Effect of Information in Bayesian Congestion Games

by

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Abstract

This thesis studies routing in a general single o-d network and information structure induced by any two heterogeneous information systems. To model the asymmetric information environment, we formulate a Bayesian congestion game, where travelers subscribing to one information system is seen as one population. We study properties of Bayesian Wardrop Equilibrium, where each population assigns their demand to routes with the lowest expected cost based on their belief. We show that if population beliefs about the state and the signal received by the other population are based on a common prior, as the population sizes change, qualitative properties of equilibrium strategies change, resulting in three distinct regimes. In the intermediate regime, the equilibrium edge load does not vary with the relative population size, and both populations face identical cost in equilibrium. In the other two regimes, the "minor" population has lower cost in equilibrium. We also introduce a metric to evaluate the impact of information. The relative population size effects the equilibrium outcome (edge load, costs) if and only if the impact of information on either population is tightly bounded by its size. Finally, we compute the bounds on the equilibrium social cost, and provide a sufficient condition for the bounds to be tight. Although we consider a more general information environment, the worst case inefficiency of equilibrium is the same as that in complete information games.

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Chapter 1

Introduction

1.1 Overview of the Problem

The recent advancements in information systems such as Google Maps/ Waze, Apple Maps, etc. allow travelers to be better informed about traffic conditions. Information systems send signals about traffic conditions to their subscribed travelers based on historical and current measurements of the network. However, signals may not be accurate about the network state due to reasons such as difficulties in data collection, noisy data source, and limitations of estimation methods. Different information systems provide signals with different accuracy, which leads to an inherently heterogeneous information structure among the travelers. Therefore, in analyzing travelers' route choice decisions, it is important to consider their private information and beliefs. This thesis addresses the following question: *How do the relative sizes of traveler populations effect the equilibrium structure and costs (both individual and social)?*

We model traffic routing under information heterogeneity as a Bayesian congestion game. Our model considers a network with single origin-destination pair. The random state of network is drawn from a set of states by a commonly known prior probability distribution. Costs on each edge are *state-dependent* increasing functions of the traffic load on that edge. The network faces non-atomic travelers with inelastic total traffic demand. There are two heterogeneous information systems, each sending a noisy signal of the state to its subscribed travelers. Travelers subscribed to one information system receive the same signal, and are modeled as one population. The joint distribution of the state and signals received by two populations is the common prior, which is known by both populations. Each population updates their beliefs about the state and the signal received by the other population from the common prior using Bayes rule. We study properties of *Bayesian Wardrop Equilibrium (BWE)*, where populations assign demand on routes with the smallest expected cost based on their private beliefs.

1.2 Related Work

We first discuss some related work that considers informational aspects in modeling route choices. The article by Liu et al. (2016) studies the effect of information heterogeneity for a two-route traffic network when a fraction of population has complete information about the traffic accidents, and the rest has no information. The paper focuses on studying a "subjective" belief structure, where travelers are only aware of the accuracy of the information system they subscribe to, but assume the other travelers are not informed. Our set up is more general as we consider any network with single o-d pair, and any two information systems with arbitrary accuracies. In our game, accuracies of both information systems are common knowledge, and the private beliefs are derived from a common prior. Notably, the recent work by Acemoglu et al. (2016) studies the effect of information heterogeneity in the case when travelers have non-identical information sets about the available edges. The authors define Informational Braess Paradox (IPB) as a phenomenon where travelers who receive information about additional routes may be worse off than those who do not. They find an intuitive suffcient and necessary condition on network topology to ensure that the IBP does not occur. Our work is complementary to Acemoglu et al. (2016) in that we adopt Bayesian framework to study the effects of noisy signals about the network state when the two different information systems introduce an asymmetric information structure in travelers' route choice. We also discover similar phenomena, where travelers with more accurate information have higher cost. Additionally, a more general set up of congestion games, where cost functions are player-specific, is studied in Milchtaich (1996). The player-specific cost functions can be a result from private beliefs or player specific preferences. Milchtaich (1996) proves the existence of pure Nash equilibrium when atomic players have the same demand. Best-response improvement path can be cyclic, which implies that there may not exist a potential function. We consider non-atomic players, thus pure equilibrium always exists. In our model, the difference in the expected costs among populations with different signals (types) is due to their private beliefs, although the preferences (or valuation of travel times) are homogeneous cross the traveler populations. A weighted potential function exists in our game since type-specific cost functions are related via the common prior.

We next review the literature on the congestion games. The well-known results include the existence of a potential function in every congestion game Rosenthal (1973), and the isomorphism between congestion and potential games Monderer and Shapley (1996). Population games with non-atomic players are shown to be the limiting case of finite player games in Sandholm (2001), and convergence results of evolutionary dynamics are provided. The paper Sorin and Wan (2015) compares the equilibria, potential functions and evolutionary dynamics of the congestion games with non-atomic players, atomic splitable players and atomic non-splittable players. Our model reduces to the classical congestion game with non-atomic players when one population takes all the traffic demand which results in homogeneous information structure.

We note that our setting is different from the literature on Bayesian congestion games with finite players. For example, the paper Gairing et al. (2005) studies a finite player Bayesian routing game with linear cost functions and type-dependent weights. The authors show that under their modeling assumptions, every weighted Bayesian congestion game has a pure Nash Equilibrium. Mixed equilibrium, social cost and computational complexity are also studied. However, their paper does not focus on effect of information structure. In more related work, the article van Heumen et al. (1996) studies an extended Bayesian potential game with finite players, in which a subset of players have access to private information. The authors show that a pure

Nash Equilibrium exists when the game has a common prior. In a follow up work Facchini et al. (1997), the authors impose additional conditions on players' utility functions to show that their game is a weighted potential game if and only if the utility function can be written as a coordination function plus a "dummy function". They also provide an example to show that pure Nash Equilibrium may not exist when there is no common prior. Our model contributes to the existing literature by incorporating the information heterogeneity to congestion games with non-atomic players, and studying the effect of the relative population size. The equilibrium characterization and the analysis of value of information in our paper cannot be obtained by straightforward application of the known results due to the key difficulty arised by the interaction of two populations with asymmetric information.

1.3 Main Contributions

Our main results are as follows: We show that the Bayesian congestion game has a weighted potential function, and any Bayesian Wardrop equilibrium is a feasible strategy profile that minimizes the potential function. As the population sizes change, qualitative properties of equilibrium strategies change, resulting in three distinct regimes. Specifically, the demand assigned to each edge in equilibrium does not depend on the population sizes in one regime, but changes as population sizes vary in the other two regimes. The intuition is that the impacts of information on populations' equilibrium strategies are fully achieved in one regime, but tightly constrained by the demand of either population in the other two regimes. A crucial step in our mathematical argument is constructing a convex optimization problem that can directly compute the aggregated demand assigned to each route by both populations in equilibrium. The equilibrium regimes are derived from the tightness of constraints in optimum, which reflect the impact of information on equilibrium strategy profiles.

We define the relative value of information as the difference in the average cost of two populations in equilibrium. For any two heterogeneous information systems, we study how relative value of information changes as the population sizes vary.

We find the connection between the relative value of information and the first order differentiation of the potential function in equilibrium with respect to the population size. By applying results of perturbation analysis in Bonnans and Shapiro (2013), Fiacco (1984), Rockafellar (1984) and Milgrom and Segal (2002), we show that the expected costs of two populations are equal in one regime, while in the other two regimes, the population with smaller demand has lower expected cost. Interestingly, the regime with equal population costs is the same regime where the equilibrium edge load does not change with the relative population size. This is because in that regime, the expected costs on the set of routes used by both populations are equal in equilibrium. However, in the other two regimes, some routes with high cost are taken by the population with higher demand but not by the population with smaller demand. Thus, the population with smaller demand has lower expected cost. Furthermore, if one population is informed, and the other population is uninformed, three regimes reduce to two, and the cost of the informed population is no higher than that of the uninformed population. Notably, the relative value of information decreases as the size of informed population gets larger. If both populations are informed, the "better informed" population can have higher cost if its size is large enough.

Finally, we study how equilibrium social cost changes as population sizes vary. We demonstrate examples where equilibrium social cost can be non-convex and non-differentiable in population sizes. We provide bounds on the equilibrium social cost, and a sufficient condition, in which the bounds are tight. Additionally, when the condition is satisfied, the equilibrium social cost attains a minimum for a continuous range of relative population size. We also present analysis on the worst case inefficiency of BWE, which is related to the well-known notion of "price of anarchy" in previous work, e.g. Roughgarden (2003), Koutsoupias and Papadimitriou (1999), Milchtaich (2004), Acemoglu and Ozdaglar (2007). Although our model considers a heterogeneous information structure, the worst-case inefficiency is the same as the complete information case in Roughgarden (2003).

The rest of the thesis is organized as follows: Chapter 2 analyzes a simple example, in which some travelers have complete information, while others are uninformed. Chapter 3 introduces the model environment and formulates the game. Next, we present equilibrium characterization in Chapter 4. Finally, the analysis of relative value of information is in Chapter 5, and the equilibrium social cost and ineffiency is studied in Chapter 6.

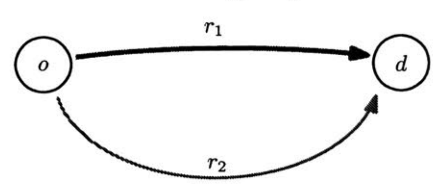
Chapter 2

Motivating Example

In this chapter, we describe a simple two route example to illustrate the main ideas in our paper. Then we analyze the equilibrium in this simple example. Finally, we study the expected population costs and the social cost in equilibrium.

2.1 Two Route Model

Consider an origin-destination pair of nodes connected by two parallel routes, r_1 and r_2. Assume that the set of network states is $\mathcal{S} = \{\mathbf{a}, \mathbf{n}\}$, where the state $\mathbf{a}$ represents an incident on route r_1, and the state $\mathbf{n}$ represents the nominal (i.e. no incident) condition. The route r_1 faces an incident with probability p. Each route's cost function (travel time) is an affine function of the flow through it, i.e.



$$c_1^s(f_1) = \begin{cases} \alpha_1^{\mathbf{a}} f_1 + b_1, & s = \mathbf{a}, \\ \alpha_1^{\mathbf{n}} f_1 + b_1, & s = \mathbf{n}. \end{cases}$$

$$c_2(f_2) = \alpha_2 f_2 + b_2.$$

For simplicity, we will assume in this example that $\alpha_1^{\mathbf{n}} < \alpha_2 < \alpha_1^{\mathbf{a}}$ and $b_1 = b_2 = b$. The network is subject to a unit demand comprising of two traveler populations $\mathcal{I} = \{1, 2\}$. The population demands are denoted by $\lambda^1 = \lambda$ and $\lambda^2 = 1 - \lambda$. Each population receives a noisy signal, t^i, of the network state from its subscribed information system. The signal space of population i is $\mathcal{T}^i = \{\mathbf{a}, \mathbf{n}\}$. We assume that population 1 receives the correct state information with probability 1 (i.e. complete information), and population 2 receives $\mathbf{a}$ or $\mathbf{n}$ with probability 0.5 independently of the state (i.e. no information).

2.2 Equilibrium

Let us study how equilibrium strategies and route flows change with respect to λ. Let $q_1^i(t^i)$ denote the demand assigned to route r_1 by population i when receiving signal t^i; the remaining demand $\lambda^i - q_1^i(t^i)$ is assigned to route r_2. Since signal t^2 is independent with states, we have $q_1^2(\mathbf{a}) = q_1^2(\mathbf{n}) \triangleq q_1^2$. Any feasible demand assignment must satisfy the constraints: $0 \leq q_1^1(t^1) \leq \lambda$ and $0 \leq q_1^2 \leq 1 - \lambda$. We represent a feasible routing strategy profile as $q_1 = (q_1^1(\mathbf{a}), q_1^1(\mathbf{n}), q_1^2)$.

The expected costs on route r for population 1 receiving signal t^1, denoted $\mathbb{E}[c_r(q)|t^1]$, can be written as follows:

$$\mathbb{E}[c_1(q)|t^1] = \begin{cases} c_1^{\mathbf{a}}\left(q_1^1(\mathbf{a}) + q_1^2\right), & t^1 = \mathbf{a}, \\ c_1^{\mathbf{n}}\left(q_1^1(\mathbf{n}) + q_1^2\right), & t^1 = \mathbf{n}. \end{cases}$$

$$\mathbb{E}[c_2(q)|t^1] = c_2(1 - q_1^1(t^1) - q_1^2), \quad \forall t^1 \in \mathcal{T}^1.$$

For population 2, the expected route costs $\mathbb{E}[c_r(q)|t^2]$ can be expressed as:

$$\mathbb{E}[c_1(q)|t^2] = p \cdot c_1^{\mathbf{a}}\left(q_1^1(\mathbf{a}) + q_1^2\right) + (1-p) \cdot c_1^{\mathbf{n}}\left(q_1^1(\mathbf{n}) + q_1^2\right),$$

$$\mathbb{E}[c_2(q)|t^2] = p \cdot c_2(1 - q_1^1(\mathbf{a}) - q_1^2) + (1-p) \cdot c_2(1 - q_1^1(\mathbf{n}) - q_1^2).$$

This routing game with asymmetric information structure admits a Bayesian Wardrop equilibrium; see Definition 1 in Sec. 3.2. Let $q_1^* = (q_1^{*1}(\mathbf{a}), q_1^{*1}(\mathbf{n}), q_1^{*2})$ denote an equilibrium. Each population with each signal can either assign all the demand on one route, or split on both routes. There are in total $3 \times 3 \times 3 = 27$ cases. We now mention the case that hold in equilibrium (see Sec. 4 for general results on equilibrium characterization).

Thus, the expected costs on both routes are equal for each population in equilirbium. Consequently, both populations split their demand on both routes. It is easy to check that all equilibria induce an identical route flow in each state.

Notice that if $\lambda \in [0, \underline{\lambda}]$, we have $q_1^{*1}(\mathbf{n}) - q_1^{*1}(\mathbf{a}) = \lambda$, i.e. population 1 shift all its demand from r_1 to r_2 on receiving incident information. However, if $\lambda \in (\underline{\lambda}, 1]$, we have $q_1^{*1}(\mathbf{n}) - q_1^{*1}(\mathbf{a}) = \underline{\lambda} < \lambda$, i.e. only part of demand is shifted to r_2 on receiving incident information.

2.3 Equilibrium Costs

Let us calculate the expected cost of each population in equilibrium. If $\lambda \in [0, \underline{\lambda})$, the expected equilibrium cost of population 1, denoted $C^{*1}(\lambda)$, can be written as follows:

$$C^{*1}(\lambda) = p\mathbb{E}[c_2(q)|\mathbf{a}] + (1-p)\mathbb{E}[c_1(q)|\mathbf{n}] = b + \bar{C} + \bar{C}^1\lambda,$$

where

$$\bar{C} = \frac{\alpha_2(\alpha_1^{\mathbf{n}}(1-p^2) + \alpha_1^{\mathbf{a}}p^2)}{\bar{\alpha}_1 + \alpha_2},$$
$$\bar{C}^1 = \frac{p(1-p)\alpha_1^{\mathbf{n}}(\alpha_1^{\mathbf{a}} + \alpha_2) + p(1-p)\alpha_2(\alpha_1^{\mathbf{n}} + \alpha_2)}{\bar{\alpha}_1 + \alpha_2}.$$

The expected costs of population 2 in equilibrium is as follows:

$$C^{*2}(\lambda) = \mathbb{E}[c_1(q)|t^2] = \mathbb{E}[c_2(q)|t^2] = b + \bar{C} + \bar{C}^2(1-\lambda),$$

where

$$\bar{C}^2 = \frac{\alpha_2 p(1-p)(\alpha_1^{\mathbf{a}} - \alpha_1^{\mathbf{n}})}{\bar{\alpha}_1 + \alpha_2}.$$

On can check that $C^{*2}(\lambda) - C^{*1}(\lambda) = \bar{C}^2(1-\lambda) - \bar{C}^1\lambda > 0$, thus the population 1 benefits from receiving information over population 2 which is uninformed. However, if $\lambda \in [\underline{\lambda}, 1]$, the expected cost of both populations are identical in equilibrium, because

First, there exsits a critical threshold fraction of population 1, denoted $\underline{\lambda}$, which distinguishes the equilibrium behavior between the cases $\lambda \in [0, \underline{\lambda})$ and $\lambda \in [\underline{\lambda}, 1]$. For this two-route example, let us define:

$$\underline{\lambda} \triangleq \alpha_2 \left(\frac{1}{\alpha_1^{\mathbf{n}} + \alpha_2} - \frac{1}{\alpha_1^{\mathbf{a}} + \alpha_2} \right).$$

On one hand, if $\lambda \in [0, \underline{\lambda})$, the routing game admits a unique equilibrium:

$$\begin{aligned} q_1^{*1}(\mathbf{a}) &= 0 \\ q_1^{*1}(\mathbf{n}) &= \lambda \\ q_1^{*2} &= \frac{\alpha_2}{\bar{\alpha}_1 + \alpha_2}(1-\lambda) - \lambda \frac{(1-p)\alpha_1^{\mathbf{n}}}{\bar{\alpha}_1 + \alpha_2} + \lambda \frac{p\alpha_2}{\bar{\alpha}_1 + \alpha_2}, \end{aligned}$$

where $\bar{\alpha}_1 = p\alpha_1^{\mathbf{a}} + (1-p)\alpha_1^{\mathbf{n}}$ is the average slope of route r_1's cost function. This equilibrium is obtained by solving the following conditions:

$$\mathbb{E}[c_1(q^*)|\mathbf{a}] > \mathbb{E}[c_2(q^*)|\mathbf{a}], \quad \mathbb{E}[c_1(q^*)|\mathbf{n}] < \mathbb{E}[c_2(q^*)|\mathbf{n}], \quad \mathbb{E}[c_1(q^*)|t^2] = \mathbb{E}[c_2(q^*)|t^2].$$

Thus, population 1 faces unequal route costs in equilibrium. Consequently, in state $\mathbf{n}$, population 1 assigns all its demand on route r_1, and in state $\mathbf{a}$, it assigns all its demand to route r_2. Population 2 splits on both routes.

