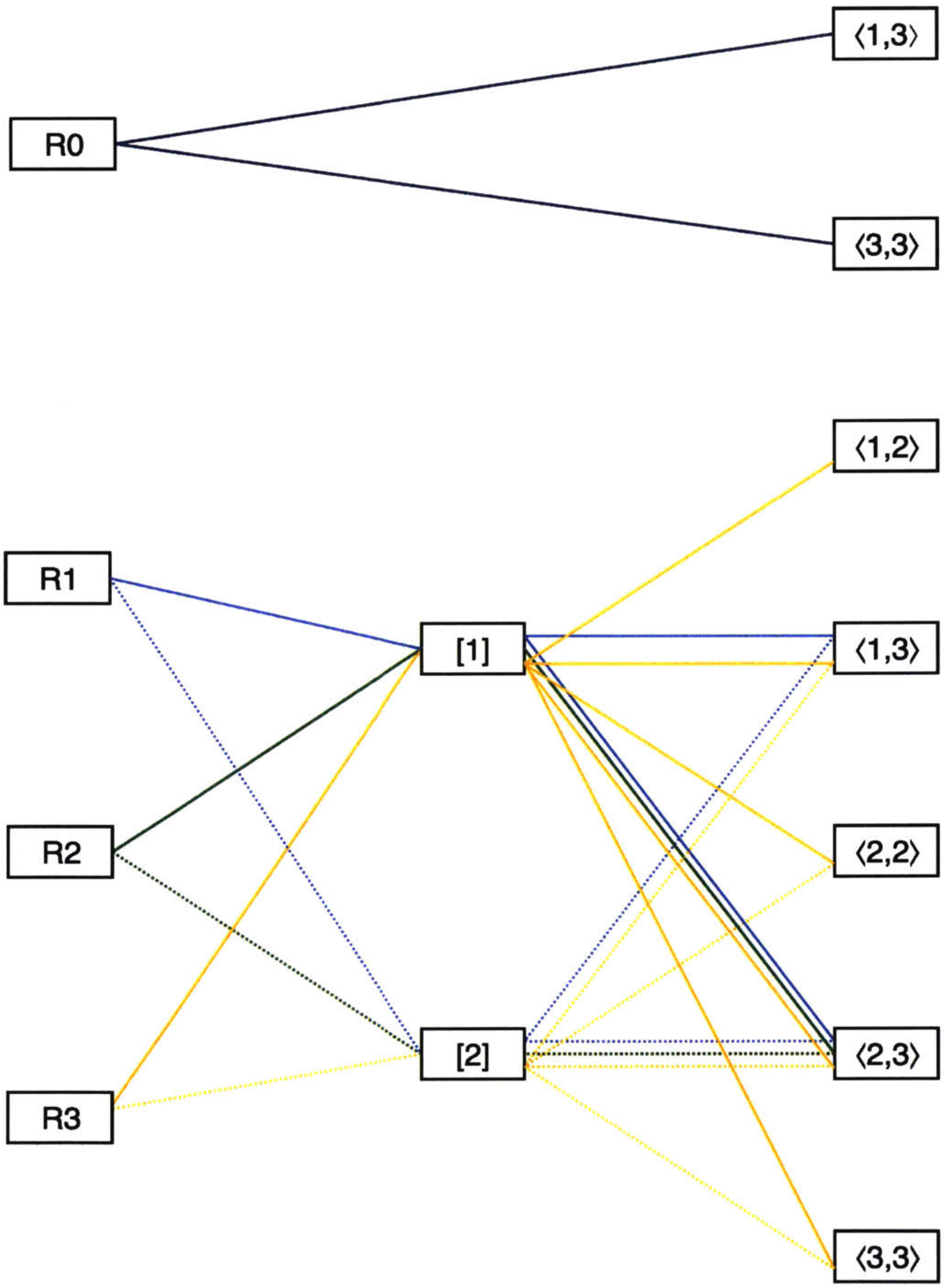
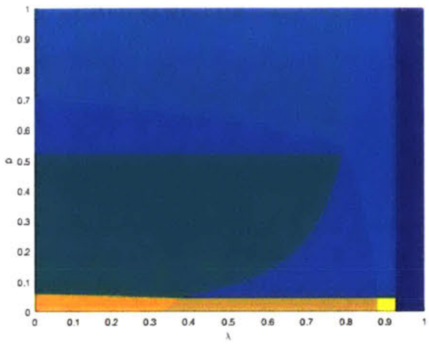


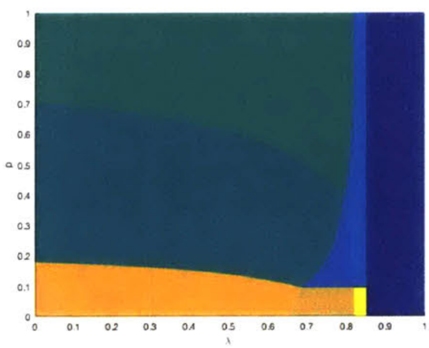
Figure 4-7: Equilibrium categorization into R0 (top) and R0' (bottom), and sub-regimes of each.

Secondly, $R\cdot[1]\langle\cdot\rangle$ exists for low values of p and λ while $R\cdot[2]\langle\cdot\rangle$ exists for high values. As λ increases, the congestion caused in the beginning of the rush hour due to type Ha commuters increases, until a point where departing at the beginning of the rush hour becomes as costly as departing at the end. At this point, equilibrium shifts from $R\cdot[1]\langle\cdot\rangle$ to $R\cdot[2]\langle\cdot\rangle$ as some type Ha players depart in an interval at the end of the rush hour. Similarly, as p increases, the departures of type L commuters shift earlier in order to avoid high chances of late arrivals, once again making it viable for some of the type Ha commuters to travel at the end of the rush hour.

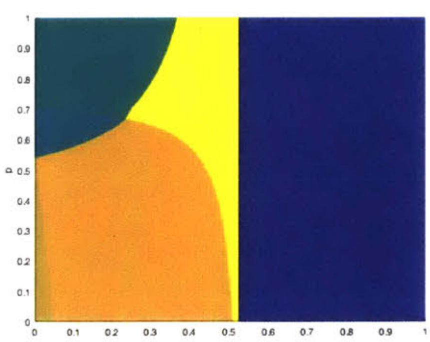
This completes the equilibrium characterization for the complete parameter space $\rho \times p \times \lambda \in (0,1] \times [0,1] \times [0,1]$. We now turn our attention to the analysis of the equilibrium costs in Chapter 5.



(a) $\rho = 0.25$



(b) $\rho = 0.5$



(c) $\rho = 0.75$

Figure 4-8: Partition of parameter space into regimes for various values of ρ. R0 is shown in dark blue, R1 in light blue, R2 in green, and R3 in yellow. The various shades of each color represent the subregimes of each; in particular, for R0', darker shades represent R·[1]⟨·⟩ while lighter shades represent R·[2]⟨·⟩.

Chapter 5

Welfare Analysis

In this chapter, we analyze how the equilibrium costs vary with the parameters of the game, in particular with the information penetration λ. In Section 5.1, we examine the populations' individual costs $\mathbb{E}[\bar{C}^H]$ and $\mathbb{E}[\bar{C}^L]$ defined in (2.18), whereas in Section 5.2, we examine the social cost $\mathbb{E}[\bar{C}]$ defined in (2.19). Our equilibrium characterization in Chapter 4 enables us to analyze the effects of changing the parameters (ρ, p, λ) on the aforementioned costs. This is because our results highlight the rich set of equilibrium behaviors and how the parameters influence the equilibrium strategies. In particular, the uniqueness of equilibrium strategies in R0' (Theorem 3) and the essential uniqueness in R0 (Theorem 2) ensure that these costs are unique for given parameter values. Although our equilibrium characterization allows us to consider the effects of changing any of the parameters, we mainly consider how the costs vary with λ for a given (ρ, p), since our main focus is the effect of information. Before starting our analysis however, we first describe how the the individual and social costs are calculated based on their definitions in Section 2.4.

Once the epoch values have been found by solving the set of linear equations described in Section 4.3.4, the state dependent cost functions C_s are calculated using (2.8). The equilibrium condition (2.14) implies that all commuters of a given type face the same expected cost, i.e. $\mathbb{E}[C^{\tau^i}(t)]$ is constant over $t \in \text{supp}(r^{\tau^i})$. This means that the expected cost faced by any type τ^i commuter is equal to the individual cost $\mathbb{E}[\bar{C}^{\tau^i}]$ for that type. Thus the individual cost $\mathbb{E}[\bar{C}^L]$ of the uninformed commuters is given by

$$\mathbb{E}[\bar{C}^L] = pC_a(t) + (1-p)C_n(t) \tag{5.1}$$

by substituting any $t \in \text{supp}(r^L)$, since that gives the expected cost faced by the the (uninformed) commuter departing at that time. Similarly, we find the individual costs $\bar{C}^{Hs}$ of type Hs commuters as $C_s(t)$ by substituting any $t \in \text{supp}(r^{Hs})$ thanks to Condition 1. The steps described above provide a shorthand method of calculating the individual costs of the three player types by considering one commuter of each type as opposed to averaging over all commuters as in (2.16) or (2.17).

Next, substituting the values for $\bar{C}^{Hs}$ into (2.18) gives us the individual cost $\mathbb{E}[\bar{C}^H]$ of population H. Finally, substituting the individual costs $\mathbb{E}[\bar{C}^H]$ and $\mathbb{E}[\bar{C}^L]$ into (2.19) gives us the social cost $\mathbb{E}[\bar{C}]$.

The benchmark we use to evaluate the effectiveness of information in reducing travel costs is the corresponding cost when no information is available, i.e. the zero information cost defined in Section 3.3. Thus all costs mentioned in this section are after being normalized with respect to the zero information cost $\mathbb{E}[\bar{C}_0]$. Note that, for a given (ρ, p), $\mathbb{E}[\bar{C}_0]$ is the social cost $\mathbb{E}[\bar{C}]$ with $\lambda = 0$.

5.1 Individual Cost

In this section we analyze how the individual costs $\mathbb{E}[\bar{C}^i]$ for each population are affected by changing the parameters (ρ, p, λ), and in particular by increasing the information penetration λ. For given values of (ρ, p), we examine how $\mathbb{E}[\bar{C}^H]$ and $\mathbb{E}[\bar{C}^L]$ vary with λ. Fig. 5-1 shows this for $\rho = 0.5$, $p = 0.25$. This figure is representative of most values of (ρ, p), and unless specifically mentioned, all features described below apply for all values.

The first notable observation is that $\mathbb{E}[\bar{C}^H]$ is monotonically increasing in λ below the threshold λ' given in (4.1), and constant thereafter. This means that as the fraction of informed commuters increases upto λ', the expected cost faced by each informed commuter also increases. As the informed fraction increases beyond λ', the cost faced by the informed commuters does not change any further. Recall from

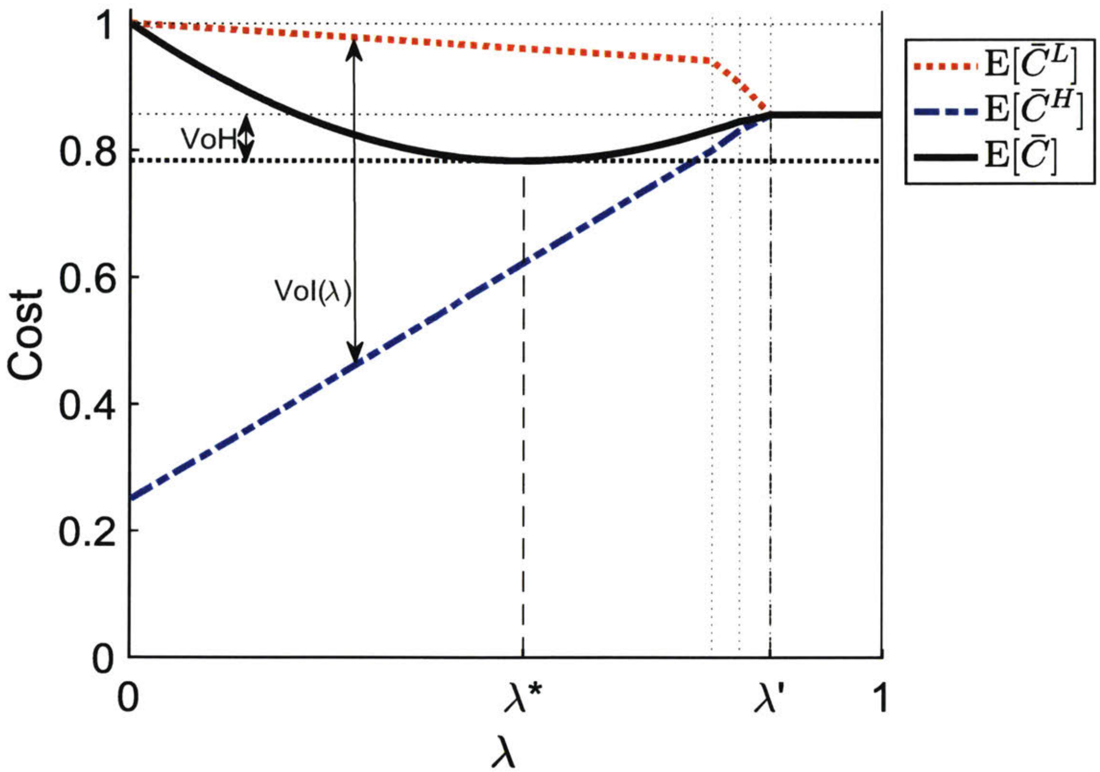


Figure 5-1: Individual and social costs versus λ for $\rho = 0.5$, $p = 0.25$.

Theorem 2 that λ' is the minimum fraction of informed commuters required for the equilibrium to lie in R0. The intuition behind $\mathbb{E}[\bar{C}^H]$ increasing with λ in R0' is that as more commuters have access to a TIS that informs them about the state, they adjust their departure times accordingly, traveling in the middle of the rush hour in state n and at the beginning (and possibly the end) in state a. The increasing number of commuters choosing to travel at these times causes more congestion in supp(r^{Hs}) and increases the average cost they face. The benefits of information for informed commuters can thus be viewed as decreasing as a whole due to congestion externalities while simultaneously being shared among a greater number of (informed) commuters. It can be seen from Fig. 5-1 that the increase in $\mathbb{E}[\bar{C}^H]$ is roughly linear with λ until λ'. For other parameter values, it may be piecewise linear with different slopes in different subregimes (see Fig. 5-2).

On the other hand, with the exception of certain values of (ρ, p) discussed below, $\mathbb{E}[\bar{C}^L]$ decreases as λ increases in R0'. This means that the benefits of some commuters being informed is seen by the uninformed commuters as well. The informed commuters, by way of their departure time decisions, impose a lesser negative externality on the uninformed commuters relative to when the informed fraction is smaller. For the parameter values $\rho = 0.5$, $p = 0.25$ in Fig. 5-1, the decrease in $\mathbb{E}[\bar{C}^L]$ is monotonic, but this is not necessarily the case.

In particular, for low values of p (upto about 0.2) and certain corresponding values of ρ, $\mathbb{E}[\bar{C}^L]$ increases in λ for some range of values of λ. In fact, $\mathbb{E}[\bar{C}^L]$ may actually increase to above 1 in these cases. See Fig. 5-2 for an example of this with $\rho = 0.4$, $p = 0.1$. In other words, under these parameter values, population H's departure time decisions make population L worse off relative to under zero information for intermediate values of λ. This is because, for these parameter values, the benefits of better decision making by the population H commuters are outweighed by the congestion externality they impose on the population L commuters. Interestingly, $\mathbb{E}[\bar{C}^L]$ increases in λ only when the equilibrium lies in R2[1]⟨2,3⟩. The main factor in this increase is the higher queuing cost faced by uninformed commuters in state a.