On the other hand, if $\lambda \in [\underline{\lambda}, 1]$, the game admits multiple equilibria. The set of equilibrium strategy profiles can be described as follows:

$$\begin{aligned} q_1^{*1}(\mathbf{a}) &= \chi, \\ q_1^{*1}(\mathbf{n}) &= \underline{\lambda} + \chi, \\ q_1^{*2} &= \frac{\alpha_2}{\alpha_1^{\mathbf{a}} + \alpha_2} - \chi, \end{aligned}$$

where $\max\left\{0, \lambda - \frac{\alpha_1^{\mathbf{a}}}{\alpha_1^{\mathbf{a}}+\alpha_2}\right\} \leq \chi \leq \min\left\{\frac{\alpha_2}{\alpha_1^{\mathbf{a}}+\alpha_2}, \lambda - \underline{\lambda}\right\}$. This set of equilibria is obtained from the following condition:

$$\mathbb{E}[c_1(q^*)|\mathbf{a}] = \mathbb{E}[c_2(q^*)|\mathbf{a}], \quad \mathbb{E}[c_1(q^*)|\mathbf{n}] = \mathbb{E}[c_2(q^*)|\mathbf{n}], \quad \mathbb{E}[c_1(q^*)|t^2] = \mathbb{E}[c_2(q^*)|t^2].$$

the expected costs for r_1 and r_2 equalize for each population:

$$C^{*1}(\lambda) = C^{*2}(\lambda) = b + \bar{C} + \frac{\bar{C}^1\bar{C}^2}{\bar{C}^1 + \bar{C}^2},$$

and consequently, the population 1 does not get any benefit of being informed about the incident on r_1. We define the relative value of information $V^*(\lambda)$ as the difference of two population costs in equilibrium:

$$V^*(\lambda) \triangleq C^{*2}(\lambda) - C^{*1}(\lambda) = \begin{cases} \frac{1}{\bar{\alpha}_1+\alpha_2} p\,(1-p)\left[\alpha_2(\alpha_1^{\mathbf{a}} - \alpha_1^{\mathbf{n}}) - \lambda(\alpha_1^{\mathbf{a}} + \alpha_2)(\alpha_1^{\mathbf{n}} + \alpha_2)\right], & \forall \lambda \in [0, \underline{\lambda}), \\ 0, & \forall \lambda \in [\underline{\lambda}, 1]. \end{cases}$$

Note that for $\lambda \in [0, \underline{\lambda})$, V decreases linearly in λ.

Finally, the equilibrium social cost is simply the average expected cost in equilibrium:

$$C^*(\lambda) = \lambda C^{*1}(\lambda) + (1-\lambda)C^{*2}(\lambda) = \begin{cases} \left(\bar{C}^1 + \bar{C}^2\right)\lambda^2 - 2\bar{C}^2\,\lambda + \bar{C} + b + \bar{C}^2, & \forall \lambda \in [0, \underline{\lambda}), \\ b + \bar{C} + \frac{\bar{C}^1\bar{C}^2}{\bar{C}^1+\bar{C}^2}, & \forall \lambda \in [\underline{\lambda}, 1]. \end{cases}$$

Note that the equilibrium social cost is a quadratic function of λ if $\lambda \in [0, \underline{\lambda})$, but does not change with $\lambda \in [\underline{\lambda}, 1]$. Furthermore, for $\lambda \in [0, \underline{\lambda})$, we can write:

$$\frac{\partial C^*(\lambda)}{\partial \lambda} = 2\left(\bar{C}^1 + \bar{C}^2\right)\lambda - 2\bar{C}^2 < 0,$$

and check that $\frac{\partial C^*(\lambda)}{\partial \lambda} = 0$ when $\lambda = \underline{\lambda}$. Thus, the social cost $C^*(\lambda)$ monotonically decreases with λ in the range $\lambda \in [0, \underline{\lambda})$, and attains the minimum value in the range $[\underline{\lambda}, 1]$. That is, increasing the share of the informed population decreases the social cost but only up to the threshold $\underline{\lambda}$; beyond this fraction, the social cost does not change.

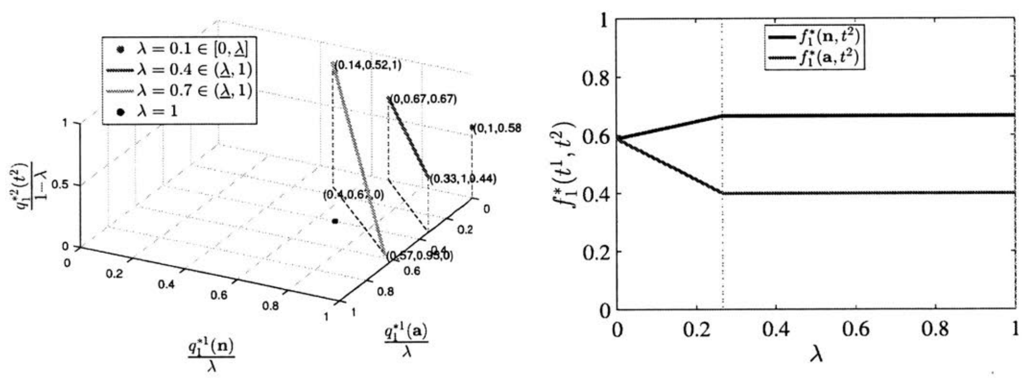
2.4 Numerical Experiment

We illustrate the abovementioned results using the network parameters in Table 2.1.

Table 2.1: Parameter values for the 2 - route example.

Symbol	Value	Units
$\alpha_1^{\mathbf{n}}$	1	$min/(veh \cdot hr^{-1})$
$\alpha_1^{\mathbf{a}}$	3	$min/(veh \cdot hr^{-1})$
α_2	2	$min/(veh \cdot hr^{-1})$
b	20	min
D	1	$10^3 veh/hr$
p	0.2	

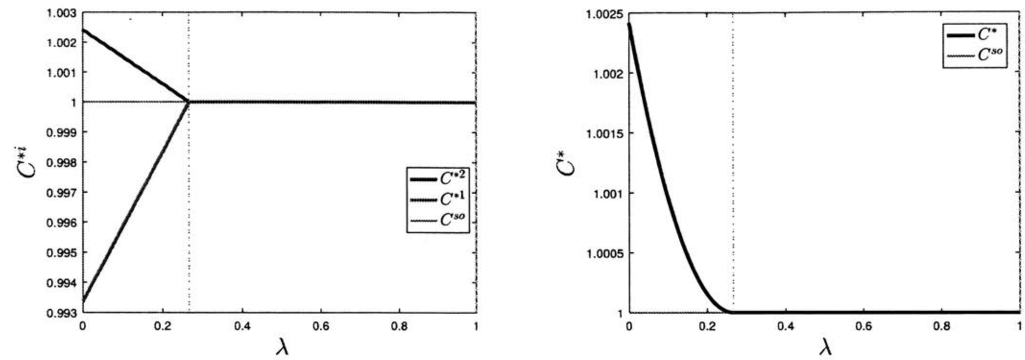
The threshold $\underline{\lambda}$ for this example is $4/15$. Fig. 2-1a shows the equilibrium strategy $(q_1^{*1}(\mathbf{a}), q_1^{*1}(\mathbf{n}), q_1^{*2})$. Fig. 2-1b shows the equilibrium flow on r_1 in each state.



(a) (b)

Figure 2-1: Equilibrium strategy and route flow.

Next, we plot the expected population costs and the social cost in equilibrium. These costs are normalized by the socially optimum cost, denoted C^{so}, which is the minimum average social cost achievable by a fully informed social planner.



(a) (b)

Figure 2-2: Equilibrium population costs and equilibrium social cost

From this simple example, we analytically characterized the effect of the relative population sizes λ on the equilibrium structure, population cost and social cost. In this article, we generalize these results to the environment in which two populations with asymmetric information about the network state but identical prefernces route their flows on a network.

Chapter 3

Model

Our modeling comprises of a network with an unknown state and state-dependent edge cost functions. Two information systems measure the network state with different levels of accuracy and induce information heterogeneity among populations of travelers with identical preferences.

3.1 Environment

We model the transportation network as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a single origin-destination pair. Let $\mathcal{V}$ denote the set of vertices, and $\mathcal{E}$ the set of edges. Let $\mathcal{R}$ denote the set of routes connecting the origin and the destination. The network is in an environment with a random state s, which is drawn by a fictitious player "Nature" from a finite set $\mathcal{S}$ according to a prior distribution $\theta \in \Delta(\mathcal{S})$. The cost function of any edge $e \in \mathcal{E}$ in state s, denoted $c_e^s(\cdot)$, is a positive, increasing[1], and differentiable function of the demand assigned to the edge e. Let $\mathcal{C}$ denote the set of cost functions of all edges in all states.

We introduce a set of two information systems, denoted $\mathcal{I} = \{1, 2\}$. Each traveler is exclusively subscribed to one information system. Travelers are modeled as a continuous player set with a fixed total demand D. We call the set of travelers

[1]Results in this article can be extended to the case with non-decreasing cost functions. For simplicity of our discussion, we focus on the case with increasing cost functions.

subscribed to the information system 1 (resp. information system 2) as population 1 (resp. population 2). Let us denote λ as the population size parameter, and denote the fraction of population i's demand as λ^i, where $\lambda^1 = \lambda$, $\lambda^2 = 1 - \lambda$. Thus, the demands of populations 1 and 2 are λD and $(1 - \lambda)$D, respectively.

Each information system $i \in \mathcal{I}$ sends a noisy signal t^i of the state s to population i. The finite signal space of information system i is denoted as $\mathcal{T}^i$. Note that $|\mathcal{T}^1|$, $|\mathcal{T}^2|$ and $|\mathcal{S}|$ need not be equal. The joint distribution of the state s and the signals t^1, t^2 is denoted $\pi \in \Delta(\mathcal{S} \times \mathcal{T}^1 \times \mathcal{T}^2)$, which satisfies the constraint that the marginal probability distribution on the state is equal to the prior distribution on state, i.e. for any $s \in \mathcal{S}$,

$$\sum_{t^1 \in \mathcal{T}^1} \sum_{t^2 \in \mathcal{T}^2} \pi(s, t^1, t^2) = \theta(s). \tag{3.1}$$

The probability that population 1 and 2 receiving signals t^1 and t^2 conditional on the state $s \in \mathcal{S}$, denoted $p(t^1, t^2|s)$, can be expressed as:

$$p(t^1, t^2|s) = \frac{\pi(s, t^{-i}, t^i)}{\theta(s)}, \quad \forall s \in \mathcal{S}, \quad \forall \left(t^1, t^2\right) \in \mathcal{T}. \tag{3.2}$$

3.2 Bayesian Congestion Game

In the framework of Bayesian games (see Harsanyi (1967)), the notion of type captures the private information received by each population. In our model, the private information of population i is the signal t^i. The type space of population i is $\mathcal{T}^i$. Based on t^i, population i generates a belief about the state s and the other population's type t^{-i}, denoted $\mu^i(s, t^{-i}|t^i) \in \Delta(\mathcal{S} \times \mathcal{T}^{-i})$.

Each population routes its demand through the network based on its belief $\mu^i(s, t^{-i}|t^i)$. Population i's strategy is a map from the type (or signal) space $\mathcal{T}^i$ to a $|\mathcal{R}|$-dimensional vector, denoted $q^i(t^i) = \left(q^i_r(t^i)\right)_{r \in \mathcal{R}}$, where $q^i_r(t^i)$ is the demand assigned by population i to route r when its type is t^i. We say that a strategy profile $q \triangleq (q^1, q^2)$ is *feasible* if it satisfies the following constraints:

$$\sum_{r\in\mathcal{R}} q_r^i(t^i) = \lambda^i D, \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}, \tag{3.3a}$$

$$q_r^i(t^i) \geq 0, \quad \forall r \in \mathcal{R}, \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}. \tag{3.3b}$$

The constraint (3.3a) ensures that the demand of each population is routed, and the constraint (3.3b) imposes that demand assigned on each route is nonnegative. Let $\mathcal{Q}^i(\lambda)$ denote the set of all feasible strategies of population i when the population size parameter is λ. From (3.3a)-(3.3b), we note that the set of feasible strategy profiles, defined as $\mathcal{Q}(\lambda) \triangleq \mathcal{Q}^1(\lambda) \times \mathcal{Q}^2(\lambda)$, is a convex polytope.

We are now ready to define the Bayesian congestion game Γ. Formally,

$$\Gamma \triangleq (\mathcal{I}, \mathcal{S}, \mathcal{T}, \mathcal{Q}(\lambda), \mathcal{C}, \mu),$$

where:

$\mathcal{I}$: Set of populations, $\mathcal{I} = \{1, 2\}$

$\mathcal{S}$: Finite set of states with prior distribution $\theta \in \Delta(\mathcal{S})$

$\mathcal{T} = (\mathcal{T}^i)_{i\in\mathcal{I}}$: Set of population type profiles (t^1, t^2).

$\mathcal{Q}(\lambda) = (\mathcal{Q}^i(\lambda))_{i\in\mathcal{I}}$: Set of feasible strategy profiles when the population size parameter is λ, with $q = (q^1, q^2) \in \mathcal{Q}(\lambda)$

$\mathcal{C} = \{c_e^s(\cdot)\}_{e\in\mathcal{E}, s\in\mathcal{S}}$: Set of state-dependent edge cost functions

$\mu = (\mu^i)_{i\in\mathcal{I}}$: μ^i is population i's belief about the state s and the other population's type t^{-i}

Importantly, our model assumes that the joint distribution π is common knowledge. In addition to π, the set of network states $\mathcal{S}$, the type sets $|\mathcal{T}^1|$ and $|\mathcal{T}^2|$, the network graph $(\mathcal{V}, \mathcal{E})$, the total demand D, the population parameter λ, and the set of

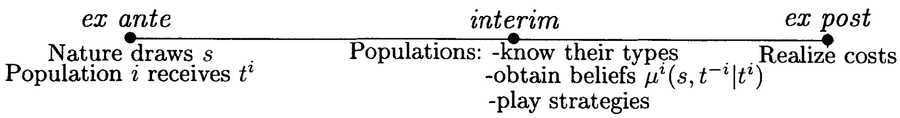


Figure 3-1: Timing of the game.

cost functions C are also common knowledge. The signal t^i is the private information of population i.

The game is played as shown in Fig 3-1. In the *ex ante* stage, Nature draws a realization of the network state $s \in \mathcal{S}$ according to the prior distribution $\theta(s)$. Signals (types) t^1 and t^2 are realized according to the conditional probability distribution $p(t^1, t^2|s)$ in (3.2). In the *interim* stage, each population forms belief $\mu^i(s, t^{-i}|t^i)$, and both populations choose their strategies q^i simultaneously.

For any $i \in \mathcal{I}$ and $t^i \in \mathcal{T}^i$, beliefs are obtained by Bayesian update on the common prior:

$$\mu^i(s, t^{-i}|t^i) = \frac{\pi(s, t^i, t^{-i})}{\Pr(t^i)}, \quad \forall s \in \mathcal{S}, \quad \forall t^{-i} \in \mathcal{T}^{-i}. \tag{3.4}$$

where $Pr(t^i) = \sum_{s \in \mathcal{S}} \sum_{t^{-i} \in \mathcal{T}^{-i}} \pi(s, t^i, t^{-i})$.

For a given strategy profile $q = (q^1, q^2) \in \mathcal{Q}(\lambda)$, the induced route flow is $f \triangleq (f_r(t^1, t^2))_{r \in \mathcal{R}, (t^1, t^2) \in \mathcal{T}}$, where $f_r(t^1, t^2)$ is the aggregated demand assigned by populations with type profile (t^1, t^2) on route r:

$$f_r(t^1, t^2) = q_r^1(t^1) + q_r^2(t^2), \quad \forall r \in \mathcal{R}, \quad \forall (t^1, t^2) \in \mathcal{T}. \tag{3.5}$$

Note that the dependence of f on q is implicit and is dropped for notational convenience.

The induced edge load is $w \triangleq (w_e(t^1, t^2))_{e \in \mathcal{E}, (t^1, t^2) \in \mathcal{T}}$, where $w_e(t^1, t^2)$ is the load on edge e assigned by populations with type profile (t^1, t^2):

$$w_e(t^1, t^2) = \sum_{r \ni e} \left(q_r^1(t^1) + q_r^2(t^2)\right) \stackrel{(3.5)}{=} \sum_{r \ni e} f_r(t^1, t^2), \quad \forall e \in \mathcal{E}, \quad \forall (t^1, t^2) \in \mathcal{T}, \tag{3.6}$$

The corresponding cost of edge $e \in \mathcal{E}$ in state $s \in \mathcal{S}$ is $c_e^s(w_e(t))$. The cost of route

$r \in \mathcal{R}$, denoted $c_r^s(q(t))$ is:

$$c_r^s(q(t)) = \sum_{e \in sr} c_e^s(w_e(t)), \quad \forall r \in \mathcal{R}, \quad \forall s \in \mathcal{S}, \quad \forall t \in \mathcal{T}. \tag{3.7}$$

The expected cost of route r based on the interim belief $\mu^i(s, t^{-i}|t^i)$ is given by:

$$\begin{aligned}
\mathbb{E}[c_r(q)|t^i] &= \sum_{s \in \mathcal{S}} \sum_{t^{-i} \in \mathcal{T}^{-i}} \sum_{e \in r} \mu^i(s, t^{-i}|t^i) c_e^s(w_e(t^i, t^{-i})) \\
&= \sum_{s \in \mathcal{S}} \sum_{t^{-i} \in \mathcal{T}^{-i}} \sum_{e \in r} \frac{\pi(s, t^i, t^{-i})}{\Pr(t^i)} c_e^s(w_e(t^i, t^{-i})), \quad \forall r \in \mathcal{R}, \quad \forall t^i \in t^i, \quad \forall i \in \mathcal{I},
\end{aligned} \tag{3.8}$$

where $w_e(t^i, t^{-i})$ is given by (3.6).

We now define the equilibrium concept of the game Γ:

Definition 1. *Bayesian Wardrop Equilibrium (BWE)*
A strategy profile $q^* : \mathcal{T} \to \mathcal{Q}(\lambda)$ is a Bayesian Wardop Equilibrium (BWE) if for any $i \in \mathcal{I}$, and any $t^i \in \mathcal{T}^i$:

$$\forall r \in \mathcal{R}, \quad q_r^{*i}(t^i) > 0 \quad \Rightarrow \quad \mathbb{E}[c_r(q^*)|t^i] \leq \mathbb{E}[c_{r'}(q^*)|t^i], \quad \forall r' \in \mathcal{R}. \tag{3.9}$$

Equivalently, in a BWE, each population i with type t^i assigns its demand only on the routes that have the smallest expected cost based on the interim belief $\mu^i(s, t^{-i}|t^i)$.

We can define *interim game*[2] $IG(\Gamma) = \left(\widehat{I}, \widehat{\mathcal{Q}}(\lambda), \widehat{C}\right)$, where:

$\widehat{\mathcal{I}} \triangleq \cup_{i \in \mathcal{I}} \mathcal{T}^i$: population set. Each type can be viewed as a population.

$\widehat{\mathcal{Q}}(\lambda) \triangleq \left(\widehat{\mathcal{Q}}^{t^i}(\lambda)\right)_{t^i \in \widehat{\mathcal{I}}}$: strategy set. Each type $t^i \in \mathcal{T}^i$ has strategy set $\widehat{\mathcal{Q}}^{t^i}(\lambda) \equiv \mathcal{Q}^i(\lambda)$.

$\widehat{C} \triangleq \left\{\widehat{c}_r^{t^i}\right\}_{t^i \in \widehat{I}, r \in \mathcal{R}}$: cost function set. The cost of type t^i taking route r is $\widehat{c}_r^{t^i}(q) \equiv \mathbb{E}[c_r(q)|t^i]$.

[2]Since the game has common prior, the *ex ante* game exists and is equivalent to the interim game.

$IG(\Gamma)$ is a congestion game with type-specific cost function $\mathbb{E}[c_r(q)|t^i]$. It is shown in Milchtaich (1996) that generally there may not exist a potential function in a congestion game with player-specific cost functions. In our model, the type-specific cost function $\mathbb{E}[c_r(q)|t^i]$ is based on t^i's belief, which is obtained from the common prior π in (3.2). We show in Sec. 4.1 that due to the existence of common prior π, the type-specific cost functions $\mathbb{E}[c_r(q)|t^i]$ are related so that a weighted potential function exists in $IG(\Gamma)$.

$IG(\Gamma)$ is a complete information population game. A strategy profile $\widehat{q}^* \in \widehat{\mathcal{Q}}(\lambda)$ is a Wardrop equilibrium of $IG(\Gamma)$, if for any $r \in \mathcal{R}$ and any $t^i \in \widehat{\mathcal{I}}$, $\widehat{q}_r^{*i}(t^i) > 0$ implies that $\widehat{c}_r^{t^i}(\widehat{q}^*)$ is the smallest among all routes, i.e. $\widehat{q}^*$ satisfies (3.9). Therefore, q^* is BWE in Γ if and only if $\widehat{q}^* = \left((q^{*1}(t^1))_{t^1 \in \mathcal{T}^1}, (q^{*2}(t^2))_{t^2 \in \mathcal{T}^2}\right)$ is a Wardrop equilibrium in $IG(\Gamma)$. In rest of the article, we will refer to the Bayesian congestion game Γ and the interim game $IG(\Gamma)$ interchangeably. We consider each type t^i as a population. Consequently, a strategy of t^i can be viewed as a $|\mathcal{R}|$-dimensional vector $q^i(t^i) = (q_r^i(t^i))_{r \in \mathcal{R}}$, and a strategy profile can be viewed as a $|\mathcal{R}| \times (|\mathcal{T}^1| + |\mathcal{T}^2|)$ dimensional vector.

Chapter 4

Equilibrium Characterization

In this chapter, we study the equilibrium structure of the game Γ. In Sec. 4.1, we show that Γ is a weighted potential game, and that any equilibrium strategy profile is an optimum of a convex optimization problem. In Sec. 4.2, we study the qualitative properties of BWE when the population size parameter λ varies from 0 to 1.

4.1 Weighted Potential Game

We adopt the definition of a weighted potential game with continuous player set from Sandholm (2001):

Definition 2. *Game Γ is a weighted potential game if there exists a continuously differentiable function $\Phi(q) : \mathcal{Q}(\lambda) \to \mathbb{R}$ and a set of positive, type-specific weights $\{\gamma(t^i)\}_{t^i \in \mathcal{T}^i, i \in \mathcal{I}}$ such that:*

$$\frac{\partial \Phi(q(t))}{\partial q_r^i(t^i)} = \gamma(t^i)\mathbb{E}[c_r(q)|t^i], \quad \forall r \in \mathcal{R}, \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}.$$

Lemma 1. *Game Γ is a weighted potential game with the weighted potential function $\Phi(q)$ as follows:*

$$\Phi\left(q\right) \triangleq \sum_{s \in \mathcal{S}} \sum_{e \in \mathcal{E}} \sum_{t^1 \in \mathcal{T}^1} \sum_{t^2 \in \mathcal{T}^2} \pi\left(s, t^1, t^2\right) \int_0^{\sum_{r \ni e} q_r^1(t^1) + q_r^2(t^2)} c_e^s(z) dz, \tag{4.1}$$

and the positive type-specific weight $\gamma(t^i) = \Pr(t^i)$ *for any* $t^i \in \mathcal{T}^i, i \in \mathcal{I}$.

Proof. To show that $\Phi(q)$ is a weighted potential function of Γ, we write the first order derivative of $\Phi(q)$ with respect to $q_r^i(t^i)$:

$$\frac{\partial \Phi(q)}{\partial q_r^i(t^i)} = \sum_{s\in\mathcal{S}} \sum_{t^{-i}\in\mathcal{T}^{-i}} \pi(s,t^i,t^{-i}) \sum_{e\in r} c_e^s\left(w_e(t^i,t^{-i})\right)$$
$$\stackrel{(3.8)}{=} \Pr(t^i)\mathbb{E}[c_r(q)|t^i], \quad \forall r \in \mathcal{R}, \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}. \tag{4.2}$$

We immediately obtain that $\Phi(q)$ satisfies Definition 2 with $\gamma(t^i) = \Pr(t^i)$, $\forall t^i \in \mathcal{T}^i, \forall i \in \mathcal{I}$. □

Note that $\Phi(q)$ is a continuous and differentiable function of q. Following (3.5) and (3.6), Φ can be equivalently expressed as a function of the induced route flow f or the induced edge load w:

$$\widehat{\Phi}(f) \triangleq \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}} \pi(s,t) \int_0^{\sum_{r\ni e} f_r(t)} c_e^s(z)dz. \tag{4.3}$$

$$\Phi(w) \triangleq \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}} \pi(s,t) \int_0^{w_e(t)} c_e^s(z)dz. \tag{4.4}$$

For any feasible strategy profile $q \in \mathcal{Q}(\lambda)$, we note that $\Phi(q) \equiv \widehat{\Phi}(f) \equiv \Phi(w)$, where f and w are the route flows and edge loads induced by q.

We will need the following lemma subsequently:

Lemma 2. $\Phi(w)$ *defined in* (4.4) *is continuously differentiable and strictly convex in* w.

Proof. Since each $c_e^s(w_e(t))$ is differentiable in $w_e(t)$, we conclude from (4.4) that $\Phi(w)$ is twice differentiable with respect to w. The first order partial derivative of $\Phi(w)$ with respect to $w_e(t)$ can be written as:

$$\frac{\partial \Phi(w)}{\partial w_e(t)} = \sum_{s\in\mathcal{S}} \pi(s,t)c_e^s\left(w_e(t)\right), \quad \forall e \in \mathcal{E}, \quad \forall t \in \mathcal{T},$$

and the second order derivative of $\Phi(w)$ can be written as follows:

$$\frac{\partial^2 \Phi(w)}{\partial w_e(t) \partial w_{e'}(t')} = \begin{cases} \sum_{s \in \mathcal{S}} \pi(s,t) \frac{dc_e^s(w_e(t))}{dw_e(t)}, & \text{if } e = e' \text{ and } t = t', \\ 0, & \text{otherwise.} \end{cases} \quad \forall e, e' \in \mathcal{E}, \quad \forall t, t' \in \mathcal{T}.$$

Since for any $e \in \mathcal{E}$, $s \in \mathcal{S}$, c_e^s is an increasing function of w_e, $\sum_{s \in \mathcal{S}} \pi(s,t) \frac{dc_e^s(w_e(t))}{dw_e(t)} > 0$. Thus, the Hessian matrix of $\Phi(w)$ is a matrix with positive elements on the diagonal and 0 for all other entities, i.e. it is positive definite. Therefore, $\Phi(w)$ is strictly convex in w. □

Theorem 1. *A strategy profile $q = (q^1, q^2)$ is a BWE if and only if it is an optimal solution of the following convex optimization problem:*

$$\begin{aligned} \min \quad & \Phi(q) \\ s.t. \quad & q \in \mathcal{Q}(\lambda), \end{aligned} \tag{OPT-$\mathcal{Q}$}$$

where $\mathcal{Q}(\lambda)$ is the set of feasible strategy profiles satisfying the constraints in (3.3).

We can write the Lagrangian of (OPT-$\mathcal{Q}$) as follows:

$$\mathcal{L}(q, \mu, \nu) = \Phi(q) + \sum_{i \in \mathcal{I}} \sum_{t^i \in \mathcal{T}^i} \mu^{t^i} \left(\lambda^i D - \sum_{r \in \mathcal{R}} q_r^i(t^i) \right) - \sum_{r \in \mathcal{R}} \sum_{i \in \mathcal{I}} \sum_{t^i \in \mathcal{T}^i} \nu_r^{t^i} q_r^i(t^i), \tag{4.5}$$

where $\mu = (\mu^{t^i})_{t^i \in \mathcal{T}^i, i \in \mathcal{I}}$ and $\nu = (\nu_r^{t^i})_{r \in \mathcal{R}, t^i \in \mathcal{T}^i, i \in \mathcal{I}}$ are Lagrange multipliers associated with the constraints (3.3a) and (3.3b), respectively.

Proof. We first show that a minimum of (OPT-$\mathcal{Q}$) is a BWE. For any optimal solution q, there must exist μ and ν such that (q, μ, ν) satisfies the following KKT conditions: KKT

$$\frac{\partial \mathcal{L}}{\partial q_r^i(t^i)} = \frac{\partial \Phi}{\partial q_r^i(t^i)} - \mu^{t^i} - \nu_r^{t^i} = 0, \qquad \forall r \in \mathcal{R}, \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}, \tag{4.6}$$

$$\nu_r^{t^i} q_r^i(t^i) = 0, \qquad \forall r \in \mathcal{R}, \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}, \tag{4.7}$$

$$\nu_r^{t^i} \geq 0, \qquad \forall r \in \mathcal{R}, \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}. \tag{4.8}$$

Using (4.2) and (4.6), we obtain that:

$$\frac{\partial \Phi(q)}{\partial q_r^i(t^i)} = \Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu^{t^i} + \nu_r^{t^i}, \quad \forall r \in \mathcal{R}, \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}.$$

From (4.7), we see that for any $r \in \mathcal{R}$, any $t^i \in \mathcal{T}^i$, and any $i \in \mathcal{I}$, if $q_r^i(t^i) > 0$, the corresponding Lagrange multiplier $\nu_r^{t^i} = 0$, and $\Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu^{t^i}$. However, if $q_r^i(t^i) = 0$, $\Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu^{t^i} + \nu_r^{t^i} \geq \mu^{t^i}$. Thus, if $q_r^i(t^i) > 0$ for a $r \in \mathcal{R}, t^i \in \mathcal{T}^i, i \in \mathcal{I}$, then:

$$\Pr(t^i)\mathbb{E}[c_r(q)|t^i] = \mu^{t^i} \leq \mu^{t^i} + \nu_{r'}^{t^i} = \Pr(t^i)\mathbb{E}[c_{r'}(q)|t^i], \quad \forall r' \in \mathcal{R},$$

which implies that $\mathbb{E}[c_r(q)|t^i] \leq \mathbb{E}[c_{r'}(q)|t^i]$ for any $r' \in \mathcal{R}$. Recalling Definition 1, we conclude that q is a BWE if it is an optimal solution of (OPT-$\mathcal{Q}$).

Next, we show that any BWE q^* is an optimal solution of (OPT-$\mathcal{Q}$). We define a pair of Lagrange multipliers $(\bar{\mu}, \bar{\nu})$, where $\bar{\mu}^{t^i} = \min_{r \in \mathcal{R}} \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i]$, and $\bar{\nu}_r^{t^i} = \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \bar{\mu}^{t^i}$. We can easily check that (4.6) and (4.8) are satisfied by such $(q^*, \bar{\mu}, \bar{\nu})$. Since q^* is a BWE, from Definition 1, if $q_r^{*i}(t^i) > 0$, $\mathbb{E}[c_r(q^*)|t^i] = \min_{r \in \mathcal{R}} \mathbb{E}[c_r(q^*)|t^i]$. Thus:

$$\bar{\nu}_r^{t^i} = \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \bar{\mu}^{t^i} = \Pr(t^i)\left(\mathbb{E}[c_r(q^*)|t^i] - \min_{r \in \mathcal{R}} \mathbb{E}[c_r(q^*)|t^i]\right) \overset{(3.9)}{=} 0.$$

Consequently, (4.7) is also satisfied by $(q^*, \bar{\mu}, \bar{\nu})$. Recall $\Phi(q) \equiv \Phi(w)$, since w is an affine function of q (see (3.6)) and $\Phi(w)$ is strictly convex in w (Lemma 2), $\Phi(q)$ is a convex function of q. Additionally, $\mathcal{Q}(\lambda)$ is a convex polytope, (OPT-$\mathcal{Q}$) is a convex problem. Thus, KKT conditions are sufficient for optimality, and q^* is an optimal solution of (OPT-$\mathcal{Q}$). □

The existence of BWE follows directly from Theorem 1. We denote the set of equilibrium strategy profiles as $\mathcal{Q}^*(\lambda)$. For any BWE $q^* \in \mathcal{Q}^*(\lambda)$, we define sets $\mathrm{M}(q^*)$ and $\mathrm{N}(q^*)$ as the set of μ^* and ν^* such that (q^*, μ^*, ν^*) satisfies the KKT conditions.

Next, we show that the equilibrium edge load w^* induced by any $q^* \in \mathcal{Q}^*(\lambda)$ is identical.

Corollary 1. *The BWE edge load w^* is unique.*

Proof. For any $q^* \in \mathcal{Q}^*(\lambda)$, $\Phi(q^*) = \Phi(w^*)$ is identical. Since $\Phi(w)$ is strictly convex in w (Lemma 2), we conclude that w^* is unique. □

Proposition 1. *For any $q^* \in \mathcal{Q}^*(\lambda)$, $\mathrm{M}(q^*)$ and $\mathrm{N}(q^*)$ are singleton sets with elements of μ^*, ν^*:*

$$\mu^{t^i *} = \min_{r \in \mathcal{R}} \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i], \quad \forall t^i \in \mathcal{T}^i, \quad \forall i \in \mathcal{I}. \tag{4.9}$$

$$\nu_r^{t^i *} = \Pr(t^i)\mathbb{E}[c_r(q^*)|t^i] - \mu^{t^i *}, \quad \forall r \in \mathcal{R}, \quad \forall t^i \in t^i, \quad \forall i \in \mathcal{I}. \tag{4.10}$$

Furthermore, both μ^* and ν^* are identical for any $q^* \in \mathcal{Q}^*(\lambda)$.

Proof. First, we argure by contradiction that *Linear Independence Constraint Qualifications* (LICQ) defined in definition 3 in Appendix A is satisfied in (OPT-$\mathcal{Q}$). We denote the set of constraints that are tight at optimum in (3.3b) as $\mathcal{B} \triangleq \{q_r^{*i}(t^i) \,|q_r^{*i}(t^i) = 0, \forall r \in \mathcal{R}, \forall t^i \in \mathcal{T}^i, \forall i \in \mathcal{I}\}$. Assume that LICQ does not hold, the set of equality constraints (3.3a) and the elements in the set $\mathcal{B}$ are linearly dependent. Since the equality constraints (3.3a) and the inequality constraints (3.3b) are each comprised of linearly independent affine functions, there must exist a type $\bar{t}^i$ such that the left hand side of the equality constraint $\sum_{r \in \mathcal{R}} q_r^{*i}(\bar{t}^i)$ is linearly dependent with the elements in the set $\mathcal{B}$, which implies that $q_r^{*i}(\bar{t}^i) \in \mathcal{B}$ for all $r \in \mathcal{R}$, i.e. $q_r^{*i}(\bar{t}^i) = 0, \forall r \in \mathcal{R}$. However, this violates the equality constraint $\sum_{r \in \mathcal{R}} q_r^{*i}(\bar{t}^i) = \lambda^i \mathrm{D}$ in (3.3a); hence we arrive at a contradiction.

We know from Proposition 10 in Appendix A that if LICQ holds, for any $q^* \in \mathcal{Q}^*$, $\mathrm{M}(q^*)$ and $\mathrm{N}(q^*)$ are singleton. From the proof of Theorem 1, we know that for any $q^* \in \mathcal{Q}^*(\lambda)$, (q^*, μ^*, ν^*) satisfies the KKT condition, where μ^*, ν^* are defined in (4.9) and (4.10). Furthermore, the equilibrium edge load induced by any $q^* \in \mathcal{Q}^*(\lambda)$ is identical (Corollary 1). From (3.8), μ^* and ν^* are identical for any $q^* \in \mathcal{Q}^*(\lambda)$. □

Proposition 1 connects the smallest expected route cost for each type t^i at equilibrium with the unique Lagrange multiplier μ^{t^i*}. This result will be used again when we discuss the relative value of information in Chapter. 5.