Having examined how the individual costs change with λ, we now turn our attention to the benefit of gaining access to information. We consider the *Relative Value of Information*, denoted VoI, to be the reduction in the travel costs resulting from its knowledge. Thus for given parameter values (ρ, p, λ), we define VoI as the difference between the individual costs faced by the two populations in equilibrium:

$$\text{VoI} = \mathbb{E}[\bar{C}^L] - \mathbb{E}[\bar{C}^H]. \tag{5.2}$$

VoI is therefore the travel cost savings that population H commuters enjoy over population L commuters due to their access to a TIS. We are mainly concerned with how VoI varies with λ for given values of (ρ, p). When we consider VoI as a function of λ, we use the notation VoI(λ) to emphasize this dependence. The main property of VoI(λ) is stated below:

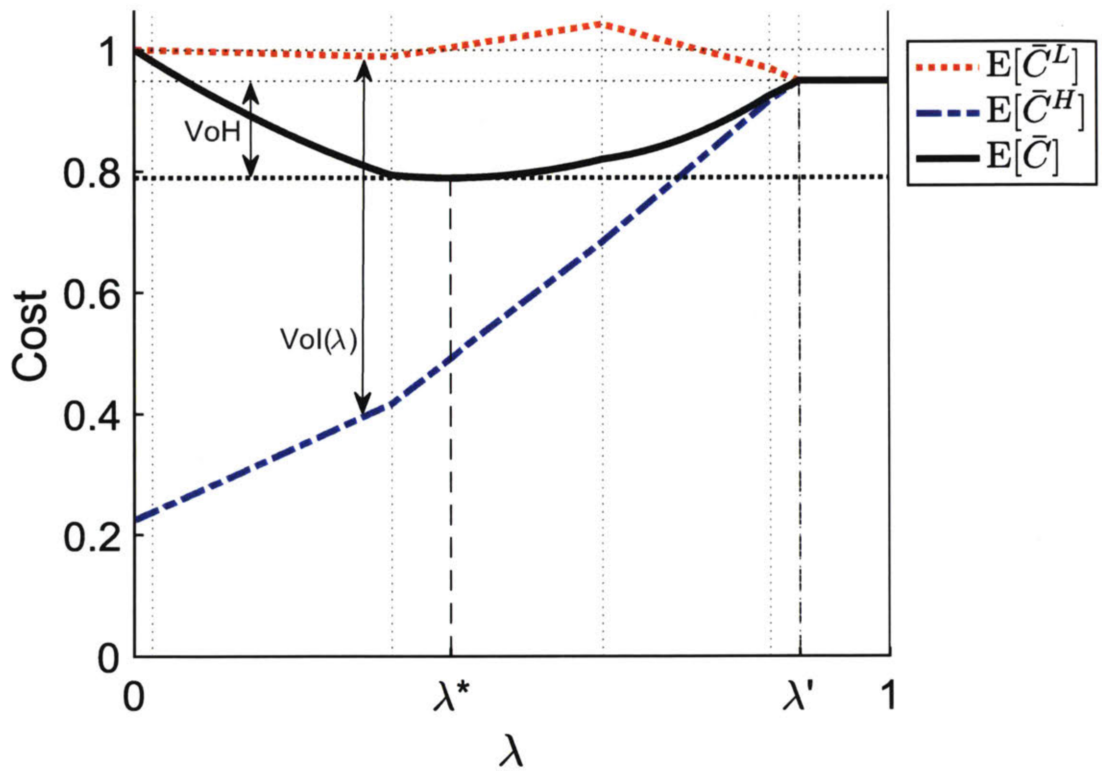


Figure 5-2: Individual and social costs versus λ for $\rho = 0.4$, $p = 0.1$, showing exceptional case where $\mathbb{E}[\bar{C}^L]$ increases with λ.

For any given (ρ, p), VoI(λ) *is positive and decreasing in* λ *in* R0' *(i.e. for* $\lambda < \lambda'$*), and* 0 *in* R0 *(i.e. for* $\lambda \geq \lambda'$*).*

This means that population H never faces a higher individual cost than population L. The non-negativity of VoI is intuitive, but not obvious. In particular, as noted in Arnott et al. (1991) there are situations in models of (symmetric) imperfect information ($\eta < 1$) in which commuters are better off when they all have no information than when they all have imperfect information. However, when the information provided is perfectly accurate (i.e. the full information case considered in Section 3.2), then commuters are better off when they have information. In our model, VoI is defined as the difference between costs faced by uninformed commuters and perfectly informed commuters in the same trip, and is never negative. We speculate that relaxing Assumption 3 and allowing the information provided to be imperfect ($\eta^H < 1$)

would result in a similar situation to Arnott et al. (1991) where VoI would indeed be negative for large λ.

For $\lambda < \lambda'$, VoI(λ) decreases due to to the changes in its components: as noted above, $\mathbb{E}[\bar{C}^H]$ is increasing in λ while $\mathbb{E}[\bar{C}^L]$ is mostly decreasing (except for low values of p coupled with certain low-moderate values of ρ). This indicates that gaining information is most beneficial when few others have it, and its benefit diminishes as it becomes more common. For $\lambda \geq \lambda'$, the two populations face the same individual costs. Recall from Section 4.2 that this is the regime R0, where, as noted in Theorem 2, $\mathbb{E}[\bar{C}^H]$ and $\mathbb{E}[\bar{C}^L]$ are equal and constant, so that VoI(λ) $= 0$. As discussed in Section 4.2, this is because in R0, type L commuters are too few to have an impact on the departure rate, and the state dependent departure rate functions r_s are equal to the state dependent departure rate functions $r_{s,1}$ under full information ($\lambda = 1$). Thus, given a state, all commuters face the same cost.

VoI(λ) falling to 0 for $\lambda \geq \lambda'$ indicates that information penetration reaches such an extent that its complete benefits are seen by even those who are not informed. In other words, the impact of information (or size of the informed population who benefit as a result of being informed relative to the uninformed population) reaches the full size of the informed population. Thus, its relative value drops to 0; a type L commuter receives no benefit by gaining access to information because the impact of information has reached its maximum level and increasing the size of the informed population further does not maintain positive VoI(λ). Thus λ' can be seen as the minimum fraction of commuters who must be informed in order for *information saturation* to occur. Alternatively, R0, which corresponds to information saturation, can be thought of as a *socially fair* outcome, since it results in equal cost for all commuters. As seen in Fig. 5-3, λ' decreases as ρ increases, indicating that when the link is more reliable (i.e. incidents are less severe), information saturation and social fairness can be achieved with a lower fraction of informed commuters.

5.2 Social Cost

We now analyze how the social cost $\mathbb{E}[\bar{C}]$ is affected by changes in the parameters (ρ, p, λ). We focus mainly on how $\mathbb{E}[\bar{C}]$ varies with increasing degrees of information penetration λ for given values of (ρ, p). Fig. 5-1 shows this for the parameter values $\rho = 0.5$, $p = 0.25$. There are three key insights about social welfare to be gained from this figure; they are described below.

First, consider the shape of the $\mathbb{E}[\bar{C}]$ curve. As λ increases from 0, $\mathbb{E}[\bar{C}]$ first decreases, then increases (except for very high values of ρ), and then remains constant for $\lambda \geq \lambda'$. $\mathbb{E}[\bar{C}]$ may have multiple local minima.

The initial decrease in $\mathbb{E}[\bar{C}]$ is due to the growing fraction of informed commuters whose cost falls significantly as they gain access to information. $\mathbb{E}[\bar{C}]$ reaches its minimum value, denoted $\mathbb{E}[\bar{C}]^*$, at a critical fraction $\lambda^* \leq \lambda'$.[1] Thus λ^* is the socially optimal fraction of informed commuters, which minimizes the social cost. If $\lambda^* < \lambda'$, as in Figs. 5-1 and 5-2, then this means that there is an increase in $\mathbb{E}[\bar{C}]$ as λ increases from λ^* to λ' (this increase may not be monotonic). This is is because the increasing concentration effects (see Ben-Akiva et al. (1996)) faced by the informed commuters outweigh the benefits of information seen by the uninformed commuters, so that the average cost faced by commuters rises. In other words, the weighted increase in $\mathbb{E}[\bar{C}^H]$ is greater than the weighted decrease in $\mathbb{E}[\bar{C}^L]$. Finally, as observed in Theorem 2, as λ increases beyond λ' (i.e. in R0), $\mathbb{E}[\bar{C}]$ remains constant since its components $\mathbb{E}[\bar{C}^H]$ and $\mathbb{E}[\bar{C}^L]$ remain constant. Recall that this is precisely the case in which VoI(λ) is 0.

Second, we can study how the value of λ^* changes with the level of reliability of the bottleneck link, i.e. the value of ρ. Intuition would suggest that it is always socially optimal to inform a large fraction of commuters, if not all. However, as shown in Fig. 5-3, for different values of p, λ^* decreases (although not monotonically) from 1 to 0 as ρ increases from 0 to 1.[2] This indicates that the more reliable the link is

[1] When there are multiple values of λ that achieve the same minimum cost $\mathbb{E}[\bar{C}]^*$, we define λ^* to be the smallest of these. It follows that $\lambda^* \leq \lambda'$.

[2] The discrete jumps in λ^* occur when the global minimum $\mathbb{E}[\bar{C}]^*$ changes from one local minimum to another.

(i.e. the less severe possible incidents are), the lower the socially optimal fraction of informed commuters is. This is important from the perspective of the design of an information distribution mechanism: if the bottleneck is unreliable, i.e. it can suffer significant capacity reductions, then it is socially optimal to inform a large fraction of commuters about the state, as intuition suggests. However, if it is more reliable, i.e. only minor capacity reductions are possible, then it is socially preferable to disseminate information to a smaller fraction of commuters.

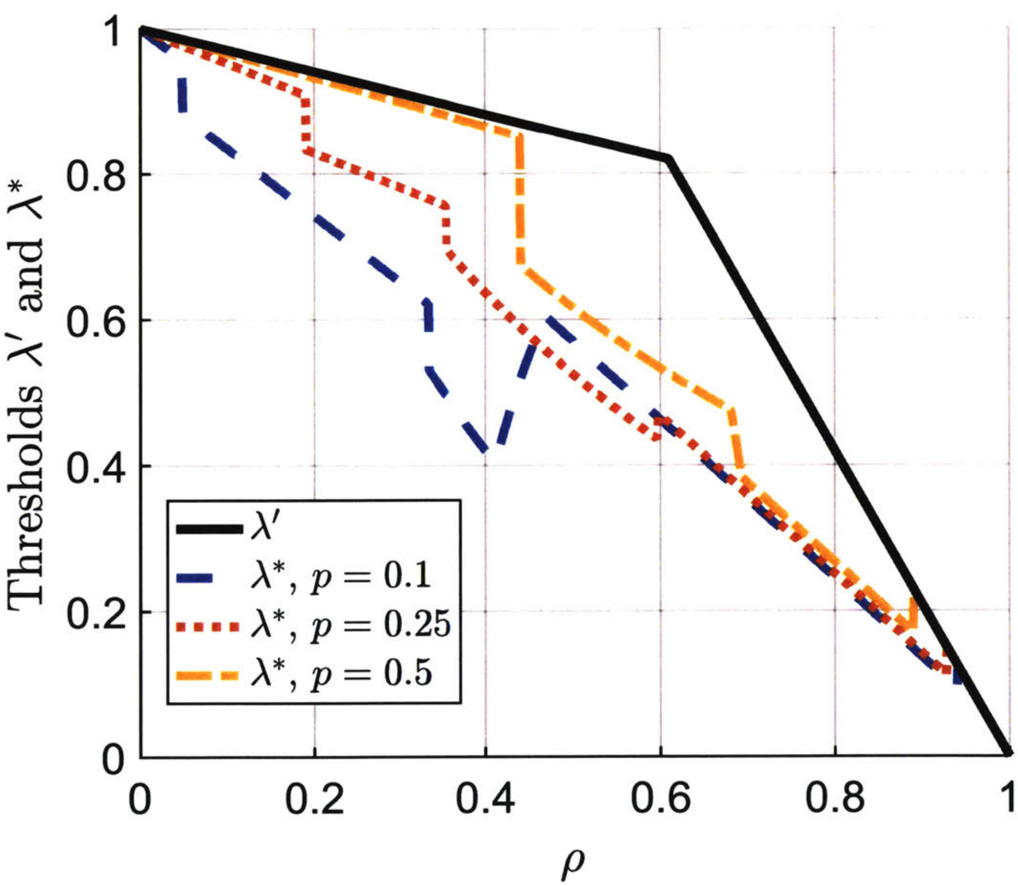


Figure 5-3: Thresholds λ' and λ^* versus ρ for different values of p.

Finally, we compare the social value of considering the heterogeneous information structure described in Chapter 2 (under Assumptions 1 to 3) as opposed to restricting ourselves only to the homogeneous information structures of full and zero information described in Section 3.2 and Section 3.3 respectively. Recall that the homogeneous information models consider only the extreme values of λ: full information refers to $\lambda = 1$ while zero information refers to $\lambda = 0$. It is shown in Arnott et al. (1988) that $\mathbb{E}[C_1] \leq \mathbb{E}[\bar{C}_0]$, i.e. that full information has a non-negative benefit in reducing the social cost. Here, we want to determine how much of a further benefit can be gained by considering the heterogeneous information structure considered in Section 2.2, i.e. by considering the full range of values $\lambda \in [0, 1]$. To do this, we define another metric called the *Value of Heterogeneity* (VoH), which is the difference between the full information cost $\mathbb{E}[C_1]$ and the optimal social cost $\mathbb{E}[\bar{C}]^*$:

$$\text{VoH} = \mathbb{E}[C_1] - \mathbb{E}[\bar{C}]^*. \tag{5.3}$$

Note that VoH is non-negative by definition because $\mathbb{E}[\bar{C}]^*$ is the minimum over a set of costs that includes $\mathbb{E}[C_1]$. Fig. 5-4 shows how VoH varies with ρ for different values of p. In all cases, VoH increases from 0 to its maximum value, which occurs for intermediate values of ρ, and then decreases to 0 again. Also notice that VoH is closely correlated to the difference between λ' and λ^* as seen in Fig. 5-3, i.e. the wider the interval $[\lambda^*, \lambda']$ in which $\mathbb{E}[\bar{C}]$ rises from $\mathbb{E}[\bar{C}]^*$ to $\mathbb{E}[C_1]$, the greater the difference between $\mathbb{E}[\bar{C}]^*$ and $\mathbb{E}[C_1]$.

We discuss the reasons behind the low values of VoH at the two extremes of ρ below (see Fig. 5-4). On one hand, for low values of ρ, representing a highly unreliable link, there are significant social cost savings by informing commuters of the state. However, almost all of these savings are achieved by the full information setting (λ^* is very close to λ'), leaving VoH small or even 0.

On the other hand, for high values of ρ, representing a reliable link, there is very little social cost savings from providing information at all. Therefore, VoH is once again close to 0 (λ^* is once again close to λ', and both are small). This is in agreement with the results of Ben-Akiva et al. (1996) who find that when day-to-day variability in traffic conditions is limited, there is little value to be gained from providing information at all. Although VoH is negligible for low and high values of ρ, there is another benefit of considering heterogeneous information. Namely, the social cost achieved when all commuters are informed (i.e. under full information) can also be achieved when fewer commuters are informed. This lower degree of information penetration might be easier to achieve in practice.

For intermediate values of ρ, however, VoH is surprisingly high, upto 20% (and λ^* is much smaller than λ'). In this case, providing all drivers with information does decrease the social cost somewhat as compared to not informing them, but there are significant further savings possible by instead disseminating information to only the optimal fraction λ^* of commuters. The peak of the VoH plot is highest for low p. This is significant because it means that the value of considering heterogeneous information is in fact highest for those scenarios which are most likely to occur in practice: moderate capacity reductions occurring with low probability. This is the key insight of our analysis.