Theorem 1 shows that the equilibrium strategy profile can be solved as the solution of a convex optimization problem (OPT-$\mathcal{Q}$). However, characterizing q^* as λ varies directly from (OPT-$\mathcal{Q}$) is difficult. On one hand, feasible set (and hence the optimal set) of (OPT-$\mathcal{Q}$) changes with λ; see (3.3a)-(3.3b). On the other hand, we observe from two route example in Chapter 2 that the induced route flow may be identical for a certain range of λ. We approach the question of characterizing $\mathcal{Q}^*(\lambda)$ by studying how the set of equilibrium route flows change with λ. We proceed in two steps: First, we represent the set of feasible route flows as a polytope. For any feasible route flow, we characterize the set of feasible strategy profiles that can induce it in Proposition 2. Next, we study how the equilibrium route flows change with λ via an auxiliary optimization problem (Proposition 3)

We define $\mathcal{F}(\lambda)\{f \in \mathbb{R}^{|\mathcal{R}|\times|\mathcal{T}|} \,|f \text{ satisfies (4.11)}\}$, where:

$$f_r(t^1,t^2) + f_r(\bar{t}^1,\bar{t}^2) = f_r(t^1,\bar{t}^2) + f_r(\bar{t}^1,t^2), \quad \forall r \in \mathcal{R}, \forall t^1, \bar{t}^1 \in \mathcal{T}^1, \text{and } \forall t^2, \bar{t}^2 \in \mathcal{T}^2, \tag{4.11a}$$

$$\sum_{r\in\mathcal{R}} f_r(t^1,t^2) = D, \quad \forall\,(t^1,t^2) \in \mathcal{T}, \tag{4.11b}$$

$$f_r(t^1,t^2) \geq 0, \quad \forall r \in \mathcal{R}, \quad \forall\,(t^1,t^2) \in \mathcal{T}, \tag{4.11c}$$

$$\mathrm{D} - \sum_{r\in\mathcal{R}} \min_{t^1\in\mathcal{T}^1} f_r(t^1,t^2) \leq \lambda \mathrm{D}, \quad \forall t^2 \in \mathcal{T}^2, \tag{4.11d}$$

$$\mathrm{D} - \sum_{r\in\mathcal{R}} \min_{t^2\in\mathcal{T}^2} f_r(t^1,t^2) \leq (1-\lambda)\,\mathrm{D}, \quad \forall t^1 \in \mathcal{T}^1. \tag{4.11e}$$

Proposition 2. *The set of feasible route flows is $\mathcal{F}(\lambda)$ defined in* (4.11). *$\mathcal{F}(\lambda)$ is a convex polytope.*

Furthermore, For a feasible route flow $f \in \mathcal{F}(\lambda)$, any feasible strategy profile q

that induces f can be written as:

$$q_r^1(t^1) = f_r(t^1, \bar{t}^2) - f_r(\bar{t}^1, \bar{t}^2) + \chi_r, \quad \forall r \in r, \quad \forall t^1 \in \mathcal{T}^1, \tag{4.12a}$$

$$q_r^2(t^2) = f_r(\bar{t}^1, t^2) - \chi_r, \quad \forall r \in \mathcal{R}, \quad \forall t^2 \in \mathcal{T}^2. \tag{4.12b}$$

where $(\bar{t}^1, \bar{t}^2)$ is any type profile in $\mathcal{T}$, and $(\chi_r)_{r\in\mathcal{R}}$ is any $|\mathcal{R}|$-dimensional vector satisfying the following constraints:

$$\sum_{r\in\mathcal{R}} \chi_r = \lambda \mathrm{D}, \tag{4.13a}$$

$$\max_{t^1\in\mathcal{T}^1} \left(f_r(\bar{t}^1, \bar{t}^2) - f_r(t^1, \bar{t}^2) \right) \leq \chi_r \leq \min_{t^2\in\mathcal{T}^2} \left(f_r(\bar{t}^1, t^2) \right), \quad \forall r \in \mathcal{R}. \tag{4.13b}$$

Proof. First, we show that any route flow $f = (f_r(t))_{r\in\mathcal{R}, t\in\mathcal{T}}$ induced by any $q \in \mathcal{Q}(\lambda)$ satisfies (4.11). Following (3.5), we obtain:

$$\begin{aligned} f_r(t^1, t^2) + f_r(\bar{t}^1, \bar{t}^2) =& q_r^1(t^1) + q_r^2(t^2) + q_r^1(\bar{t}^1) + q_r^2(\bar{t}^2) \\ =& f_r(t^1, \bar{t}^2) + f_r(\bar{t}^1, t^2), \quad \forall r \in \mathcal{R}, \ \forall t^1, \bar{t}^1 \in \mathcal{T}^1, \text{and } \forall t^2, \bar{t}^2 \in \mathcal{T}^2, \end{aligned}$$

which implies that f satisfies (4.11a). From (3.3a) and (3.3b), f must satisfy (4.11b) and (4.11c). Additionally, for any $t^2 \in \mathcal{T}^2$:

$$\mathrm{D} - \sum_{r\in\mathcal{R}} \min_{t^1\in\mathcal{T}^1} f_r(t^1, t^2) \overset{(3.5)}{=} \mathrm{D} - \sum_{r\in\mathcal{R}} q_r^2(t^2) - \sum_{r\in\mathcal{R}} \min_{t^1\in\mathcal{T}^1} q_r^1(t^1) \overset{(3.3a)}{=} \lambda\mathrm{D} - \sum_{r\in\mathcal{R}} \min_{t^1\in\mathcal{T}^1} q_r^1(t^1) \leq \lambda\mathrm{D}.$$

Therefore, f satisfies (4.11d). Analogously, f must also satisfy (4.11e). Thus, the route flow f induced by any $q \in \mathcal{Q}(\lambda)$ satisfies the constraints (4.11a)-(4.11e).

Next, we show that for any f satisfies (4.11), there exists a feasible strategy profile $q \in \mathcal{Q}(\lambda)$ that induces it. Since f satisfies (4.11), for any type profile $(\bar{t}^1, \bar{t}^2) \in \mathcal{T}$, consider any χ satisfies (4.13):

$$\sum_{r\in\mathcal{R}} \chi_r \overset{(4.13a)}{=} \lambda\mathrm{D} \overset{(4.11d)}{\geq} \mathrm{D} - \sum_{r\in\mathcal{R}} \min_{t^1\in\mathcal{T}^1} f_r(t^1, t^2) = \sum_{r\in\mathcal{R}} \max_{t^1\in\mathcal{T}^1} \left(f_r(\bar{t}^1, \bar{t}^2) - f_r(t^1, \bar{t}^2) \right).$$

Analogously, we can obtain $\sum_{r\in\mathcal{R}} \chi_r \leq \sum_{r\in\mathcal{R}} \min_{t^2\in\mathcal{T}^2} \left(f_r(\bar{t}^1, t^2) \right)$. Additionally, from

(4.11a), we have:

$$
\begin{aligned}
& f_r(\bar{t}^1, t^2) + f_r(t^1, \bar{t}^2) - f_r(\bar{t}^1, \bar{t}^2) \overset{(3.5)}{=} q_r^1(t^1) + q_r^2(t^2) \overset{(3.3b)}{\geq} 0, && \forall r \in \mathcal{R}, t^1 \in \mathcal{T}^1, t^2 \in \mathcal{T}^2 \\
\Rightarrow \quad & f_r(\bar{t}^1, t^2) \geq f_r(\bar{t}^1, \bar{t}^2) - f_r(t^1, \bar{t}^2), && \forall r \in \mathcal{R}, t^1 \in \mathcal{T}^1, t^2 \in \mathcal{T}^2. \\
\Rightarrow \quad & \min_{t^2 \in \mathcal{T}^2} f_r(\bar{t}^1, t^2) \geq \max_{t^1 \in \mathcal{T}^1} \left(f_r(\bar{t}^1, \bar{t}^2) - f_r(t^1, \bar{t}^2) \right), && \forall r \in \mathcal{R}.
\end{aligned}
$$

Thus, the set of χ satisfying (4.13) is non-empty. For any $\hat{\chi}$ satisfying (4.13), consider the corresponding $\hat{q}$ in (4.12). $\hat{q}^1$ satisfies (3.3a):

$$
\sum_{r \in \mathcal{R}} \hat{q}_r^1(t^1) = \sum_{r \in \mathcal{R}} \left(f_r(t^1, \bar{t}^2) - f_r(\bar{t}^1, \bar{t}^2) + \hat{\chi}_r \right) \overset{(4.11b)}{=} \sum_{r \in \mathcal{R}} \hat{\chi}_r \overset{(4.13a)}{=} \lambda \mathrm{D}, \quad \forall t^1 \in \mathcal{T}^1, \tag{4.14}
$$

and $\hat{q}^1$ satisfies (3.3b):

$$
\hat{q}_r^1(t^1) = f_r(t^1, \bar{t}^2) - f_r(\bar{t}^1, \bar{t}^2) + \hat{\chi}_r \overset{(4.13b)}{\geq} 0, \quad \forall r \in \mathcal{R}, \quad \forall t^1 \in \mathcal{T}^1. \tag{4.15}
$$

Similarly, we can show that $\hat{q}^2$ also satisfies (3.3a) and (3.3b). Thus, $\hat{q}$ is feasible.

To check that $\hat{q}$ induces the route flow f:

$$
\hat{q}_r^1(t^1) + \hat{q}_r^2(t^2) = f_r(t^1, \bar{t}^2) - f_r(\bar{t}^1, \bar{t}^2) + f_r(\bar{t}^1, t^2) \overset{(4.11a)}{=} f_r(t^1, t^2), \quad \forall r \in \mathcal{R}, t^1 \in \mathcal{T}^1, t^2 \in \mathcal{T}^2.
$$

Therefore, for any f satisfies constraints (4.11), there exists feasible strategy profiles that can induce f.

Actually, (4.11d) and (4.11e) are equivalent to the following affine constraints:

$$
\begin{aligned}
1 - \frac{1}{\mathrm{D}} \left(\sum_{r \in \mathcal{R}} f_r(t_{k_1^r}^1, t^2) \right) &\leq \lambda, \quad \forall t_{k_1^r}^1 \in \mathcal{T}^1, \quad t^2 \in \mathcal{T}^2. \\
1 - \frac{1}{\mathrm{D}} \left(\sum_{r \in \mathcal{R}} f_r(t^1, t_{k_2^r}^2) \right) &\leq 1 - \lambda, \quad \forall t_{k_2^r}^2 \in \mathcal{T}^2, \quad t^1 \in \mathcal{T}^1.
\end{aligned}
$$

Thus, $\mathcal{F}$ is a convex polytope.

Finally, we show that for any feasible route flow $f \in \mathcal{F}(\lambda)$, the set of feasible strategy that can induce f is in (4.12). For any route $r \in \mathcal{R}$, the system of equations in (3.5) contains $|\mathcal{T}^1| \times |\mathcal{T}^2|$ equations in $|\mathcal{T}^1| + |\mathcal{T}^2|$ variables, $\{q_r^1(t^1)\}_{t^1 \in \mathcal{T}^1}$, $\{q_r^2(t^2)\}_{t^2 \in \mathcal{T}^2}$. For any given $\bar{t}^1 \in \mathcal{T}^1, \bar{t}^2 \in \mathcal{T}^2$, the following equations are linearly independent:

$$\begin{aligned} q_r^1(t^1) + q_r^2(\bar{t}^2) &= f_r(t^1, \bar{t}^2), \quad \forall t^1 \in \mathcal{T}^1. \\ q_r^1(\bar{t}^1) + q_r^2(t^2) &= f_r(\bar{t}^1, t^2), \quad \forall t^2 \in \mathcal{T}^2 \setminus \{\bar{t}^2\}. \end{aligned} \tag{4.16}$$

Any constraint for that r in (3.5) can be derived from (4.16):

$$\begin{aligned} q_r^1(t^1) + q_r^2(t^2) &= \left(q_r^1(t^1) + q_r^2(\bar{t}^2)\right) + \left(q_r^1(\bar{t}^1) + q_r^2(t^2)\right) - \left(q_r^1(\bar{t}^1) + q_r^2(\bar{t}^2)\right) \\ &\overset{(4.16)}{=} f_r(t^1, \bar{t}^2) + f_r(\bar{t}^1, t^2) - f_r(\bar{t}^1, \bar{t}^2) \\ &\overset{(4.11a)}{=} f_r(t^1, t^2), \quad \forall t^1 \in \mathcal{T}^1, t^2 \in \mathcal{T}^2. \end{aligned}$$

Thus, for any r, (3.5) contains $|\mathcal{T}^1| + |\mathcal{T}^2| - 1$ linearly independent equations and $|\mathcal{T}^1| + |\mathcal{T}^2|$ variables. From the rank-nullity theorem, the dimension of its null space is 1. Any solution of (3.5) can be expressed as (4.12), where $\bar{t}^1 \in \mathcal{T}^1$, $\bar{t}^2 \in \mathcal{T}^2$ are any given type of population 1 and 2, and $\chi_r \in \mathbb{R}$ is a free variable.

From (4.14) and (4.15), to ensure that q is a feasible strategy profile, $(\chi_r)_{r \in \mathcal{R}}$ must satisfy (4.13). We know from previous discussion that when $f \in \mathcal{F}(\lambda)$, such χ exists. Consequently, the set of feasible strategies that can induce f is in (4.12).

Any given $(\bar{t}^1, \bar{t}^2) \in \mathcal{T}$ determines a set of basis in (3.5). The set of feasible strategies represented by (4.12) is identical for any chosen $(\bar{t}^1, \bar{t}^2)$. □

The constraints (4.11) can be understood as follows: (4.11a) captures that aggregated flows assigned by the same set of types are equal. (4.11b) ensures that all the demand D is routed, and (4.11c) guarantees that the demand assigned to any route is nonnegative. (4.11a), (4.11b) and (4.11c) are unrelated to λ.

Next, we interpret (4.11d)-(4.11e). For any $q \in \mathcal{Q}(\lambda)$, we define $\mathcal{J}^i(q)$ as follows:

$$\mathcal{J}^i(q) \triangleq \lambda^i \mathrm{D} - \sum_{r \in \mathcal{R}} \min_{t^i \in \mathcal{T}^i} q_r^i(t^i) \overset{(3.3a)}{=} \sum_{r \in \mathcal{R}} \max_{t^i \in \mathcal{T}^i} \left(q_r^1(\bar{t}^i) - q_r^i(t^i)\right),$$

where $\bar{t}^i$ is any type in $\mathcal{T}^i$.

$\mathcal{J}^i(q)$ is the summation over all r of the maximum difference in the demand assigned to route r by type $\bar{t}^i$ comparing to any type $t^i \in \mathcal{T}^i$. $\mathcal{J}^i(q)$ evaluates the impact of signals on population i's strategy.

$\mathcal{J}^i(q)$ can be rewritten in terms of the induced flow f:

$$\begin{aligned}\widehat{\mathcal{J}}^i(f) \triangleq \mathcal{J}^i(q) &= \sum_{r\in\mathcal{R}} \max_{t^i\in\mathcal{T}^i}(f_r(\bar{t}^i, \bar{t}^{-i}) - f_r(t^i, \bar{t}^{-i})), \\ &= \mathrm{D} - \sum_{r\in\mathcal{R}} \min_{t^i\in\mathcal{T}^i} f_r(t^i, \bar{t}^{-i})),\end{aligned} \tag{4.17}$$

where $(\bar{t}^i, \bar{t}^{-i})$ is any type profile in $\mathcal{T}$. Thus, constraints (4.11d) and (4.11e) can be restated as:

$$\widehat{\mathcal{J}}^1(f) \leq \lambda \mathrm{D}, \tag{4.11d'}$$

$$\widehat{\mathcal{J}}^2(f) \leq (1-\lambda)\mathrm{D}. \tag{4.11e'}$$

Constraints (4.11d') and (4.11e') ensure that the impact of signals on each population's strategy ($\widehat{\mathcal{J}}^i(f)$) is bounded by its demand.

We note two cases: On one hand, if $q_r^i(t^i)$ is identical for all $t^i \in \mathcal{T}^i$, the information received by population i does not effect population i's strategy, then $\mathcal{J}^i(f)$ is 0. On the other hand, note that for any $i \in \mathcal{I}$:

$$\mathcal{J}^i(q) = \lambda^i \mathrm{D} \quad \Leftrightarrow \quad \min_{t^i\in\mathcal{T}^i} q_r^i(t^i) = 0, \quad \forall r \in \mathcal{R}. \tag{4.19}$$

The second case plays a key role in distinguishing equilibrium regimes in Sec. 4.2.

We next present the auxiliary optimization problem for solving equilibrium route flows.

Proposition 3. *Any feasible route flow $f \in \mathcal{F}(\lambda)$ is a BWE route flow if and only if* *f is an optimal solution of the following convex optimization problem:*

$$\begin{aligned} \min \quad & \widehat{\Phi}(f) \\ s.t. \quad & f \in \mathcal{F}(\lambda), \end{aligned} \qquad \text{(OPT-}\mathcal{F}\text{)}$$

where $\widehat{\Phi}(f)$ is given by (4.3)*, and $\mathcal{F}(\lambda)$ is the set of feasible route flow vectors defined by* (4.11).