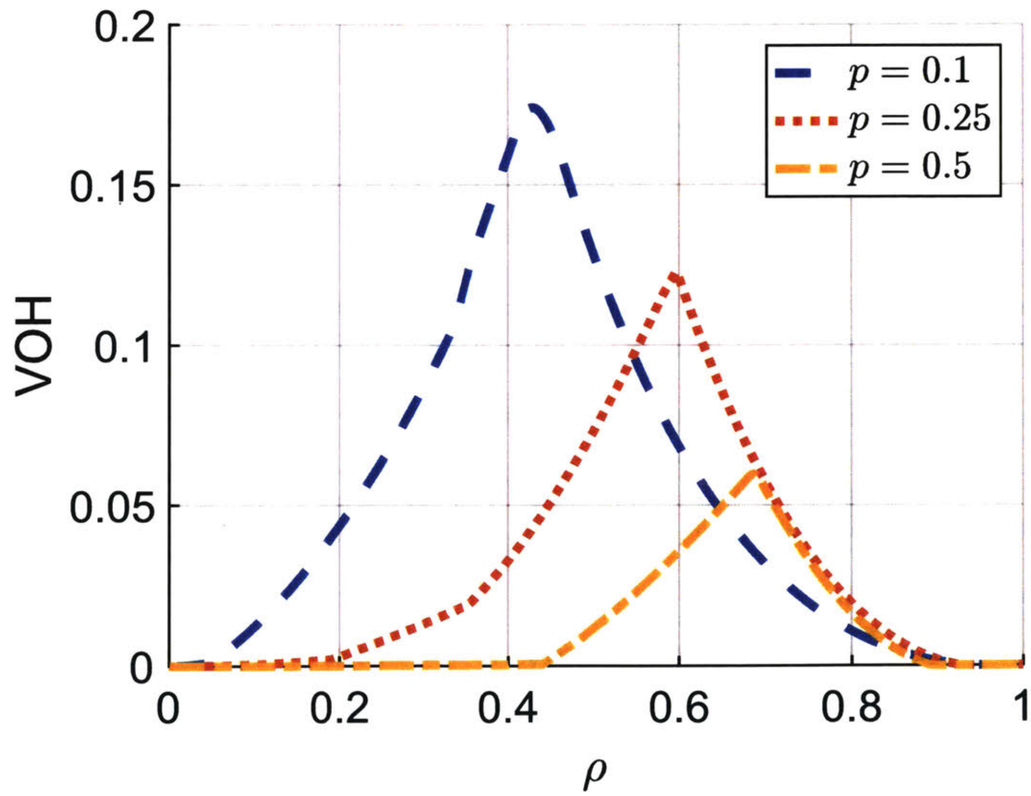


Figure 5-4: VoH (Value of Heterogeneity) versus ρ for different values of p.

Chapter 6

Conclusion

6.1 Summary

In this thesis, we study the effects of heterogeneous information on traffic congestion when commuters must choose when to travel across an incident prone bottleneck link. We consider a Bayesian congestion game in which one population has access to a Traveler Information System (TIS) that informs them of the state of the link, while the other does not. This game is an extension of the well known bottleneck model of traffic congestion proposed by Vickrey (1969). We provide a full characterization of the equilibrium of the game for the complete range of values of link reliability, incident probability, and information penetration, revealing rich equilibrium behavior. Equilibrium strategies can be broadly categorized into two distinct regimes, both of which can be refined further based on certain qualitative features. We show that when the fraction of informed commuters is higher than a particular threshold, the populations' equilibrium strategies are non-unique, and members of the two populations face the same cost. However, the aggregate departure rate function is unique and remains unchanged as the informed fraction grows further. On the other hand, when the informed fraction is below the threshold, the populations' strategies are unique, and informed commuters face lower costs than uninformed commuters. Thus, in both cases, gaining access to information is never detrimental; we find that the Value of Information (VoI) is positive and decreasing in the informed fraction upto the aforementioned threshold, and 0 thereafter. In terms of the effect of the informed fraction on the average (social) cost faced by commuters, we find that this cost is minimized when the fraction is below 1. This indicates that heterogeneous information can be socially preferable to informing all commuters. We quantify this by means of a metric called Value of Heterogeneity (VoH), and find that it is significantly large (upto 20%) when incidents are moderately intense and relatively rare. Finally, we find that the optimal fraction of informed commuters is lower when the link is more robust to capacity loss under incidents, and vice versa.

6.2 Future Work

We note that our model is simplistic and its findings cannot be presumed to hold for real-world scenarios without further verification. There are several directions in which this model can be extended to gain better insight into the effect of heterogeneous information on traffic congestion:

- Our model assumes that the interim beliefs are derived from a common prior (see Assumption 2). That is, each population has knowledge about how the types of other populations are distributed. It needs to be seen whether the qualitative features of the equilibrium still hold without the existence of common prior. This entails considering subjective commuter beliefs, which is important since, in practice, commuters may not know the accuracy of the TISs used by others.

- Our model places rather stringent assumptions on the accuracies of the two TISs. In particular, we assume that one TIS is perfectly accurate while the other is completely uninformative (Assumption 3). We argue that this scenario is of practical relevance since some commuters may choose not to use TISs, making them effectively "uninformed", whereas those who do use TISs have access to near-perfect information due to the accuracy of modern TISs. Nevertheless, it would be useful to know whether the results concerning the value of information would change if these assumptions were relaxed. In particular, Arnott et al.

(1991) find that while perfect (homogeneous) information is welfare improving, imperfect information, even if it is highly accurate, may be detrimental to welfare. Similarly, Wu et al. (2017) find that a population with access to more accurate (but imperfect) information may be worse off than others. However, as noted earlier, we believe that relaxing this assumption to consider the general case $0.5 < \eta^L < \eta^H < 1$ makes the equilibrium structure significantly more complicated, and may not be analytically tractable. We expect that the distinctions between R0 and R0' will continue to exist in equilibrium for this case, but that there will also be a regime with negative value of information.

- Our model considers the simplified case of a single route. This isolated analysis may not reflect the effects of information on a traffic network; it would be necessary to extend the analysis to a network comprising of multiple routes in order to claim that it is applicable to urban traffic. Such an extension of the bottleneck model has been studied in Yang and Meng (1998). However, given that this would most likely make the equilibrium strategies intractable, the immediate extension would be to consider two routes in parallel.

- Our model assumes that demand is fixed and that capacity can take only two values. Better insights could be gained by considering an arbitrary joint distribution on demand and capacity, as in Arnott et al. (1991).

- Our model is also "static" in the sense that it assumes that information is available to commuters before they begin their journeys, and that the state remains constant during the travel period. Since TISs provide real-time information and state can change within a travel period (e.g. an accident occurring at the peak traffic time), it would be desirable to relax these assumptions, such as in Zhang et al. (2010).

It is worth noting that several of these extensions have been proposed in earlier literature on the bottleneck model. Indeed our work is a response to one of the proposed extensions, that information heterogeneity should be considered. However, to gain better insights on traffic congestion in practical scenarios, it is necessary to incorporate other aspects of real-world traffic along with information heterogeneity. Despite the considerable advancements in TIS technology, accuracy, and adoption, it is still very much an open question as to how their use affects commuters' decisions and thus their travel costs. Along with the development of more sophisticated models, empirical studies using real-world data are also needed in order to verify the effects found in the existing literature.

Notational Glossary

Cost Parameters

D	total demand
α	unit cost of travel time
β	unit cost of early arrival
γ	unit cost of late arrival

States

s	(generic) state of nature
n	nominal state
a	incident state
S	set of states
p	probability of state a occurring
θ	prior distribution of states

Capacities

c	deterministic capacity
c_s	capacity in state s
ρ	ratio of c_a to c_n

TISs

i	(generic) Traveler Information Service (TIS)
I	set of TISs
H	high accuracy TIS
L	low accuracy TIS
η^i	accuracy of TIS i
τ^i	signal given by TIS i
$\mathrm{\tau}^i$	set of possible signals for TIS i

Populations

i	(generic) commuter population
I	set of populations
λ^i	fraction of commuters in population i
τ^i	(generic) type of population i
τ	type profile of populations
$\mathcal{T}^i$	set of possible types of population i
$\mathcal{T}$	set of possible type profiles

Beliefs

μ^i	belief of population i about s and τ^{-i}
μ	belief profile of populations
π	joint probability distribution of s and τ

Strategies

σ^i	strategy of population i
σ	strategy profile of populations
Σ^i	set of possible strategies of population i
Σ	set of possible strategy profiles

Times

t	departure time (variable)
t^*	ideal arrival time
t_0	first departure time
t_1	last departure time
$t_{0,s}$	first departure time in state s
$t_{1,s}$	last departure time in state s
$t_0^{\tau^i}$	first departure time of type τ^i commuters
$t_1^{\tau^i}$	last departure time of type τ^i commuters
$t_{0'}^{Ha}$	first departure time of type Ha commuters in their first interval of departure

$t_{1'}^{Ha}$	last departure time of type Ha commuters in their first interval of departure
$t_{0''}^{Ha}$	first departure time of type Ha commuters in their second interval of departure
$t_{1''}^{Ha}$	last departure time of type Ha commuters in their second interval of departure
t_f	last arrival time
$t_{f,s}$	last arrival time in state s
$\tilde{t}$	departure time for arrival at t^*
$\tilde{t}_s$	departure time for arrival at t^* in state s
$\hat{t}$	queue dissipation time
$\hat{t}_s$	queue dissipation time in state s
$\hat{t}_{n'}$	dissipation time of first queue in state s when there are two queues
$q_s(.)$	queuing time function in state s
$T(t)$	latest time before time t with no queue

Departure Rates

$r(.)$	departure rate
r_e	departure rate before $\tilde{t}$ in deterministic case
r_l	departure rate after $\tilde{t}$ in deterministic case
$r_s(.)$	(aggregate) departure rate in state s
$r_{s,e}$	departure rate before $\tilde{t}$ in state s in full information case
$r_{s,l}$	departure rate after $\tilde{t}$ in state s in full information case
$r^{\tau^i}(.)$	departure rate of type τ^i commuters
$r_{s,1}(.)$	departure rate in state s under full information
$r^{\tau^i}(.)$	departure rate of type τ^i commuters
$r^{\sigma(\tau)}(.)$	aggregate departure rate of type profile τ
$\mathcal{F}$	set of possible type-dependent departure rate functions $r^{\sigma(\tau)}(.)$

Informed Fractions

λ	information penetration
λ'	threshold information penetration between R0' and R0
λ^*	optimal information penetration

Costs

$C_s(.)$	cost function in state s
$\mathcal{C}$	set of state dependent cost functions
$\bar{C}_s^{\tau^i}$	average cost of type τ^i commuters in state s
$\bar{C}_s$	average cost in state s
$\mathbb{E}[C^{\tau^i}(.)]$	expected cost function of type τ^i commuters
$\mathbb{E}[\bar{C}^{\tau^i}]$	expected average cost of type τ^i commuters
$\mathbb{E}[\bar{C}^i]$	expected average (individual) cost of population i
$\mathbb{E}[\bar{C}]$	social cost
$\mathbb{E}[\bar{C}]^*$	optimal social cost
C	deterministic cost (social cost in deterministic case)
$\mathbb{E}[C_1]$	full information cost (social cost in full information case)
$\mathbb{E}[\bar{C}_0]$	zero information cost (social cost in zero information case)

Thresholds

ϕ_{12}	threshold between R1 and R2
ϕ_{23}	threshold between R2 and R3
ϕ_{AB}	threshold between RA and RB

Appendix A

Equilibrium Under Zero Information

This appendix derives the equilibrium structure under the zero information setting described in Section 3.3. We first list some properties of the equilibrium structure. We then use these properties to prove Theorem 1. Finally, we show how the equilibrium strategy is derived.

Proposition 1. *The following properties hold in equilibrium:*

1. $q_n(t) < q_a(t) \; \forall \; t \in (t_0, t_{f,a})$
2. $\tilde{t}_a < \tilde{t}_n$
3. $\hat{t}_n < \hat{t}_a$
4. $\hat{t}_a = t_{f,a}$
5. $t_0 < t^* < \hat{t}_a$
6. $t^* \leq t_1 \leq \hat{t}_a$

Proof.

1. The departure rate function and thus the inflow into the bottleneck is identical for both states, and, since $c_n > c_a$, the outflow from the bottleneck is higher in state n. Thus it follows that the queuing time, will always be shorter in state n.

2. This follows directly from the property above. Since the queuing time is always longer in state a, departure must be earlier in state a than in state n in order to have an identical arrival time, in this case, t^*.

3. Again, it follows directly from the first property that the queue will dissipate earlier in state n (if it exists in that state at all), giving $\hat{t}_n < \hat{t}_a$. (note that in R1, $\hat{t}_n = t_0$ by default)

4. By contradiction:

   Suppose $\hat{t}_a < t_{f,a}$. Then, since there is no queue at $t_{f,a}$, the commuter arriving at $t_{f,a}$ would also have departed at $t_{f,a}$. Thus he would incur a larger total cost than the commuter who departed and arrived at $\hat{t}_a$, which is inconsistent with equilibrium. The other alternate, $\hat{t}_a > t_{f,a}$, meaning that the queue dissipates after the last arrival, is clearly nonsensical.

5. By contradiction:

   Suppose $t_0 \geq t^*$ or $\hat{t}_a \leq t^*$. Consider a commuter who departs at t^*. He faces no queue and no scheduling cost, thus incurring him a total cost of 0, which is inconsistent with equilibrium. Thus t^* lies in the interior of the rush hour.

6. By contradiction:

   Suppose $t_1 < t^*$. Consider a commuter who departs at time $t = t_1 + \delta < t^*$, where δ is small. If there is a queue, he faces a smaller queuing cost and identical arrival time as the commuter who departs at t_1. If there is no queue, he faces no queuing cost and a smaller scheduling cost the commuter who departs at t_1, since he arrives closer to t^*. Thus, in both cases, he incurs a smaller total cost, which is inconsistent with equilibrium. Thus $t_1 \geq t^*$.

   Conversely, suppose $t_1 > \hat{t}_a$. Consider the commuter who departs at $t = t_1 - \delta > \hat{t}_a$. He incurs the same queuing cost (0) but arrives closer to t^* than the commuter departing at t_1. Thus he incurs a smaller total cost, which is inconsistent with equilibrium. Thus $t_1 \leq \hat{t}_a$. Combining these results, we get

$t^* \le t_1 \le \hat{t}_a$, i.e. the last departure occurs between t^* and $\hat{t}_a$. Note that due to the previous property, at least one of the inequalities is strict.

□

We now prove the claims made in Theorem 1.

Proof. We begin by proving that the regimes described in Theorem 1 are the only ones that can exist in equilibrium. First, we must define the different types of experiences commuters can face while traveling. For each state s, a commuter departing at a given time will either arrive early ($t \le \tilde{t}_s$) or late ($t > \tilde{t}_s$), and either face a queue ($t < \hat{t}_s$) or not ($t \ge \hat{t}_s$). Thus for a given state, a commuter faces one of the following experiences:

- **eq**: early arrival and queue
- **ez**: early arrival and no queue
- **lq**: late arrival and queue
- **lz**: late arrival and no queue

Over the two-state set S, a commuter's experience can be represented as tuples of the above mentioned types, representing the conditions he would face if he departed at the same time t in both states. For example, (ez,lq) represents experiencing early arrival and no queue in state n and late arrival and a queue in state a. Thus there are $4 \times 4 = 16$ types of experiences, not all of which can exist in equilibrium. Furthermore, we use (·,lq), for example, to refer to experiencing late arrival and a queue in state a without any restriction on the experience in state n. Table A.1 lists whether each of these experiences can exist in equilibrium.

(eq,eq)	✓	(ez,eq)	✓	(lq,eq)	✗	(lz,eq)	✗
(eq,ez)	✗	(ez,ez)	✗	(lq,ez)	✗	(lz,ez)	✗
(eq,lq)	✓	(ez,lq)	✓	(lq,lq)	✓	(lz,lq)	✓
(eq,lz)	✗	(ez,lz)	✗	(lq,lz)	✗	(lz,lz)	✗

Table A.1: Existence of all possible experiences in equilibrium

We now argue why the experiences marked with a cross in Table A.1 cannot exist in equilibrium. First, note that under the 4th property in Proposition 1, the rush hour is $[t_O, \hat{t}_a]$. Since queuing starts at t_O in state a (to assume otherwise would clearly contradict the equilibrium condition), there is no time in the rush hour in which there is no queue in state a. This rules out the experiences in the 2nd and 4th rows of Table A.1. Secondly, under the 1st property in Proposition 1, there is no possibility of arriving earlier in state a than in state n for any departure time. Thus the experiences (lq,eq) and (lz,eq) are also impossible in equilibrium.

Next, for the remaining experience, we assert that certain conditions must be met by the order in which they occur in equilibrium. These conditions are shown in Table A.2. The table also shows the intervals corresponding to each of these experiences, the value of r in each of these intervals (proved later in this Appendix), and the regimes in which they exist in equilibrium. The experience which exists in equilibrium as the 1st experience in R2, for example, is denoted R2.1, and similarly for the other regimes.