Proof. We first show that in any BWE, the route flow f^* is an optimal solution of (OPT-$\mathcal{F}$). By definition, f^* is induced by an equilibrium strategy profile $q^* \in \mathcal{Q}^*(\lambda)$, and from Theorem 1, $\widehat{\Phi}(f^*) = \Phi(q^*) = \min_{q \in \mathcal{Q}} \Phi(q)$. Now assume that $\widehat{\Phi}(f^*) > \min_{f \in \mathcal{F}} \widehat{\Phi}(f)$, then there exists a feasible route flow $f \in \mathcal{F}(\lambda)$, which is induced by a feasible strategy $q \in \mathcal{Q}(\lambda)$, such that $\Phi(q) = \widehat{\Phi}(f) < \widehat{\Phi}(f^*) = \Phi(q^*)$. This is a contradiction, because by Theorem 1, q^* must minimize $\Phi(q)$. Thus, f^* minimizes $\widehat{\Phi}(f)$.

Next, we show that any optimal solution of (OPT-$\mathcal{F}$) say f^* is an equilibrium route flow. Since $f^* \in \mathcal{F}(\lambda)$, it can be induced by a feasible strategy profile $q \in \mathcal{Q}(\lambda)$. Now assume that q is not a BWE, then there exists an equilibrium $q^{*'} \in \mathcal{Q}(\lambda)$ and an induced flow $f^{*'} \in \mathcal{F}(\lambda)$, such that $\widehat{\Phi}(f^{*'}) = \Phi(q^{*'}) < \Phi(q) = \widehat{\Phi}(f^*)$. This contradicts the fact that f^* minimizes $\widehat{\Phi}(f)$.

To sum up, $f \in \mathcal{F}(\lambda)$ is BWE route flow if and only if f minimizes $\widehat{\Phi}(f)$. □

Proposition 3 forms the basis of our further investigation. Note that among the constraints (4.11) which define the set of feasible route flows, $f \in \mathcal{F}(\lambda)$, only (4.11d) and (4.11e) depend on the relative populationm size λ. We show in section 4.2 that the tightness of these constraints leads to the qualitatively different equilibrium regimes (Theorem 2).

4.2 Equilibrium Regimes

In this section, we study how BWE changes when the relative population size λ changes. Let $f^\dagger$ denote an optimal solution of a simpler optimization problem:

$$\begin{aligned} \min \quad & \widehat{\Phi}(f) \\ s.t. \quad & \text{(4.11a), (4.11b) and (4.11c)} \end{aligned}$$

That is $f^\dagger$ minimizes potential function $\widehat{\Phi}(f)$ and satisfies the constraints that are independent of λ. Following the same analysis in Corollary 1, since $\Phi(w)$ is strictly convex in w, any $f^\dagger$ induces identical edge load $w^\dagger$. The optimal solution set, denoted $\mathcal{F}^\dagger$, can be written as:

$$\mathcal{F}^\dagger = \left\{ f \,\middle|\, \begin{array}{l} f \text{ satisfies (4.11a), (4.11b) and (4.11c).} \\ \sum_{r \ni e} f_r(t) = w_e^\dagger(t), \quad \forall e \in \mathcal{E}, \quad t \in \mathcal{T}. \end{array} \right\} \tag{4.20}$$

Recall the two-route example in Chapter 2, there is a range of λ such that the equilibrium route flow does not change with λ. We will show next that we can find a range of λ such that the equilibrium edge load does not change with λ, and $w^*(\lambda) = w^\dagger$.

From (4.20), $\mathcal{F}^\dagger$ is a bounded polytope. We define $\underline{\lambda}$ and $\bar{\lambda}$ as follows:

$$\underline{\lambda} \triangleq \frac{1}{\mathrm{D}} \min_{f^\dagger \in \mathcal{F}^\dagger} \left\{ \widehat{\mathcal{J}}^1(f^\dagger) \right\}, \tag{4.21}$$

$$\bar{\lambda} \triangleq \frac{1}{\mathrm{D}} \max_{f^\dagger \in \mathcal{F}^\dagger} \left\{ \mathrm{D} - \widehat{\mathcal{J}}^2(f^\dagger) \right\}. \tag{4.22}$$

Since $\mathcal{F}^\dagger$ is a bounded polytope, and $\widehat{\mathcal{J}}^i(f^\dagger)$ is continuous in $f^\dagger$, we know that $\underline{\lambda}$ and $\bar{\lambda}$ can be attained on the set $\mathcal{F}^\dagger$. From (4.17), (4.21) is equivalent to a linear program as follows:

$$
\begin{aligned}
\underline{\lambda} &= \frac{1}{D} \min z \\
s.t. \quad \mathrm{D} - &\left(\sum_{r \in \mathcal{R}} f_r^\dagger (t_{k_1^r}^1, t^2) \right) \leq z, \quad \forall t_{k_1^r}^1 \in \mathcal{T}^1, \quad t^2 \in \mathcal{T}^2, \\
&f^\dagger \in \mathcal{F}^\dagger.
\end{aligned}
$$

Similarly, $\bar{\lambda}$ can also be obtained from a linear program.

The next lemma shows that $\underline{\lambda}$ and $\bar{\lambda}$ are valid thresholds of λ.

Lemma 3. $0 \leq \underline{\lambda} \leq \bar{\lambda} \leq 1$.

Proof. First, we show that $\underline{\lambda}, \bar{\lambda} \in [0,1]$. Since $\underline{\lambda}$ is attainable on the set $\mathcal{F}^\dagger$, there exists $f^\dagger \in \mathcal{F}^\dagger$ such that:

$$
\underline{\lambda} = \frac{1}{\mathrm{D}} \widehat{\mathcal{J}}^1(f^\dagger) \overset{(4.17)}{=} \frac{1}{\mathrm{D}} \left(\mathrm{D} - \sum_{r \in \mathcal{R}} \min_{t^1 \in \mathcal{T}^1} f_r^\dagger \left(t^1, t^2\right) \right) \geq \frac{1}{\mathrm{D}} \left(\mathrm{D} - \sum_{r \in \mathcal{R}} f_r^\dagger \left(t^1, t^2\right) \right) = 0,
$$

Similarly, we can check that $\bar{\lambda} \leq 1$.

Next, we show that $\underline{\lambda} \leq \bar{\lambda}$. For any $f^\dagger \in \mathcal{F}^\dagger$, we obtain:

$$
\begin{aligned}
\bar{\lambda} &\overset{(4.22)}{\geq} 1 - \frac{1}{\mathrm{D}} \widehat{\mathcal{J}}^2(f^\dagger) \overset{(4.17)}{=} \frac{1}{\mathrm{D}} \sum_{r \in \mathcal{R}} \min_{t^2 \in \mathcal{T}^2} f_r^\dagger(\bar{t}^1, t^2) \\
&\geq \frac{1}{\mathrm{D}} \sum_{r \in \mathcal{R}} \max_{t^1 \in \mathcal{T}^1} \left(f_r^\dagger(\bar{t}^1, \bar{t}^2) - f_r^\dagger(t^1, \bar{t}^2) \right) \overset{(4.17)}{=} \frac{1}{\mathrm{D}} \widehat{\mathcal{J}}^1(f^\dagger) \overset{(4.21)}{\geq} \underline{\lambda}.
\end{aligned}
$$

Thus, $0 \leq \underline{\lambda} \leq \bar{\lambda} \leq 1$. □

The thresholds $\underline{\lambda}, \bar{\lambda}$ divide the range $[0,1]$ into three regimes, we denote $[0, \underline{\lambda})$, $[\underline{\lambda}, \bar{\lambda}]$ and $(\bar{\lambda}, 1]$ as regime Λ_1, Λ_2 and Λ_3 respectively. The next result shows that these regimes are distinguished based on whether or not the constraints (4.11d) and (4.11e) are tight in equilibrium.

Theorem 2. *The set of equilibrium route flow $\mathcal{F}^*(\lambda)$ for λ in regime Λ_1 and Λ_3 are* *as follows:*

$$\mathcal{F}^*(\lambda) = \left\{ f \;\middle|\; \begin{array}{lll} & f \text{ minimizes } \widehat{\Phi}(f), & \\ s.t. & (4.11a), (4.11b), (4.11c), (4.11d) & \text{if } \lambda \text{ is in regime } \Lambda_1, \\ & (4.11a), (4.11b), (4.11c), (4.11e) & \text{if } \lambda \text{ is in regime } \Lambda_3. \end{array} \right\} \tag{4.23}$$

Furthermore, in regime Λ_1 *and* Λ_3*, constraints* (4.11d) *and* (4.11e) *are tight in equilibrium respectively. Additionally, in regime* Λ_2*,* $\mathcal{F}^*(\lambda) \subset \mathcal{F}^\dagger$*, where* $\mathcal{F}^\dagger$ *is given by* (4.20).

Proof. [Regime Λ_1]: First, we show by contradiction that the constraints (4.11d) are tight for any equilibrium route flow. Assume that for a $\lambda \in [0, \underline{\lambda})$, there exists an equilibrium route flow $f^*(\lambda)$ such that (4.11d) is not tight. From Proposition 3, $f^*(\lambda)$ is an optimal solution of (OPT-$\mathcal{F}$). Since (OPT-$\mathcal{F}$) is a convex optimization problem, $f^*(\lambda)$ is still an optimal solution if we eliminate (4.11d). Note that there exists $f^\dagger \in \mathcal{F}^\dagger$ such that $\bar{\lambda}$ is attained in (4.22). We obtain:

$$\begin{aligned} 1 - \frac{1}{\mathrm{D}}\widehat{\mathcal{J}}^2(f^\dagger) = \bar{\lambda} &\overset{\text{(Lemma 3)}}{\geq} \underline{\lambda} \overset{\text{(Regime } \Lambda_1)}{>} \lambda. \\ \Rightarrow \qquad \frac{1}{\mathrm{D}}\widehat{\mathcal{J}}^2(f^\dagger) &< 1 - \lambda. \end{aligned}$$

Thus, $f^\dagger$ minimizes $\widehat{\Phi}(f)$ and satisfies constraints (4.11a), (4.11b), (4.11c) and (4.11e). Additionally, $\widehat{\Phi}(f^\dagger) = \widehat{\Phi}(f^*(\lambda))$. Following Corollary 1, $f^\dagger$ and $f^*(\lambda)$ must induce the same equilibrium edge load vector $w^\dagger$. Recall the definition of $\mathcal{F}^\dagger$ in (4.20), $f^*(\lambda) \in \mathcal{F}^\dagger$. From (4.21), we have:

$$\underline{\lambda} \leq \frac{1}{\mathrm{D}}\widehat{\mathcal{J}}1(f^*(\lambda)).$$

Recall our assumption that (4.11d) is not tight, we obtain:

$$\frac{1}{\mathrm{D}}\widehat{\mathcal{J}}^1(f^*(\lambda)) < \lambda < \underline{\lambda} \leq \frac{1}{D}\widehat{\mathcal{J}}^1(f^*(\lambda)),$$

which is a contradiction. Thus, (4.11d) is tight in equilibrium for any λ in regime Λ_1.

Finally, since (4.11d) is tight in equilibrium, we have:

$$\widehat{\mathcal{J}}^2(f^*(\lambda)) \overset{(4.17)}{=} \sum_{r\in\mathcal{R}} \max_{t^2\in\mathcal{T}^2} \left(f_r^*(\bar{t}^1,\bar{t}^2) - f_r^*(\bar{t}^1,t^2)\right) \leq \sum_{r\in\mathcal{R}} \min_{t^1\in\mathcal{T}^1} f_r^*(t^1,\bar{t}^2) = \mathrm{D} - \widehat{\mathcal{J}}^1(f^*(\lambda)) = (1-\lambda)\mathrm{D}.$$

Thus, (4.11e) is satisfied when (4.11d) is tight, and hence can be drop without changing the optimal solution set in (OPT-$\mathcal{F}$).

[Regime Λ_2]: Now we define two additional thresholds $\underline{\lambda}'$ and $\bar{\lambda}'$ as:

$$\underline{\lambda}' \triangleq \frac{1}{\mathrm{D}} \max_{f^\dagger\in\mathcal{F}^\dagger} \left\{\widehat{\mathcal{J}}^1(f^\dagger)\right\}.$$
$$\bar{\lambda}' \triangleq 1 - \frac{1}{\mathrm{D}} \max_{f^\dagger\in\mathcal{F}^\dagger} \left\{\widehat{\mathcal{J}}^2(f^\dagger)\right\}.$$

From the definition of $\underline{\lambda}$ and $\bar{\lambda}$, we can check that $\underline{\lambda} \leq \underline{\lambda}'$, and $\bar{\lambda}' \leq \bar{\lambda}$. We now consider any $\lambda \in [\underline{\lambda}, \underline{\lambda}']$. Since the set $\mathcal{F}^\dagger$ in (4.20) is a bounded polytope, and $\underline{\lambda}$, $\underline{\lambda}'$ are the minimum and maximum of the continuous function $\widehat{\mathcal{J}}^1(f^*(\lambda))$ on $\mathcal{F}^\dagger$, we know from mean value theorem that we can find a $f^\dagger \in \mathcal{F}^\dagger$ satisfying:

$$\lambda = \frac{1}{\mathrm{D}}\widehat{\mathcal{J}}^1(f^\dagger),$$

that is (4.11d) is tight, and such $f^\dagger$ also satisfies constraints (4.11e). By definition of $f^\dagger$, $f^\dagger$ minimizes $\widehat{\Phi}(f)$ and satisfies constraints (4.11). Thus, $f^\dagger$ is an equilibrium route flow, $\mathcal{F}^*(\lambda) \cap \mathcal{F}^\dagger \neq \emptyset$. Since the equilibrium edge load vector is unique, and the edge load induced by $f^\dagger$ is $w^\dagger$, we must have $w^*(\lambda) = w^\dagger$. Furthermore, from (4.20), $\mathcal{F}^\dagger$ includes all route flows that can induce $w^\dagger$ and satisfy (4.11a)-(4.11c). Therefore, $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^\dagger$ for any $\lambda \in [\underline{\lambda}, \underline{\lambda}']$. Similarly, we can argue that for any $\lambda \in [\bar{\lambda}', \bar{\lambda}]$, $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^\dagger$.

Next, we consider two cases: If $\underline{\lambda}' \geq \bar{\lambda}'$, our previous discussion has cover all $\lambda \in [\underline{\lambda}, \bar{\lambda}]$. If $\underline{\lambda}' < \bar{\lambda}'$, consider any $\lambda \in (\underline{\lambda}', \bar{\lambda}')$, any $f^\dagger \in \mathcal{F}^\dagger$ satisfies constraints (4.11d) and (4.11e), hence is equilibrium route flow. $\mathcal{F}^*(\lambda) = \mathcal{F}^\dagger$ for $\lambda \in (\underline{\lambda}', \bar{\lambda}')$. Consequently, for both cases, we obtain that $\mathcal{F}^*(\lambda) \subset \mathcal{F}^\dagger$ in regime Λ_2.

[Regime Λ_3]: Analogous to the proof given for regime Λ_1, we can argue that (4.11e) are tight in any equilibrium for any λ in regime Λ_3, and (4.11d) can be dropped from the constraint set. □

In regime Λ_1, the impact of information on population 1's strategy in equilibrium is maximal, $\mathcal{J}^1(q^*) = \widehat{\mathcal{J}}^1(f^*) = \lambda\mathrm{D}$. $\mathcal{J}^1(q^*)$ increases as the demand of population 1 increases. In regime Λ_3, the impact on population 2's strategy is maximal $\mathcal{J}^2(q^*) = \widehat{\mathcal{J}}^2(f^*) = (1-\lambda)\mathrm{D}$. $\mathcal{J}^2(q^*)$ increases as the demand of population 2 increases. The tightness of constraints (4.11d)-(4.11e) results in different mototonicity of the value of potential function in equilibrium, and different qualitative properties of both equilibrium edge loads and equilibrium strategy profiles in different regimes, which are shown next.

We define $\Psi(\lambda)$ as the value of potential function in equilibrium. $\Psi(\lambda)$ is the optimal value of $\Phi(q)$.

$$\Psi(\lambda) \triangleq \Phi(q^*(\lambda)). \tag{4.24}$$

Proposition 4. *In regime Λ_1, $\Psi(\lambda)$ monotonically decreases with λ. In regime Λ_2, $\Psi(\lambda)$ does not change with λ. In regime Λ_3, $\Psi(\lambda)$ monotonically increases with λ.*

Proof. [Regime Λ_1]: From Theorem 2, constraint (4.11d) is tight in equilibrium. Consider λ' and λ such that $0 \leq \lambda' < \lambda < \underline{\lambda}$. Any equilibrium route flows $f^*(\lambda')$ and $f^*(\lambda)$ satisfy:

$$\frac{1}{\mathrm{D}}\widehat{\mathcal{J}}^1(f^*(\lambda')) = \lambda' < \lambda = \frac{1}{\mathrm{D}}\widehat{\mathcal{J}}^1(f^*(\lambda)).$$

$f^*(\lambda')$ satisfies constraints (4.11a)-(4.11e), thus is a feasible solution of optimization problem in (4.23) with parameter λ in regime Λ_1. However, $f^*(\lambda')$ is not an optimal solution when the population size parameter is λ, because constraint (4.11d) is not tight. Since $f^*(\lambda)$ is an optimal solution, we must have $\Psi(\lambda') = \widehat{\Phi}(f^*(\lambda)) > \Psi(\lambda)$.

[Regime Λ_2]: From Theorem 2, $\mathcal{F}^*(\lambda) \subseteq \mathcal{F}^\dagger$ for any $\lambda \in [\underline{\lambda}, \bar{\lambda}]$. Since equilibrium edge load vector is unique, $w^*(\lambda) = w^\dagger$. $\Psi(\lambda) = \Phi(w^\dagger)$, which does not change with λ.

[Regime Λ_3]: Following similar analysis in regime Λ_1, $\Psi(\lambda)$ monotonically increases with λ in regime Λ_3. □

Corollary 2. *In equilibrium, the edge load vector $w^*(\lambda)$ does not change with λ if and only if λ is in regime Λ_2. Furthermore, in regime Λ_2, $w^*(\lambda) = w^\dagger$.*

Proof. From the analysis in Proposition 4, we know that $w^*(\lambda) = w^\dagger$ in regime Λ_2.