Experience	Interval	$r(t)$	Condition(s)	Existence in Regimes
(eq,eq)	$[t_O, \tilde{t}_a]$	$\frac{\alpha}{(\alpha-\beta)\frac{p}{c_a}+(\alpha-\beta)\frac{1-p}{c_n}}$	must be 1st	R2.1, R3.1
(ez,eq)	$[t_O, \tilde{t}_a]$	$\frac{c_a(p\alpha+(1-p)\beta)}{p(\alpha-\beta)}$	must be 1st	R1.1
(eq,lq)	$[\tilde{t}_a, \min\{\tilde{t}_n, \hat{t}_a\}]$	$\frac{\alpha}{(\alpha+\gamma)\frac{p}{c_a}+(\alpha-\beta)\frac{1-p}{c_n}}$	must be after (eq,eq)	R2.2, R3.2
(ez,lq)	$[\max\{\hat{t}_n, \tilde{t}_a\}, \tilde{t}_n]$	$\frac{c_a(p\alpha+(1-p)\beta)}{p(\alpha+\gamma)}$	must be after (ez,eq) or (lq,eq)	R1.2, R2.3
(lq,lq)	$[\tilde{t}_n, \hat{t}_n]$	$\frac{\alpha}{(\alpha+\gamma)\frac{p}{c_a}+(\alpha+\gamma)\frac{1-p}{c_n}}$	must be after (lq,eq)	R3.3
(lz,lq)	$[\max\{\tilde{t}_n, \hat{t}_n\}, \hat{t}_a]$	$\max\{\frac{c_a(p\alpha-(1-p)\gamma)}{p(\alpha+\gamma)}, 0\}$	must be last, and after (ez,lq) or (lz,lq)	R1.3, R2.4, R3.4

Table A.2: Conditions on existence of experiences in equilibrium, and regimes in which they exist

First, note that if a queue builds up in a given state, it starts at t_O. Thus the first experience in any regime must be (eq,eq) or (ez,eq). In R2 and R3, where the queue builds up in both states, the former applies, while in R1, where the queue doesn't build up in state n, the latter applies. Note that (ez,eq) cannot follow (eq,eq) for the following reason: if a queue builds up in state n, it is because $r(t) > c_n$ during that experience.[1] Thus for the queue to dissipate, a decrease in $r(t)$ is needed, which can only occur when the experience changes. The only possible reason for a transition from (eq,eq) other than dissipation of the queue in state n is the transition of arrival time in state a from early to late, i.e. a transition to (eq,lq). Thus, (eq,eq) must necessarily be followed by (eq,lq), as is the case in both R2 and R3. Since (eq,eq) was the only possible precursor to (ez,eq), it follows that (ez,eq) must be the first experience whenever it occurs. Now, consider that the only possible reason for a transition from (ez,eq) is similarly the transition of arrival time in state a from early to late, i.e. a transition to (ez,lq) as is the case in R1. From (eq,lq) however, one of two transitions can occur. One possibility is that the queue in state n dissipates before the arrival time in state n transitions from early to late. This is the case in R2, and leads to a transition to (ez,lq). The alternate is that the arrival time in state n transitions from early to late first, as in R3, which leads to a transition to (lq,lq). Next, note that the only possible transition after (ez,lq) is the arrival time in state n moving from early to late, as is the case in both R1 and R2, resulting in a transition to (lz,lq). Similarly, the only possible transition from (lq,lq) is the dissipation of the queue in state n, as is the case in R3, again resulting in a transition to (lz,lq). Finally, recall from Proposition 1 that $\hat{t}_n < \hat{t}_a$ and $t^* < \hat{t}_a$. These conditions, when combined, ensure that the experience (lz,lq) must exist. Thus (lz,lq) is necessarily the terminal experience for any regime in equilibrium, since no further transitions are possible.

A close examination of these arguments reveals that there are only 3 distinct sequences of experiences. Each of these sequences correspond to one of R1, R2 and R3, as shown in Table A.3, completing the proof that these are the only possible regimes.

We round off the proof that by describing why each of R1, R2 and R3 has two further types: A and B. Note that the sequence of experiences is same in the re-

[1]Note that this reasoning assumes that r is constant for a given experience, which will be proved later in this Appendix.

spective regime types A and B of each of R1, R2 and R3, and the only difference is that there are no departures under the last experience (lz,lq) in each of the B type regimes. Recall from Proposition 1 that $t_1 > t^*$, and note that in each regime, the last experience lies fully after t^* whereas at least part of the second last experience lies before t^*. Thus, since $r(t)$ is constant during an experience, it cannot be 0 in any experience except the last one, as that would contradict the above mentioned property. On the other hand, $r(t)$ may or may not be 0 in the last experience. These two possibilities correspond to RB and RA respectively. Note that in the former case, although there are no new departures in the last experience, the queue still exists in state a, and so the experience is still part of the rush hour.

Sequence of experiences	Corresponding Regime
(ez,eq) $\rightarrow$ (ez,lq) $\rightarrow$ (lz,lq)	R1
(eq,eq) $\rightarrow$ (eq,lq) $\rightarrow$ (ez,lq) $\rightarrow$ (lz,lq)	R2
(eq,eq) $\rightarrow$ (eq,lq) $\rightarrow$ (lq,lq) $\rightarrow$ (lz,lq)	R3

Table A.3: Sequence of experiences for each regime

Finally, to conclude the proof, we show how ϕ_{12} is derived.[2] The equilibrium shifts from R1 to R2 when the rate required to maintain equal cost in the experience (ez,eq) becomes larger than c_n. Thus, at the threshold, $c_n = \dfrac{c_a(p\alpha + (1-p)\beta)}{p(\alpha - \beta)}$. Solving this for p gives the expression for ϕ_{AB} given in Theorem 1.

□

The departure rate $r(t)$ for each of the experiences mentioned in Table A.2 is derived below. The equilibrium condition (2.14) implies that the expected cost $\mathbb{E}[C(t)]$ faced by commuters is constant over the rush hour:

$$\frac{\mathrm{d}\mathbb{E}[C(t)]}{\mathrm{d}t} = 0, \forall t \in \mathrm{supp}(r), \tag{A.1}$$

where, using (2.13), $\mathbb{E}[C(t)]$ is given by:

$$\mathbb{E}[C(t)] = (1-p)C_n(t) + pC_a(t). \tag{A.2}$$

[2]Recall from Section 3.3 that ϕ_{23} and ϕ_{AB} are mentioned in Arnott et al. (1988).

The expression for $C_s(t)$ depends on the experience at time t. For each experience, substituting (2.7) into C_n and C_a, and setting the derivative of (A.2) to 0 gives the departure rate $r(t)$. As an example, $r(t)$ is derived below for the experience (ez,lq):

$$\mathbb{E}[C(t)] = p(\alpha q_a(t) + \gamma(q_a(t) + t - t^*)) + (1-p)\beta(t - t^*).$$

Taking the derivative and setting it to 0, we get:

$$\frac{\mathrm{d}\mathbb{E}[C(t)]}{\mathrm{d}t} = p(\alpha + \gamma)\frac{\mathrm{d}q_a(t)}{\mathrm{d}t} + p\gamma - (1-p)\beta = 0.$$

Substituting $\dfrac{\mathrm{d}q_a(t)}{\mathrm{d}t} = \dfrac{r(t) - c_a}{c_a}$ and rearranging, we get:

$$r(t) = \frac{c_a(p\alpha + (1-p)\beta)}{p(\alpha + \gamma)}.$$

For the experience (lz,lq), (A.2) can give a negative value. In this case it is not possible to satisfy (A.1), so there cannot be any departures, and $r(t) = 0$. This is precisely what leads to the RA-RB distinction. When $r(t) = 0$ for the experience (lz,lq), then $t_1 < \hat{t}_a$ (RB), and there exists a non-degenerate interval $(t_1, \hat{t}_a]$ in the rush hour with no departures. On the other hand, when $r(t) > 0$ for this experience (RA), then $t_1 = \hat{t}_a$ and there are departures throughout the rush hour.

Table A.2 gives the values of $r(t)$ for each experience that commuters face in equilibrium, whereas Table A.3 gives the order in which these experiences exist in each regime. To complete the derivation of the equilibrium strategy, it remains to be shown how the epoch values that delimit the experiences are derived. They are derived by solving a set of linear equations as explained below.

First, since the bottleneck operates at capacity throughout the rush hour in state a, we can write, analogously to (3.2):

$$\hat{t}_a - t_0 = \frac{D}{c_a}. \tag{A.3}$$

Next, by definition of the pivot time $\tilde{t}_a$ in state a:

$$q_a(\tilde{t}_a) = \int_{t_0}^{\tilde{t}_a} \frac{r(t) - c_a}{c_a} \, \mathrm{d}t = t^* - \tilde{t}_a. \tag{A.4}$$

In R3, there is an equivalent equation for the pivot time $\tilde{t}_n$ in state n. In R1 and R2 however, $\tilde{t}_n = t^*$, so the additional equation is not required.