In regime Λ_1 or regime Λ_3, since $\Psi(\lambda)$ changes with λ, and $\Phi(w)$ is strictly convex with w, $w^*(\lambda)$ must change with λ. □

The next proposition shows that the tightness of constraints (4.11d) (resp. (4.11e)) results in one-to-one correspondence between q^* and f^* in regime Λ_1 (resp. Λ_3).

Proposition 5. *In regimes Λ_1 or Λ_3, any equilibrium route flow vector $f^* \in \mathcal{F}^*(\lambda)$ is induced by a unique equilibrium strategy profile q^*. Specifically,*

- *In regime Λ_1,*

$$\begin{aligned} q_r^{*1}(t^1) &= f_r^*(t^1,t^2) - \min_{t^1 \in \mathcal{T}^1} f_r^*(t^1,t^2), \quad \forall r \in \mathcal{R}, \quad \forall t^1 \in \mathcal{T}^1, \quad \forall t^2 \in \mathcal{T}^2, \\ q_r^{*2}(t^2) &= \min_{t^1 \in \mathcal{T}^1} f_r^*(t^1,t^2), \quad \forall r \in \mathcal{R}, \quad \forall t^2 \in \mathcal{T}^2. \end{aligned} \tag{4.25}$$

- *In regime Λ_3,*

$$\begin{aligned} q_r^{*1}(t^1) &= \min_{t^2 \in \mathcal{T}^2} f_r^*(t^1,t^2) \quad \forall r \in \mathcal{R}, \quad \forall t^1 \in \mathcal{T}^1, \\ q_r^{*2}(t^2) &= f_r^*(t^1,t^2) - \min_{t^2 \in \mathcal{T}^2} f_r^*(t^1,t^2), \quad \forall r \in \mathcal{R}, \quad \forall t^1 \in \mathcal{T}^1, \quad \forall t^2 \in \mathcal{T}^2. \end{aligned} \tag{4.26}$$

Proof. Recall from Proposition 2, for any equilibrium load vector $f^* \in \mathcal{F}^*(\lambda)$ and any given $\bar{t}^1 \in \mathcal{T}^1$, $\bar{t}^2 \in \mathcal{T}^2$, equilibrium strategy profiles that can induce $f^*(\lambda)$ can be written in (4.12). Since constraint (4.11d) is tight in equilibrium (Theorem 2) in regime Λ_1, there is unique χ that satisfies (4.13), $\chi = \left(\max_{t^1 \in \mathcal{T}^1} \left(f_r^*(\bar{t}^1,\bar{t}^2) - f_r^*(t^1,\bar{t}^2)\right)\right)_{r \in \mathcal{R}}$. Following (4.12), the corresponding strategy profile is:

$$\begin{aligned} q_r^{*1}(t^1) &= f_r^*(t^1,\bar{t}^2) - f_r^*(\bar{t}^1,\bar{t}^2) + \max_{t^1 \in \mathcal{T}^1} \left(f_r^*(\bar{t}^1,\bar{t}^2) - f_r^*(t^1,\bar{t}^2)\right) \\ &= f_r^*(t^1,\bar{t}^2) - \min_{t^1 \in \mathcal{T}^1} f_r^*(t^1,\bar{t}^2), \end{aligned}$$

which is identical for any given $\bar{t}^2 \in \mathcal{T}^2$. Analogously, $q_r^{*2}(t^2)$ can be written as:

$$\begin{aligned}
q_r^{*2}(t^2) &= f_r^*(\bar{t}^1, t^2) - \max_{t^1 \in \mathcal{T}^1} \left(f_r^*(\bar{t}^1, \bar{t}^2) - f_r^*(t^1, \bar{t}^2)\right) \\
&= f_r^*(\bar{t}^1, t^2) - f_r^*(\bar{t}^1, \bar{t}^2) + \min_{t^1 \in \mathcal{T}^1} f_r^*(t^1, \bar{t}^2) \\
&\overset{(3.5)}{=} \left(q_r^{*1}(\bar{t}^1) + q_r^{*2}(t^2)\right) - \left(q_r^{*1}(\bar{t}^1) + q_r^{*2}(\bar{t}^2)\right) + \left(\min_{t^1 \in \mathcal{T}^1} q_r^{*1}(t^1) + q_r^{*2}(\bar{t}^2)\right) \\
&= q_r^{*2}(t^2) + \min_{t^1 \in \mathcal{T}^1} q_r^{*1}(t^1) \\
&\overset{(3.5)}{=} \min_{t^1 \in \mathcal{T}^1} f_r^*(t^1, t^2),
\end{aligned}$$

Following similar analysis, in regime Λ_3, χ is unique, and q^* can be written in (4.26).

□

Proposition 5 can be simplified as follows when the origin and destination are connected by a parallel route network.

Corollary 3. *$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a parallel route network. For any $\lambda \in [0, 1]$, the equilibrium route flow f^* is unique. In regime Λ_1 and Λ_3, the equilibrium strategy q^* is unique.*

Proof. For a parallel route network, we immediately obtain the uniqueness of f^* from Corollary 1. The uniqueness of q^* in regime Λ_1 and Λ_3 follows from Theorem 5 immediately. □

We now have a general discussion on equilibrium properties in each regime. From Theorem 2, we know in regimes Λ_1, $\widehat{\mathcal{J}}^1(f^*) = \mathcal{J}^1(q^*) = \lambda \mathrm{D}$. The impact of information is maximal on the strategy of population 1 in equilibrium. Each equilibrium route flow can only be induced by a unique equilibrium strategy profile. From (4.19), we obtain $\min_{t^1 \in \mathcal{T}^1} q_r^{*i}(t^i) = 0$ for any $r \in \mathcal{R}$. That is for any route $r \in \mathcal{R}$, there exists at least one type t^1, which does not assign demand on r. Recall the two-route example in Chapter 2, we can see that the impact of information on population 1's equilibrium strategy is indeed maximized, since population 1 switches all the demand from r_1 to r_2 when signal changes from $\mathbf{n}$ to $\mathbf{a}$. Following similar analysis, in regime

Λ_3, $\widehat{\mathcal{J}}^2(f^*) = \mathcal{J}^2(q^*) = (1-\lambda)\mathrm{D}$. The impact of information on population 2's equilibrium strategy is maximized, and for any route $r \in \mathcal{R}$, there exists at least one type t^2, which does not assign demand on r. On the contrary, in regime Λ_2, the impact of information is not necessarily tightly bounded by the demand in equilibrium. As we can see in the two-route example, $\widehat{\mathcal{J}}^1(f^*) = \mathcal{J}^1(q^*) = \underline{\lambda} < \lambda$, there are multiple equilibria that induces an identical equilibriuim route flow.

The threshold $\underline{\lambda}$ is the minimum fraction of population 1 at which the impact of the received signal on population 1's strategy is fully achieved in equilibrium. For $\lambda < \underline{\lambda}$, $\mathcal{J}^1(q^*)$ linearly increases with λ, i.e. $\mathcal{J}^1(q^*) = \lambda\mathrm{D}$. Analogously, $1-\bar{\lambda}$ is the minimum fraction of population 2 at which the impact of the received signal on population 2's strategy is fully achieved in equilibrium. For $\lambda > \underline{\lambda}$, $\mathcal{J}^2(q^*)$ linearly decreases with λ (increases with $1-\lambda$), i.e. $\mathcal{J}^2(q^*) = (1-\lambda)\mathrm{D}$. For $\lambda \in [\underline{\lambda}, \bar{\lambda}]$, the impacts of the received signal on both populations' strategies are fully achieved in equilibrium.

We continue to study the two route example in Chapter 2. Consider two information systems such that $p^1(t^1 = \mathbf{a}|s = \mathbf{a}, t^2) = p^1(t^1 = \mathbf{n}|s = \mathbf{n}, t^2) = 0.8$; $p^2(t^2 = \mathbf{a}|s = \mathbf{a}, t^1) = p^2(t^2 = \mathbf{n}|s = \mathbf{n}, t^1) = 0.6$. The rest parameters are the same as the parameters in Table 2.1. Fig. 4-1a shows that the optimal value of the potential function decreases in regime Λ_1, does not change in regime Λ_2, and increases in regime Λ_3. Fig. 4-1b shows the equilibrium route flow split fraction (the ratio of the demand assigned to each route and the total demand) does not change with λ if and only if λ is in regime Λ_2. The thresholds are $\underline{\lambda} = 0.1382$, $\bar{\lambda} = 0.9693$.

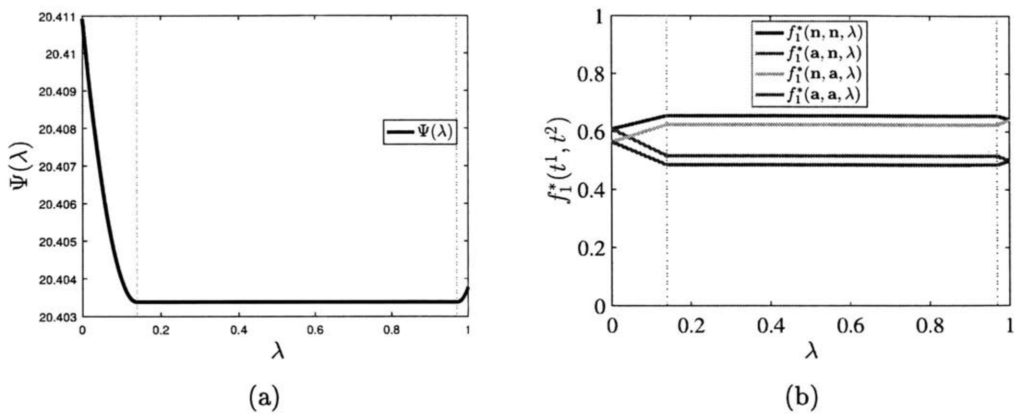


Figure 4-1: Optimal value of potential function $\Psi(\lambda)$ and equilibrium route flow

Chapter 5

Relative Value of Information

In this chapter, we study how does difference between the average costs of two populations in equilibrium changes with the relative population size. In section 5.1, we study the special case when one population is uninformed.

We define the *BWE population cost* as the expected equilibrium cost of travelers of a given population:

$$C^{*i}(\lambda) \triangleq \frac{1}{\lambda^i \mathrm{D}} \sum_{t^i \in \mathcal{T}^i} \Pr(t^i) \sum_{r \in \mathcal{R}} \mathbb{E}[c_r(q^*)|t^i] q_r^{*i}(t^i) \overset{(3.9)}{=} \sum_{t^i \in \mathcal{T}^i} \min_{r \in \mathcal{R}} \Pr(t^i) \mathbb{E}[c_r(q^*)|t^i], \quad (5.1)$$

where the last equation is from Definition 1, each type t^i only assigns demand on routes which incur the lowest expected cost $\mathbb{E}[c_r(q^*)|t^i]$. Since all equilibrium strategy profiles induce identical equilibrium edge load, thus incur identical cost (Corollary 1), we immediately obtain that $C^{*i}(\lambda)$ is identical in all equilibria.

We define the *relative value of information* $V^*(\lambda)$ as the difference between equilibrium cost of population 1 and 2:

$$V^*(\lambda) \triangleq C^{*2}(\lambda) - C^{*1}(\lambda).$$

$V^*(\lambda)$ evaluates the value of information provided by information system 1 relative to that provided by information system 2.

Theorem 3. *Given any two information systems, the relative value of information*

$V^*(\lambda)$ is non-increasing in λ. Furthermore, $V^*(\lambda) > 0$ in regime Λ_1, $V^*(\lambda) = 0$ in regime Λ_2, and $V^*(\lambda) < 0$ in regime Λ_3.

Proof. We first prove that $\Psi(\lambda)$ is differentiable in λ. $\Phi(q)$ and constraints (3.3a)-(3.3b) are differentiable with respect to q and λ. The set of equilibrium strategy profiles $\mathcal{Q}^*(\lambda)$ is nonempty and bounded. For any $q^* \in \mathcal{Q}^*(\lambda)$, the Lagrange multipliers (μ^*, ν^*) in (4.9) and (4.10) are bounded. From Proposition 12, we know that $\Psi(\lambda)$ is directionally differentiable. We obtain the right-hand-side derivative, denoted $\Psi'(\lambda+)$ as follows:

$$\Psi'(\lambda+) = \lim_{\epsilon \to 0^+} \frac{\Psi(\lambda+\epsilon) - \Psi(\lambda)}{\epsilon} = D_{z=1}\Psi(\lambda) \overset{(A.3)}{=} \min_{q^* \in \mathcal{Q}^*(\lambda)} \max_{\substack{(\mu^*,\nu^*) \\ \in (M(q^*), N(q^*))}} \frac{\partial L(q^*, \mu^*, \nu^*)}{\partial \lambda}$$

$$\overset{(4.5)}{=} \min_{q^* \in \mathcal{Q}^*(\lambda)} \max_{\substack{(\mu^*,\nu^*) \\ \in (M(q^*), N(q^*))}} \left(\sum_{t^1 \in \mathcal{T}^1} \mu^{*t^1} - \sum_{t^2 \in \mathcal{T}^2} \mu^{*t^2} \right) \mathrm{D},$$

where $D_{z=1}\Psi(\lambda)$ is the directional derivative defined in Definition 4 in Appendix A for the direction $z = 1$. Analogously, given direction $z = -1$, the left derivative $\Psi'(\lambda-)$ can be written as:

$$\begin{aligned}
\Psi'(\lambda-) &= \lim_{\epsilon \to 0^-} \frac{\Psi(\lambda+\epsilon) - \Psi(\lambda)}{\epsilon} = -D_{z=-1}\Psi(\lambda) \\
&\overset{(A.3)}{=} -\left(\min_{q^* \in \mathcal{Q}^*(\lambda)} \max_{\substack{(\mu^*,\nu^*) \\ \in (M(q^*), N(q^*))}} -\frac{\partial L(q^*, \mu^*, \nu^*)}{\partial \lambda} \right) \\
&= \max_{q^* \in \mathcal{Q}^*(\lambda)} \min_{\substack{(\mu^*,\nu^*) \\ \in (M(q^*), N(q^*))}} \frac{\partial L(q^*, \mu^*, \nu^*)}{\partial \lambda} \\
&\overset{(4.5)}{=} \max_{q^* \in \mathcal{Q}^*(\lambda)} \min_{\substack{(\mu^*,\nu^*) \\ \in (M(q^*), N(q^*))}} \left(\sum_{t^1 \in \mathcal{T}^1} \mu^{*t^1} - \sum_{t^2 \in \mathcal{T}^2} \mu^{*t^2} \right) \mathrm{D}.
\end{aligned}$$

Proposition 1 ensures that sets $M(q^*), N(q^*)$ are singletons. Corollary 1 shows that the cost of all BWE is identical. $\Psi'(\lambda+)$ is equal to $\Psi'(\lambda-)$, and can be written as follows:

$$\Psi^{'}(\lambda+) = \Psi^{'}(\lambda-) = \left(\sum_{t^1 \in \mathcal{T}^1} \mu^{*t^1} - \sum_{t^2 \in \mathcal{T}^2} \mu^{*t^2} \right) \mathrm{D}$$

Therefore, $\Psi(\lambda)$ is differentiable. The derivative of $\Psi(\lambda)$ in λ, denoted $\Psi^{'}(\lambda)$, is as follows:

$$\begin{aligned}
\Psi^{'}(\lambda) &\overset{(4.9)}{=} \left(\sum_{t^1 \in \mathcal{T}^1} \min_{r \in \mathcal{R}} \Pr(t^1)\mathbb{E}[c_r(q^*)|t^1] - \sum_{t^2 \in \mathcal{T}^2} \min_{r \in \mathcal{R}} \Pr(t^2)\mathbb{E}[c_r(q^*)|t^2] \right) \mathrm{D} \\
&\overset{(5.1)}{=} \left(C^{*1}(\lambda) - C^{*2}(\lambda)\right) \mathrm{D} = -V^*(\lambda) \cdot \mathrm{D}, \\
\Rightarrow \quad V^*(\lambda) &= -\frac{\Psi^{'}(\lambda)}{\mathrm{D}}.
\end{aligned}$$

From Proposition 4, $\Psi(\lambda)$ decreases with λ in regime Λ_1, does not change in regime Λ_2 and increases in regime Λ_3. Therefore, $V^*(\lambda) > 0$ in regime Λ_1, $V^*(\lambda) = 0$ in regime Λ_2, and $V^*(\lambda) < 0$ in regime Λ_3

Finally, recall the convex optimization problem (OPT-$\mathcal{Q}$). The constraints (3.3a) and (3.3b) are affine in both q and λ. $\Phi(q)$ is convex in q, and independent with λ. λ takes value in the convex domain $[0, 1]$. The problem satisfies the conditions of Proposition 11 in Appendix A, thus we obtain that $\Psi(\lambda)$ is convex in λ. $\Psi^{'}(\lambda)$ is non-decreasing in λ, and we conclude that the relative value of information $V^*(\lambda)$ is non-increasing in λ. □

We consider the two route example. The two information systems are $p^1(t^1 = \mathbf{a}|s = \mathbf{a}, t^2) = p^1(t^1 = \mathbf{n}|s = \mathbf{n}, t^2) = 0.8$; $p^2(t^2 = \mathbf{a}|s = \mathbf{a}, t^1) = p^2(t^2 = \mathbf{n}|s = \mathbf{n}, t^1) = 0.6$. That is the probability of population 1 to receive the correct state is 0.8, and the probability for population 2 is 0.6. Other parameters are the same in Table 2.1. Fig. 5-1 shows the equilibrium population cost . The thresholds are $\underline{\lambda} = 0.1382$, $\bar{\lambda} = 0.9693$.

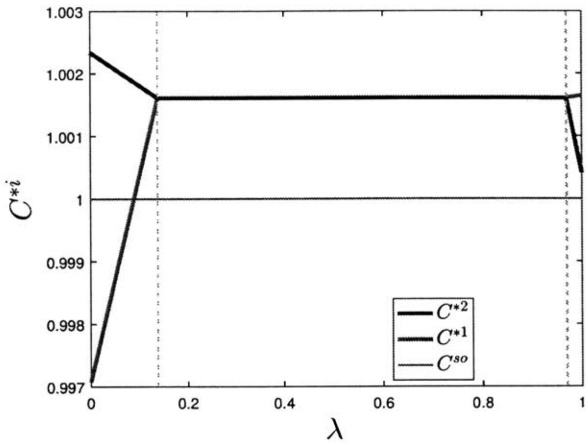


Figure 5-1: Equilibrium population costs.

Notice that when $\lambda > \bar{\lambda}$, population 1, which is better informed has higher cost than population 2, which is less informed. We say that information has negative value if the better informed population is worse off.

Recall that when $\lambda > \bar{\lambda}$, $\mathcal{J}^2(q^*) = (1-\lambda)\mathrm{D}$. From (4.19), for each $r \in \mathcal{R}$, there exists at least one $\bar{t}^2 \in \mathcal{T}^2$ such that $\bar{t}^2$ does not assign demand on route r. The set of routes, on which type $\bar{t}^2$ assign demand, must have lower cost than that of r. Therefore, the intuition behind the negative value of information is that the demand of the less informed population is low, such that the less informed population only assigns demand on routes with lower cost given the stragety of the more informed population.

5.1 Non-negative Relative Value of Information

we next provide a sufficient condition, in which negative value of information does not happen.

Proposition 6. *If population 2 is not informed, i.e. for any $t^1 \in \mathcal{T}^1$ and $s \in \mathcal{S}$, $Pr(t^2|s,t^1) = Pr(t^2)$, $\forall t^2 \in \mathcal{T}^2$, then $\bar{\lambda} = 1$. Thus, $V(\lambda) \geq 0$.*

Proof. We first show that the interim beliefs $\mu(s,t^1|t^2)$ in (3.4) are identical for any

$t^2 \in \mathcal{T}^2$:

$$\mu(s, t^1|t^2) = \frac{\pi(s, t^1, t^2)}{Pr(t^2)} = \frac{Pr(t^2|s, t^1) \cdot Pr(s, t^1)}{Pr(t^2)}$$
$$= Pr(s, t^1) = \sum_{t^2 \in \mathcal{T}^2} \pi(s, t^1, t^2), \quad \forall s \in \mathcal{S}, \forall t^1 \in \mathcal{T}^1, \forall t^2 \in \mathcal{T}^2.$$

Therefore, for any equilibrium strategy profile q^*, $q^*(t^2)$ is identical across all $t^2 \in \mathcal{T}^2$. $\mathcal{J}^2(q^*) = \widehat{\mathcal{J}}^2(f^*) = 0$ for any λ. From the definition of $\bar{\lambda}$ in (4.22), for any $f^\dagger \in \mathcal{F}^\dagger$,

$$\bar{\lambda} \overset{(4.22)}{=} \frac{1}{\mathrm{D}} \max_{f^\dagger \in \mathcal{F}^\dagger} \{\mathrm{D} - \widehat{\mathcal{J}}^2(f^\dagger)\} = 1.$$

Following Theorem 3, we obtain $V^*(\lambda) \geq 0$ for any λ. □

When population 2 has no information about the state, signals have no effect on population 2's strategy, i.e. $\mathcal{J}^2(q^*) = 0$. Regime Λ_3 degenerates to a singleton $\{1\}$. In regime Λ_1, the informed population has lower cost than the uninformed population. When the fraction of informed population exceeds $\underline{\lambda}$, two populations have the same cost.

Proposition 6 shows that population 2 being uninformed is a sufficient condition for $\bar{\lambda} = 1$. However, it is not a necessary condition. Fig. 5-2 shows equilibrium population cost for the case where $\mathrm{D} = 10$, $p = 0.2$, $c_1^n(l) = l + 15$, $c_1^a(l) = 3l + 15$, $c_2(l) = 20l + 30$; $p^1(t^1 = \mathbf{a}|s = \mathbf{a}, t^2) = p^1(t^1 = \mathbf{n}|s = \mathbf{n}, t^2) = 0.8$; $p^2(t^2 = \mathbf{a}|s = \mathbf{a}, t^1) = p^2(t^2 = \mathbf{n}|s = \mathbf{n}, t^1) = 0.6$. The thresholds are $\underline{\lambda} = 0.0227$, $\bar{\lambda} = 1$.

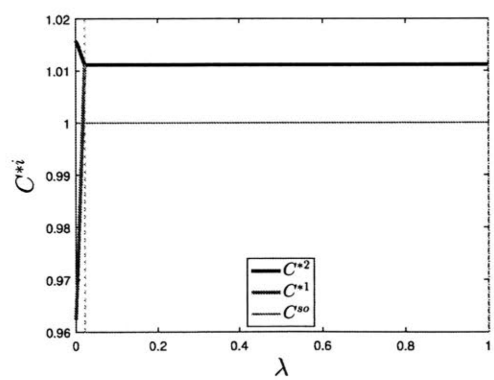


Figure 5-2: Equilibrium population cost $C^{*i}(\lambda)$

Since the slope of population 2 is sufficiently high, the expected cost of route 2 is higher than that of route 1 for both $t^2 = \mathbf{a}$ and $t^2 = \mathbf{n}$. Although population 2 is informed, since both types of population 2 route all demand on r_1, signal t^2 has no impact on the strategy of population 2 in equilibrium. We obtain $\bar{\lambda} = 1$.

Chapter 6

Equilibrium Social Cost

In this chapter, we study how equilibrium social cost changes when the relative population size changes. We provide bounds of equilibrium social cost in Sec. 6.1, and study the worst case inefficiency of equilibrium in Sec. 6.2

For any feasible strategy profile, the average social cost $C(q)$ is the expected cost incurred by a traveler of any population across all network states:

$$C(q) \triangleq \frac{1}{\mathrm{D}} \sum_{i \in \mathcal{I}} \sum_{t^i \in \mathcal{T}^i} \Pr(t^i) \sum_{r \in \mathcal{R}} \mathbb{E}[c_r(q)|t^i] q_r^i(t^i). \tag{6.1}$$

We define the equilibrium social cost $C^*(\lambda)$ as the average social cost in equilibrium:

$$C^*(\lambda) \triangleq \frac{1}{\mathrm{D}} \sum_{i \in \mathcal{I}} \sum_{t^i \in \mathcal{T}^i} \Pr(t^i) \sum_{r \in \mathcal{R}} \mathbb{E}[c_r(q^*)|t^i] q_r^{*i}(t^i).$$

$C^*(\lambda)$ is identical for any BWE.

We first present an example to show that the social cost can be non-differentiable and non-convex in λ. The game setup is the same as the two-route example in Chapter 2. ($\theta(a) = 0.6$, $\mathrm{D} = 3$, $c_1^n(l) = l + 15$, $c_1^a(l) = 3l + 15$, $c_2(l) = 2l + 20$; $p^1(t^1 = \mathbf{a}|s = \mathbf{a}, t^2) = p^1(t^1 = \mathbf{n}|s = \mathbf{n}, t^2) = 0.8$; $p^2(t^2 = \mathbf{a}|s = \mathbf{a}, t^1) = p^2(t^2 = \mathbf{n}|s = \mathbf{n}, t^1) = 0.6$). Fig. 6-1 shows the equilibrium social cost. The thresholds for this example are $\underline{\lambda} = 0.2088$, $\bar{\lambda} = 0.9769$.

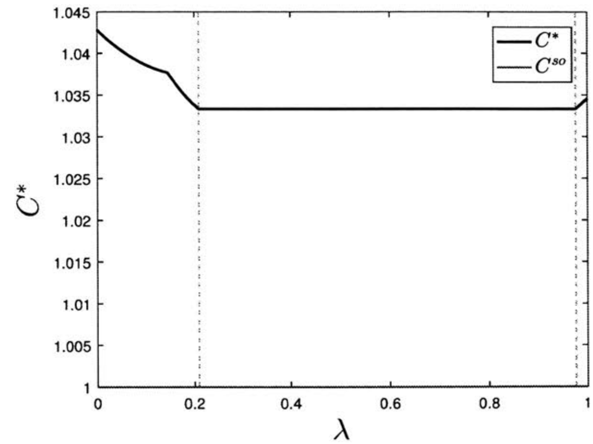


Figure 6-1: Expected social cost $C^*(\lambda)$

Since the equilibrium social cost is non-convex and non-differentiable with respect to λ, it is difficult to study the property of equilibrium social cost in general. In this section, we are interested in studying the bounds on the equilibrium social cost, and the bound on the inefficiency of equilibrium.

6.1 Bounds on Equilibrium Social Cost

Consider the socially optimal strategy $q^{opt}(\lambda)$, i.e. the feasible strategy that minimizes the social cost.

$$C^{opt}(\lambda) \triangleq \min_{q \in \mathcal{Q}(\lambda)} C(q) = C(q^{opt}(\lambda)). \tag{6.2}$$

Naturally, $C^{opt}(\lambda) \leq C^*(\lambda)$ for all λ.

We study how $C^{opt}(\lambda)$ changes with λ via a modified game, Γ^m, which is defined similarly to Γ, except with edge cost function $\tilde{c}_e^s$ as follows:

$$\tilde{c}_e^s(w_e) = \frac{\partial \left(w_e \cdot c_e^s(w_e)\right)}{\partial w_e} = c_e^s(w_e) + w_e \cdot \frac{\partial c_e^s(w_e)}{\partial w_e}.$$

$\tilde{c}_e^s(w_e)$ is the maginal cost function of edge e in the original game Γ. Γ^m is also a weighted potential game. Let $\Phi^m(q)$ denote the weighted potential function of Γ^m, and

$\Psi^m(\lambda)$ as the optimal value of the potential function of Γ^m, $\Psi^m(\lambda) \triangleq \min_{q\in\mathcal{Q}(\lambda)} \Phi^m(q)$.

Lemma 4. *The socially optimal strategy $q^{opt}(\lambda)$ is the BWE strategy of the modified game. Furthermore, $C^{opt}(\lambda) = \frac{1}{\mathrm{D}}\Psi^m(\lambda)$.*

Proof. Proof of Lemma 4. Following (4.1), the potential function of Γ^m, denoted $\Phi^m(q)$, can be written as:

$$\begin{aligned}
\Phi^m(q) &= \sum_{s\in\mathcal{S}} \sum_{e\in\mathcal{E}} \sum_{t^1\in\mathcal{T}^1} \sum_{t^2\in\mathcal{T}^2} \pi(s,t^1,t^2) \int_0^{\sum_{r\ni e} q_r^1(t^1)+q_r^2(t^2)} \tilde{c}_e^s(z)dz \\
&= \sum_{s\in\mathcal{S}} \sum_{e\in\mathcal{E}} \sum_{t^1\in\mathcal{T}^1} \sum_{t^2\in\mathcal{T}^2} \pi(s,t^1,t^2) c_e^s\left(w_e(t^1,t^2)\right) \cdot w_e(t^1,t^2) \\
&= \sum_{i\in\mathcal{I}} \sum_{t^i\in\mathcal{T}^i} \Pr(t^i) \sum_{r\in\mathcal{R}} \mathbb{E}[c_r(q)|t^i] q_r^i(t^i) \\
&\overset{(6.1)}{=} \mathrm{D}\cdot C(q).
\end{aligned}$$

The equilibrium of the modified game minimizes the potential function $\Phi^m(q)$, thus minimizes $C(q)$. Therefore, $C^{opt}(\lambda) = \frac{1}{\mathrm{D}}\Psi^m(\lambda)$. Any social optimal strategy $q^{opt}(\lambda)$ is a BWE for the marginal game Γ^m. □

All our results so far are applicable for the modified game Γ^m. Following similar discussion in Sec. 4.2, there exists thresholds $0 \leq \underline{\lambda}^m \leq \bar{\lambda}^m \leq 1$, which creates three regimes $\Lambda_1^m = [0, \underline{\lambda}^m)$, $\Lambda_2^m = [\underline{\lambda}^m, \bar{\lambda}^m]$, and $\Lambda_3^m = (\bar{\lambda}^m, 1]$.

Corollary 4. *For any λ, $C^{opt}(\lambda) \leq C^*(\lambda)$. Furthermore, $C^{opt}(\lambda)$ decreases with λ in Λ_1^m, does not change in Λ_2^m, and increases in Λ_3^m*

This result directly follows from Proposition 4 and Lemma 4.

We define the minimum equilibrium social cost, denoted C^*_{min} as follows:

$$C^*_{min} \triangleq \min_{\lambda\in[0,1]} C^*(\lambda).$$

We define Λ_{min} as the set of λ that attains C^*_{min}. Recall Corollary 2, $w^*(\lambda)$ does not change with λ in regime Λ_2, thus $C^*(\lambda)$ does not change. Similarly, $C^{opt}(\lambda)$ does not change with λ in regime Λ_2^m of the modified game.

Proposition 7. *The minimum BWE social cost C^*_{min} satisfies:*

$$C^{opt}_{\Lambda_2^m} \leq C^*_{min} \leq C^*_{\Lambda_2},$$

where $C^*_{\Lambda_2}$ is the equilibrium social cost for λ in regime Λ_2, and $C^{opt}_{\Lambda_2^m}$ is the optimal social cost for λ in regime Λ_2^m.

Proof. By the definition of C^*_{min}, we know that $C^*_{min} \leq C^*_{\Lambda_2}$. From Corollary 4, the minimum of $C^{opt}(\lambda)$ is achieved for λ in Λ_2^m. Thus,

$$C^*_{min} \geq C^{opt}(\lambda_{min}) \geq C^{opt}_{\Lambda_2^m}.$$

□

We next show that the bounds presented in Corollary 4 and Proposition 7 are tight.

Proposition 8. *If cost functions can be written as:*

$$c_e^s(w_e) = h_e^s(w_e) + \beta_e^s, \quad \forall e \in \mathcal{E}, s \in \mathcal{S},$$

where h_e^s is a homogeneous function with degree $k > 1$, and $\{\beta_e^s\}_{e \in \mathcal{E}, s \in \mathcal{S}}$ satisfies:

$$\sum_{e \in r} \beta_e^s = \beta^s, \quad \forall s \in \mathcal{S}, \quad \forall r \in \mathcal{R}.$$

then $C^{opt}(\lambda) = C^*(\lambda)$. Additionally, $C^*_{\Lambda_2} = C^*_{min} = C^{opt}_{\Lambda_2^m}$. Minimum equilibrium social cost is achieved in regime Λ_2, and it is equal to the socially optimal cost.

Proof. Since $h_e^s(\cdot)$ is a k-th order homogeneous function of w_e, we know from Euler's homogeneous function theorem that the marginal cost function can be written as:

$$z \cdot \frac{dh_e^s(z)}{dz} = k \cdot h_e^s(z).$$

Thus, the marginal cost function can be expressed as:

$$\tilde{c}_e^s(w_e) = h_e^s(w_e) + \beta_e^s + w_e \cdot \frac{\partial h_e^s(w_e)}{\partial w_e} = (k+1) \cdot h_e^s(w_e) + \beta_e^s.$$

For any feasible strategy profile $q \in \mathcal{Q}$, the expected social cost $C(q)$ in (6.1) can be written as:

$$\begin{aligned}
C(q) &= \frac{1}{\mathrm{D}} \sum_{e\in\mathcal{E}} \sum_{t^2\in\mathcal{T}^2} \sum_{t^1\in\mathcal{T}^1} \sum_{s\in\mathcal{S}} \pi(s,t^1,t^2) \int_0^{\sum_{r\ni e} q_r^1(t^1)+q_r^2(t^2)} (k+1)\cdot h_r^s(z)dz \\
&+ \frac{1}{\mathrm{D}} \sum_{e\in\mathcal{E}} \sum_{t^2\in\mathcal{T}^2} \sum_{t^1\in\mathcal{T}^1} \sum_{s\in\mathcal{S}} \pi(s,t^1,t^2) \int_0^{\sum_{r\ni e} q_r^1(t^1)+q_r^2(t^2)} \beta_e^s dz \\
&= \frac{k+1}{\mathrm{D}} \sum_{e\in\mathcal{E}} \sum_{t^2\in\mathcal{T}^2} \sum_{t^1\in\mathcal{T}^1} \sum_{s\in\mathcal{S}} \pi(s,t^1,t^2) \int_0^{\sum_{r\ni e} q_r^1(t^1)+q_r^2(t^2)} \left(h_r^s(z) + \beta_e^s\right) dz \\
&- \frac{k}{\mathrm{D}} \sum_{e\in\mathcal{E}} \sum_{t^2\in\mathcal{T}^2} \sum_{t^1\in\mathcal{T}^1} \sum_{s\in\mathcal{S}} \pi(s,t^1,t^2) \int_0^{\sum_{r\ni e} q_r^1(t^1)+q_r^2(t^2)} \beta_e^s dz \\
&= \frac{(k+1)}{\mathrm{D}} \Phi(q) - \frac{k}{\mathrm{D}} \sum_{s\in\mathcal{S}} \sum_{t\in\mathcal{T}} \pi(s,t) \cdot \left(\sum_{r\in\mathcal{R}} \left(\sum_{e\in r} \beta_e^s \right) \cdot f_r(t) \right). \qquad (6.3)
\end{aligned}$$

Following (6.3), when the free flow travel time is identical across all routes, $C(q)$ can be simplified as:

$$C(q) = \frac{(k+1)}{\mathrm{D}} \Phi(q) - k \cdot \left(\sum_{s\in\mathcal{S}} \left(\sum_{t^1\in\mathcal{T}^1} \sum_{t^2\in\mathcal{T}^2} \pi(s,t^1,t^2) \right) \cdot \beta^s \right) \overset{(3.1)}{=} \frac{(k+1)}{\mathrm{D}} \Phi(q) - k \cdot \left(\sum_{s\in\mathcal{S}} \theta(s) \cdot \beta^s \right).$$

Since the equilibrium strategy profile q^* minimizes $\Phi(q)$, q^* also minimizes $C(q)$. Thus, $C^*(\lambda) = C^{opt}(\lambda)$. The minimim equilibrium social cost is achieved in regime Λ_2. Therefore, $C^*_{\Lambda_2} = C^*_{min} = C^{opt}_{\Lambda_2^m}$. □

Fig. 6-2 shows the equilibrium social cost $C^*(\lambda)$ and $C^{opt}(\lambda)$ ($\mathrm{D} = 10$, $c_1^n(l) = l+15$, $c_1^a(l) = 3l+15$, $c_2(l) = 2l+20$; $p^1(t^1 = \mathbf{a}|s = \mathbf{a}, t^2) = p^1(t^1 = \mathbf{n}|s = \mathbf{n}, t^2) = 0.8$; $p^2(t^2 = \mathbf{a}|s = \mathbf{a}, t^1) = p^2(t^2 = \mathbf{n}|s = \mathbf{n}, t^1) = 0.6$). The thresholds are $\underline{\lambda} = 0.1521$, $\bar{\lambda} = 0.9662$. $C^{opt}(\lambda)$ is a lower bound of $C^*(\lambda)$. λ_{min} is in regime Λ_1.