Finally, by definition of the queue dissipation time $\hat{t}_a$ in state a:

$$q_a(\hat{t}_a) = \int_{t_0}^{\hat{t}_a} \frac{r(t) - c_a}{c_a} \, \mathrm{d}t = 0. \tag{A.5}$$

In R2 and R3, there is an equivalent equation for the pivot time $\tilde{t}_n$ in state n. In R1 however, there is no queue, so $\hat{t}_n$ is not defined and the additional equation is not required.

Thus there are 3 equations and 3 unique epochs in R1: t_0, $\tilde{t}_a$, and $\hat{t}_a$. In R2, there is an additional unique epoch ($\hat{t}_n$) and a corresponding equation. In R3, there is a further unique epoch ($\tilde{t}_n$) and a further corresponding equation. Therefore in each regime there is a perfectly defined linear system which can be solved to give the values of the epochs (including t_0, which is used to calculate $\mathbb{E}[\bar{C}_0]$).

Appendix B

Proof of Condition 4

Intuitively, it is clear that a commuter who departs before (resp. after) the interval of queueing in state a unnecessarily incurs a greater scheduling cost than she would incur by departing at the time the queue starts (resp. dissipates). Furthermore, since departures are more spread out in state a than in state n, this commuter would also be facing an unnecessarily high cost in state n, contradicting the equilibrium condition.

Formally, this condition can be shown to hold separately for R0 and R0'. In R0, the aggregate departure rate function in state a is $r_{a,1}$, as shown in Theorem 2. This function results in a queue throughout the rush hour, as explained in Section 3.1.

For R0', we show that state a can be partitioned into three sets of times, and that the queue is non-zero in each of them. First, from Condition 3, there is necessarily a non-zero queue during the interior of $\text{supp}(r^{Ha})$. Next, the queue is also non-zero during the set $\{t : t \in (t_0^{Ha}, t_1^{Ha}) \setminus \text{supp}(r^{Ha})\}$. This is because if any (necessarily type L) commuter departing before (resp. after) $\tilde{t}_a$ in this set faced zero queuing cost, her total cost would be lower than the scheduling (and hence total) cost faced by the (type Ha) commuter departing at t_0^{Ha} (resp. t_1^{Ha}), contradicting Condition 2. In R·[2]⟨·⟩, the rush hour is $[t_0^{Ha}, t_1^{Ha}]$, so the two aforementioned sets of times comprise the entire rush hour, and the condition follows. On the other hand, R·[1]⟨·⟩ only exists for $p \leq \phi_{AB}$, i.e. in RB, so that $t_1^L = t_1^{Hn}$. Thus $\text{supp}(r^L)$ consists of the intervals $[t_0^L, t_0^{Hn}]$ and $[t_0^{Hn}, t_1^{Hn}]$. In the latter interval, the cost faced by all commuters is equal (Condition 9), which requires a continuously changing queuing cost, which is only possible if the queue is non-zero. In the former interval, if the queue is non-zero at any time before $\tilde{t}_a$, the commuter departing at that time faces a lower scheduling and hence total cost than type Ha commuters, contradicting Condition 2. On the other hand, if the queue is non-zero at any time after $\tilde{t}_a$, it is not possible to have a continuously decreasing queuing cost in $[t_0^{Hn}, t_1^{Hn}]$ as would be required by Condition 9.

Appendix C

Proof of Theorem 4

Proof. We derive the rates in the following order: first we calculate $r^{Ha}(t)$, then $r^{L}(t)$ for $t \in \text{supp}(r^{Hn})$, then $r^{Hn}(t)$ and finally $r^{L}(t)$ for $t \in \text{supp}(r^{L}) \setminus \text{supp}(r^{Hn})$.

1. $r^{Ha}(t)$

   First, note that, as seen in section Section 3.1, the aggregate departure rate required to maintain equal realized cost over time in state s is given by:

$$r_s(t) = \begin{cases} \dfrac{\alpha c_s}{\alpha - \beta} & t \leq \tilde{t}_s, \\ \\ \dfrac{\alpha c_s}{\alpha + \gamma} & t > \tilde{t}_s. \end{cases} \tag{C.1}$$

   Condition 1 implies that (C.1) must hold in each state s for $t \in \text{supp}(r^{Hs})$. Under Condition 7, type Ha commuters do not depart concurrently with other commuters, so $r^{Ha}(t) = r_a(t)\ \forall t \in \text{supp}(r^{Ha})$. Thus $r^{Ha}(t)$ is given by (C.1). Note that, due to Condition 3, $\text{supp}(r^{Ha})$ can only include intervals of type (·,eq) and (·,lq). Furthermore, under Condition 7, $r^{Ha}(t)$ is only dependent on the interval type in state a, not state n. Combining these arguments results in the values given in Table 4.3.

2. $r^{L}(t)$ **for** $t \in \text{supp}(r^{Hn})$

   Next, note that Condition 9 implies that (C.1) must also hold in state a for $t \in$

supp(r^{Hn}). Under Condition 7 only type L commuters depart in this interval, so $r^L(t) = r_a(t)\ \forall t \in$ supp(r^{Hn}). Thus $r^L(t)$ for $t \in$ supp(r^{Hn}) is given by (C.1). Once again, due to Condition 3, supp(r^{Hn}) can only include intervals of type (eq,·) and (lq,·). The values of $r^L(t)$ for $t \in$ supp(r^{Hn}) are given in the last two rows of Table 4.4.

3. $r^{Hn}(t)$

   Next we use $r^L(t)$ determined above to find $r^{Hn}(t)$. Using the fact that $r^{Hn}(t) = r_n(t) - r^L(t)$ and substituting in the values of $r_n(t)$ from (C.1) and $r^L(t)$ from Table 4.4 for each of the possible experiences (eq,eq), (eq,lq), and (lq,lq), we get the values given in Table 4.2.

4. $r^L(t)$ **for** $t \in \text{supp}(r^L) \setminus \text{supp}(r^{Hn})$

   Finally, we show how $r^L(t)$ is derived for $t \in \{\text{supp}(r^L) \setminus \text{supp}(r^{Hn})\}$. The equilibrium condition (2.14) implies that:

$$\frac{\mathrm{d}\mathbb{E}[C^L(t)]}{\mathrm{d}t} = 0, \forall t \in \text{supp}(r^L), \tag{C.2}$$

where, using (2.13), $\mathbb{E}[C^L(t)]$ is given by:

$$\mathbb{E}[C^L(t)] = (1-p)C_n(t) + pC_a(t). \tag{C.3}$$

The expression for $C_s(t)$ depends on the experience at time t. For each experience, substituting (2.7) into C_n and C_a, and setting the derivative of (C.3) to 0 gives the departure rate $r^L(t)$. As an example, $r^L(t)$ is derived below for the experience (ez,lq):

$$\mathbb{E}[C^L(t)] = p(\alpha q_a(t) + \gamma(q_a(t) + t - t^*)) + (1-p)\beta(t - t^*).$$

Taking the derivative and setting it to 0, we get:

$$\frac{\mathrm{d}\mathbb{E}[C^L(t)]}{\mathrm{d}t} = p(\alpha+\gamma)\frac{\mathrm{d}q_a(t)}{\mathrm{d}t} + p\gamma - (1-p)\beta = 0.$$

Substituting $\frac{\mathrm{d}q_a(t)}{\mathrm{d}t} = \frac{r^L(t) - c_a}{c_a}$ and rearranging, we get:

$$r^L(t) = \frac{c_a(p\alpha + (1-p)\beta)}{p(\alpha+\gamma)}.$$

For the experience (lz,lq), (C.3) can give a negative value. In this case it is not possible to satisfy (C.2), so there cannot be any departures, and $r^L(t) = 0$. This is precisely what leads to Distinction 4. When $r^L(t) = 0$ in the interval (lz,lq), then $t_1^L = t_1^{Hn}$ (RB), and there exists an interval in the rush hour with no departures but in which arrivals do occur (in state a). On the other hand, when $r^L(t) > 0$ in this interval (RA), then $t_1^L > t_1^{Hn}$ and there are departures throughout the rush hour.

Table 4.4 gives the values of $r^L(t)$ for each experience that type L commuters face in equilibrium. Which of these experiences they face for a particular set of parameter values depends on which subregime the equilibrium happens to fall into for those parameter values. For example, by recalling the definitions of R1 and R3, one can see that the experience (ez,eq), the first type mentioned in Table 4.4, can occur in the former but not in the latter.

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