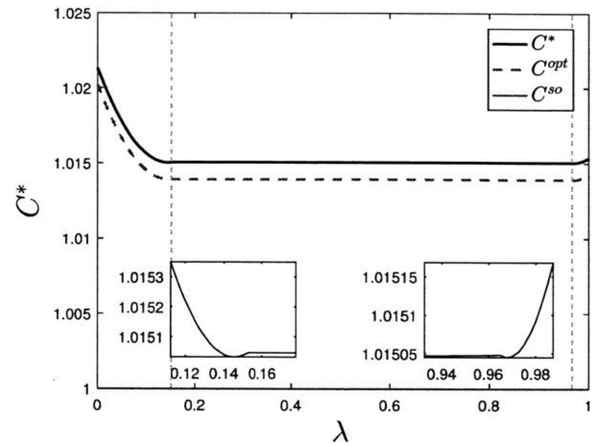


Figure 6-2: BWE social cost $C^*(\lambda)$ bounded by $C^{opt}(\lambda)$.

Fig. 6-3 presents the case where the free flow travel time on two routes are identical. (D = 1, $c_1^n(l) = l + 20$, $c_1^a(l) = 3l + 20$, $c_2(l) = 2l + 20$; $p^1(t^1 = \mathbf{a}|s = \mathbf{a}, t^2) = p^1(t^1 = \mathbf{n}|s = \mathbf{n}, t^2) = 0.8$; $p^2(t^2 = \mathbf{a}|s = \mathbf{a}, t^1) = p^2(t^2 = \mathbf{n}|s = \mathbf{n}, t^1) = 0.6$). The thresholds are $\underline{\lambda} = 0.1382$, $\bar{\lambda} = 0.9693$. The bound $C^{opt}(\lambda)$ is tight, and C^*_{min} is achieved in the full range of regime Λ_2.

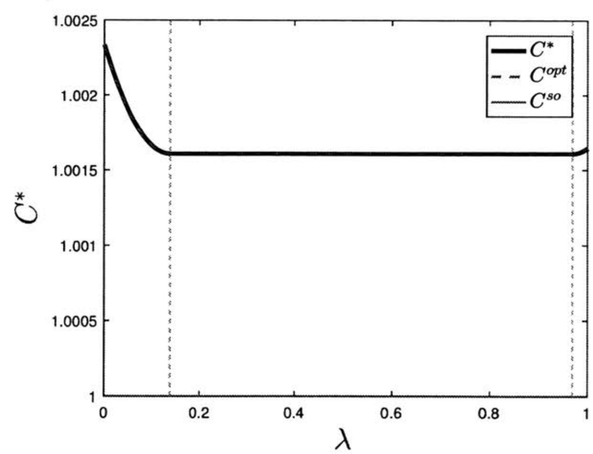


Figure 6-3: BWE social cost $C^*(\lambda)$

6.2 Bounds on the Inefficiency

We evaluate the inefficiency of BWE by the worst case of fraction $\frac{C^*(\lambda)}{C^{opt}(\lambda)}$. Since the equilibrium cost is unique, $\frac{C^*(\lambda)}{C^{opt}(\lambda)}$ is identical in all equilibria. Proposition 13 from Roughgarden and Tardos (2004) presents the bound of efficiency loss of Wardrop Equilibrium with complete information. We next show that the results in Proposition 13 hold in our game with heterogeneous information structure.

Define the pigou bound $\kappa(C)$ as:

$$\kappa(C) = \sup_{c \in C} \sup_{x \geq 0, y \geq 0} \frac{y \cdot c(y)}{x \cdot c(x) + (y - x) \cdot c(y)}, \tag{6.4}$$

where C is the set of edge cost function $\{c_e^s(w_e)\}_{e \in \mathcal{E}, s \in \mathcal{S}}$.

Proposition 9. *For any given λ, we have $\frac{C^*(\lambda)}{C^{opt}(\lambda)} \leq \kappa(C)$, and the bound is tight. If the edge cost functions $c_e^s(w_e)$ are affine functions for all $e \in \mathcal{E}, s \in \mathcal{S}$, we have $\frac{C^*(\lambda)}{C^{opt}(\lambda)} \leq \frac{4}{3}$.*

Proof. Proof of Proposition 9. Following Definition 1, we obtain:

$$\sum_{r \in \mathcal{R}} q_r^{*i}(t^i)\mathbb{E}[c_r(q^*)|t^i] = \lambda^i \mathrm{D} \cdot \min_{r \in \mathcal{R}} \mathbb{E}[c_r(q^*)|t^i] \leq \sum_{r \in \mathcal{R}} q_r^{opti}(t^i)\mathbb{E}[c_r(q^*)|t^i], \quad \forall t^i \in \mathcal{T}^i, \forall i \in \mathcal{I}. \tag{6.5}$$

Define f^{opt} as the route flow vector induced by optimal social cost strategy q^{opt}, and

w^{opt} as the induced edge load vector. Following (6.5), we have:

$$
\begin{aligned}
&\sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}}\pi(s,t)w_e^*(t)c_e^s(w_e^*(t)) = \sum_{s\in\mathcal{S}}\sum_{r\in\mathcal{R}}\sum_{t\in\mathcal{T}}\pi(s,t)f_r^*(t)\left(\sum_{e\in r}c_e^s\left(w_e^*(t)\right)\right)\\
=&\sum_{t^1\in\mathcal{T}^1}\Pr(t^1)\left(\sum_{r\in\mathcal{R}}q_r^{*1}(t^1)\mathbb{E}[c_r(q^*)|t^1]\right)+\sum_{t^2\in\mathcal{T}^2}\Pr(t^2)\left(\sum_{r\in\mathcal{R}}q_r^{*2}(t^2)\mathbb{E}[c_r(q^*)|t^2]\right)\\
\overset{(6.5)}{\leq}&\sum_{t^1\in\mathcal{T}^1}\Pr(t^1)\left(\sum_{r\in\mathcal{R}}q_r^{opt1}(t^1)\mathbb{E}[c_r(q^*)|t^1]\right)+\sum_{t^2\in\mathcal{T}^2}\Pr(t^2)\left(\sum_{r\in\mathcal{R}}q_r^{opt2}(t^2)\mathbb{E}[c_r(q^*)|t^2]\right)\\
=&\sum_{s\in\mathcal{S}}\sum_{r\in\mathcal{R}}\sum_{t\in\mathcal{T}}\pi(s,t)f_r^{opt}(t)\left(\sum_{e\in r}c_e^s(w_e^*(t))\right)=\sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}}\pi(s,t)w_e^{opt}(t)c_e^s(w_e^*(t)).
\end{aligned}
\tag{6.6}
$$

This leads to:

$$
\begin{aligned}
C^{opt}(\lambda) &= \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}}\pi(s,t)c_e^s\left(w_e^{opt}(t)\right)w_e^{opt}(t)\\
&=\sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}}\pi(s,t)\left[c_e^s\left(w_e^*(t)\right)w_e^*(t)\cdot\frac{w_e^{opt}(t)c_e^s\left(w_e^{opt}(t)\right)+\left(w_e^*(t)-w_e^{opt}(t)\right)c_e^s\left(w_e^*(t)\right)}{c_e^s\left(w_e^*(t)\right)w_e^*(t)}\right]\\
&+\sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}}\pi(s,t)\left(w_e^{opt}(t)-w_e^*(t)\right)c_e^s\left(w_e^*(t)\right)\\
&\overset{(6.6)}{\geq}\left(\sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}}\pi(s,t)c_e^s(w_e^*(t))w_e^*(t)\right)\\
&\cdot\left[\min_{s\in\mathcal{S},e\in\mathcal{E},t\in\mathcal{T}}\frac{w_e^{opt}(t)c_e^s\left(w_e^{opt}(t)\right)+\left(w_e^*(t)-w_e^{opt}(t)\right)c_e^s\left(w_e^*(t)\right)}{c_e^s\left(w_e^*(t)\right)w_e^*(t)}\right]\\
&\overset{(6.4)}{\geq}\frac{C^*(\lambda)}{\kappa(C)}.
\end{aligned}
$$

Thus, $\frac{C^*(\lambda)}{C^{opt}(\lambda)} \leq \kappa(C)$. Consider affine cost functions, for any $e \in \mathcal{E}$ and $s \in \mathcal{S}$, assuming $c_e^s(w_e) = \alpha_e^s w_e + \beta_e^s$, we have:

$$
\begin{aligned}
&w_e^{opt}(t)\left(c_e^s\left(w_e^*(t)\right)-c_e^s\left(w_e^{opt}(t)\right)\right)\\
=&\alpha_e^s w_e^{opt}(t)\left(w_e^*(t)-w_e^{opt}(t)\right)\leq\frac{1}{4}\alpha_e^s w_e^*(t)^2\\
\leq&\frac{1}{4}w_e^*(t)c_e^s\left(w_e^*(t)\right).
\end{aligned}
$$

Thus, we obtain:

$$
\begin{aligned}
C^*(\lambda) &= \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}} \pi(s,t) w_e^*(t) c_e^s(w_e^*(t)) \\
&\overset{(6.6)}{\leq} \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}} \pi(s,t) w_e^{opt}(t) c_e^s(w_e^*(t)) \\
&= \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}} \pi(s,t) w_e^{opt}(t) c_e^s(w_e^{opt}(t)) + \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}} \pi(s,t) w_e^{opt}(t) \left(c_e^s(w_e^*(t)) - c_e^s(w_e^{opt}(t))\right) \\
&\overset{(6.7)}{\leq} \sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}} \pi(s,t) w_e^{opt}(t) c_e^s(w_e^{opt}(t)) + \frac{1}{4}\sum_{s\in\mathcal{S}}\sum_{e\in\mathcal{E}}\sum_{t\in\mathcal{T}} \pi(s,t) w_e^*(t) c_e^s(w_e^*(t)) \\
&= C^{opt}(\lambda) + \frac{1}{4} C^*(\lambda).
\end{aligned}
$$

Therefore, $\frac{C^*(\lambda)}{C^{opt}(\lambda)} \leq \frac{4}{3}$. Since the game with one population, one state is a special case of our game, the bound is tight follows Proposition 13. □

Chapter 7

Conclusion

7.1 Summary of the Results

This thesis models network routing in a heterogeneous information environment as a Bayesian congestion game. We study how does Bayesian Wardrop equilibrium change when the relative size of two populations with asymmetric information changes. Our results hold for a general single o-d network with state dependent increasing edge cost functions and an information environment induced by any two different information systems with a common prior. Due to the existence of common prior, the Bayesian congestion game has a weighted potential function, and the set of Bayesian Wardrop equilibria can be solved as the optimal solution of a convex optimization program. The equilibrium edge load is unique. We show that the qualitative properties of equilibrium strategies change as the relative population size changes, resulting in three distinct regimes. Interestingly, the equilibrium edge load does not vary with the relative population size, and both populations face identical cost in equilibrium if and only if the relative population size is in the intermediate regime. In the other two regimes, the "minor" population has lower cost in equilibrium. Additionally, we define a metric to evaluate the impact of information. The impact of information on the minor population is tightly bounded by its size in the two side regimes, which results in the effect of relative population size on equilibrium edge load and costs. Finally, we provide the bounds on the equilibrium social cost, and a sufficient condition for the bounds to be tight. We obtain the same worst case inefficiency of equilibrium as the well-known results in the literature for complete information games.

The results in the thesis are applicable to games with more than two populations. Three equilibrium regimes exist when we perturb the relative population size between any two populations, and keep the sizes of the remaining populations as constants. Analogously, there is an intermediate region of population sizes, where the equilibrium edge load does not depend on the population sizes, and the costs of all populations are identical in equilibrium. If one population is uninformed, all the other populations have no higher cost than the uninformed population regardless of the population sizes. Furthermore, our results can be extended to the case where the edge cost functions are non-increasing instead of increasing. The equilibrium edge load may not be unique, but the cost on each edge is unique in equilibrium. The equilibrium characterization, impact of information and cost analysis follow directly from our discussion in Chapter 4- Chapter 6. The expression of regime thresholds do not change, but the computation is more complex, since the edge load in the intermediate regime may not be unique.

7.2 Future Work

Based upon the current results, we propose the following directions for future work: First, common prior may not exist in real world applications as populations may not know the accuracy of the information received by others. It would be interesting to study how robust our results in the information environment without common prior.

Another extension of our work is to take into account the off-equilibrium strategy and the learning procedure. Using a dynamic model, we could study if travelers can learn towards equilibrium when the game is repeated and the realization of cost is stochastic.

Appendix A

Review of Perturbation Analysis

We provide a brief review on results of perturbation analysis of optimization problem. For detailed discussion, see Fiacco and Kyparisis (1986), Fiacco (2009), Milgrom and Segal (2002), Bonnans and Shapiro (2013) and Wachsmuth (2013). We first define the parametric nonlinear optimization problem as follows:

$$\begin{aligned} &\min_q \Phi(q,\lambda) \\ s.t.\quad & g_i(q,\lambda) \geq 0, \quad i = 1,\dots,m, \\ & h_j(q,\lambda) = 0 \quad j = 1,\dots,p, \\ & \lambda \in [0,1], \end{aligned} \tag{A.1}$$

The Lagrange function associated with (A.1) is:

$$L(q,\mu,\nu,\lambda) = \Phi(q,\lambda) - \sum_{i=1}^{m} \mu_i g_i(q,\lambda) + \sum_{j=1}^{p} \nu_j h_j(q,\lambda). \tag{A.2}$$

We define the set of optimal solutions as $\mathcal{Q}^*(\lambda)$. For any optimal solution $q^* \in \mathcal{Q}^*(\lambda)$, we define $M(q^*)$ and $N(q^*)$ as the set of Lagrange multiplier μ^* and ν^* associated with the optimal solution q^*.

We first review the constraint qualification that ensures uniqueness of Lagrange multipliers at optimum.

Definition 3. *(Wachsmuth (2013))* Linear independence constraint qualification, *i.e.*

$LICQ(q,\lambda)$, holds at q if the set of vectors $\{\nabla_q g_i(q,\lambda) | g_i(q,\lambda) = 0, i = 1, \ldots m\}$ and $\{\nabla_q h_j(q,\lambda)\}_{j=1,\ldots,p}$ are linearly independent.

Proposition 10. *(Theorem 2 in Wachsmuth (2013)) For any optimal solution $q^* \in \mathcal{Q}^*(\lambda)$, if $LICQ(q^*,\lambda)$ holds, $M(q^*)$ and $N(q^*)$ are singleton sets.*

The next two propositions show the convexity and directional derivative of $\Psi(\lambda)$.

Definition 4. *The directional derivative of $\Psi(\lambda)$ at λ in direction z is:*

$$D_z\Psi(\lambda) = \lim_{\epsilon \to 0^+} \frac{\Psi(\lambda + \epsilon z) - \Psi(\lambda)}{\epsilon}.$$

Proposition 11. *(Corollary 2.2 in Fiacco and Kyparisis (1986)) The optimal value function $\Psi(\lambda)$ is convex with respect to λ if $\Phi(q,\lambda)$ and $-g_i(q,\lambda)$ are jointly convex in q and λ for any $i = 1, \ldots, m$, $h_j(q,\lambda)$ is linearly affine in q for any $j = 1, \ldots, p$.*

Proposition 12. *(Proposition 6 in Fiacco (2009)) Assume that $\Phi(q,\lambda)$ is convex in q for each $\lambda \in [0,1]$, and the problem functions are once continuously differentiable in q and λ. If $\lambda \in (0,1)$ and the set of points satisfying KKT condition is nonempty and bounded, then in a neighborhood of λ, $\Psi(\lambda)$ is continuously and directionally differentiable in λ and in any direction z:*

$$D_z\Psi(\lambda) = \min_{q^* \in \mathcal{Q}^*(\lambda)} \max_{\substack{(\mu^*,\nu^*) \\ \in (M(q^*),N(q^*))}} \nabla_\lambda L(q^*,\mu^*,\nu^*,\lambda) z, \tag{A.3}$$

where $M(q^*)$ and $N(q^*)$ are the sets of Lagrange multiplier μ^* and ν^* associated with the optimal solution $q^* \in \mathcal{Q}^*(\lambda)$.

Appendix B

Review of Price of Anarchy

We provide a brief characterization of efficiency bound of Wardrop Equilibrium. For detailed analysis, see Roughgarden and Tardos (2004). Define the pigou bound $\kappa(C)$ as:

$$\kappa(C) = \sup_{c \in C} \sup_{q \geq 0, y \geq 0} \frac{y \cdot c(y)}{q \cdot c(q) + (y - q) \cdot c(y)},$$

where C is the set of cost functions.

Proposition 13. *Roughgarden and Tardos (2004) The worst case ratio of equilibrium cost and social optimal cost* $\frac{C^*}{C^{opt}(\lambda)} \leq \kappa(C)$*, and the bound is tight. If* C *contains all affine functions, we have* $\frac{C^*}{C^{opt}(\lambda)} \leq \frac{4}{3}$.

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